USING CHAINS OF BOXES TO RECOGNIZE STAIRCASE STARSHAPED SETS IN $\mathbb{R}^d$

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ABSTRACT. Let $\mathcal{C}$ be a finite family of boxes in $\mathbb{R}^d, d \geq 3$, with $S = \cup\{C : C \text{ in } \mathcal{C}\}$ connected and $p \in S$. Assume that, for every geodesic chain D of $\mathcal{C}$-boxes containing p, each coordinate projection $\pi(D)$ of D is staircase starshaped with $\pi(p) \in \text{Ker } \pi(D)$. Then S is staircase starshaped and $p \in \text{Ker } S$. For n fixed, $1 \leq n \leq d-2$, an analogous result holds for composites of n coordinate projections of D into $(d-n)$-dimensional flats.

1. INTRODUCTION

We begin with some definitions from [2]. A set B in $\mathbb{R}^d$ is called a *box* if and only if B is a convex polytope (possibly degenerate) whose edges are parallel to the coordinate axes. A set S in $\mathbb{R}^d$ is an *orthogonal polytope* if and only if S is a connected union of finitely many boxes. Let λ be a simple polygonal path in $\mathbb{R}^d$ whose edges are parallel to the coordinate axes. For x, y in S, the path λ is called an *$x - y$ path* in S if and only if λ lies in S and has endpoints x and y. In this case, $\lambda(x, y)$ will represent the path λ, ordered from x to y. The path λ is an *$x - y$ geodesic* in S if and only if λ is an $x - y$ path of minimal length in S. (Clearly an $x - y$ geodesic need not be unique.) The path $\lambda(x, y)$ is a *staircase path* (or simply a *staircase*) if and only if no two of its edges have opposite directions. That is, for each standard basis vector $e_i, 1 \leq i \leq d$, all the edges of $\lambda(x, y)$ parallel to e_i have the same direction. For convenience of notation, we use e_i or $-e_i$ to indicate the associated direction. Clearly if $\lambda(x, y)$ is a staircase path in S, then λ is an $x - y$ geodesic.

For points x and y in a set S, we say *x sees y* (x is *visible* from y) *via staircase paths* if and only if there is a staircase path in S that contains both x and y. A set S is *staircase convex (orthogonally convex)* if and only if, for every pair x, y in S, x sees y via staircase paths. Similarly, a set S is *staircase starshaped (orthogonally starshaped)* if and only if, for some point

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p in S, p sees each point of S via staircase paths. The set of all such points p is the *staircase kernel* of S, denoted Ker S.

We will use a few standard terms from graph theory. For $F = \{C_1, \ldots, C_n\}$ a finite collection of distinct sets, the *intersection graph* G of F has vertex set $\{c_1, \ldots, c_n\}$. Further, for $1 \leq i < j \leq n$, the points c_i, c_j determine an edge in G if and only if the corresponding sets C_i, C_j in F have a nonempty intersection. A graph G is a *tree* if and only if G is connected and acyclic. A sequence $v_1, \ldots, v_n$ of vertices in G is a *walk* if and only if each consecutive pair v_i, v_{i+1} determines an edge of $G, 1 \leq i \leq k-1$. A walk is a *path* if and only if its points are distinct.

For $B_1, \ldots, B_n$ a collection of distinct boxes in $\mathbb{R}^d$, we say that their union is a *chain* of boxes (relative to our ordering) if and only if the intersection graph of $\{B_1, \ldots, B_n\}$ is the path $b_1, \ldots, b_n$ (where b_i represents the set B_i in the intersection graph, $1 \leq i \leq n$). That is, relative to our labeling, for $1 \leq i < j \leq k, B_i \cap B_j \neq \emptyset$ if and only if $j = i+1$. Finally, for $\mathcal{C}$ a finite family of boxes in $\mathbb{R}^d$ and $S = \cup\{C : C \text{ in } \mathcal{C}\}$, a chain A of boxes from $\mathcal{C}$ is called a *geodesic chain in* S if and only if, for some x, y in A and some $x - y$ geodesic $\lambda(x, y)$ in S, A contains $\lambda(x, y)$ and no subchain of A contains $\lambda(x, y)$. Certainly if A is a geodesic chain in S, so are its subchains.

Many results in convexity that involve the usual notion of visibility via straight line segments have interesting analogues that instead use the idea of visibility via staircase paths. For instance, the familiar Krasnosel'skii theorem [7] says that, for a nonempty compact set S in the plane, S is starshaped via segments if and only if every three points of S see via segments in S a common point. In the staircase analogue [1], for a nonempty simply connected orthogonal polygon S in $\mathbb{R}^2, S$ is staircase starshaped if and only if every two points of S see via staircase paths in S a common point. Further, in an interesting study involving rectilinear spaces, Chepoi [3] has generalized the planar result to any set $S = \cup\{C : C \text{ in } \mathcal{C}\}$, where $\mathcal{C}$ is a finite family of distinct boxes in $\mathbb{R}^d$ whose corresponding intersection graph is a tree. In [2], related results are established for such a set S, and some of these results concern chains of boxes and their projections into appropriate hyperplanes. Here we remove the requirement that the intersection graph of $\mathcal{C}$ be a tree and obtain sufficient conditions for set S to be staircase starshaped with a specified point p in its kernel.

We will use the following terminology. We call each of the hyperplanes $\{(x_1, \ldots, x_d) : x_i = 0\}, 1 \leq i \leq d,$ a *coordinate hyperplane*. Similarly, any intersection of coordinate hyperplanes will be a *coordinate flat*. Any projection of $\mathbb{R}^d$ onto a coordinate hyperplane will be a *coordinate projection*. For $d \geq 3, n$ any fixed integer, $1 \leq n \leq d-2$, and $\pi_1, \ldots, \pi_n$ any n distinct coordinate projections, we say that the composite function $\psi = \pi_1 \cdot \ldots \cdot \pi_n$

mapping $\mathbb{R}^d$ onto a coordinate $(d-n)$- flat is an n- *composite projection.* Readers may refer to Valentine [9], to Lay [8], to Danzer, Grünbaum, Klee [4], and to Eckhoff [5] for discussions concerning visibility via straight line segments and starshaped sets. Readers may refer to Harary [6] for information on intersection graphs, trees, and other graph theoretic concepts.

2. THE RESULTS.

In the paper, we refer to several results from [2]. For completeness, we include these as Results 1, 2, and 3 below.

Result 1 [2, Lemma 2]. *For $d \geq 2$ and for each $i, 1 \leq i \leq d$, let π_i denote the coordinate projection from $\mathbb{R}^d$ onto the coordinate hyperplane $\{(x_1, \ldots, x_d) : x_i = 0\}$. Let A, C be boxes in $\mathbb{R}^d$. If $A \cap C = \emptyset$, then for at least $d-1$ of the projections $\pi_i, 1 \leq i \leq d, \pi_i(A) \cap \pi_i(C) = \emptyset$.*

Result 2 [2, Theorem 3]. *Let $A \equiv B_1 \cup \ldots \cup B_k$ be a chain of boxes in $\mathbb{R}^d, d \geq 3$. The chain A is staircase convex if and only if, for every subchain D of A, each projection of D into a coordinate hyperplane is staircase convex.*

Result 3 [2, Corollary 3.1]. *For $d \geq 3$, let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \ldots \cup C_n$. The set S is staircase convex if and only if, for every chain A of boxes in $\mathcal{C}$, each projection of A into a coordinate hyperplane is staircase convex.*

We begin our discussion with some easy observations.

Observation 1. Let $\mathcal{C}$ be a finite family of distinct boxes in $\mathbb{R}^d$, with $S = \cup\{C : C \text{ in } \mathcal{C}\}$, and let $p, q \in S$. If S contains a $p-q$ staircase, then every $p-q$ geodesic in S is a staircase.

Observation 2. If A, B are boxes in $\mathbb{R}^d$ and $A \cap B \neq \emptyset$, then $A \cup B$ is staircase convex.

Lemma 1. *Let $\mathcal{C}$ be a finite family of distinct boxes in $\mathbb{R}^d$, with $S = \cup\{C : C \text{ in } \mathcal{C}\}$ connected. For every p, q in S, there exist a $p-q$ geodesic λ_0 in S and a corresponding subcollection $\{C_1, \ldots, C_{k_0}\}$ of $\mathcal{C}$ such that $\lambda_0 \subseteq C_1 \cup \ldots \cup C_{k_0}$ and $C_1 \cup \ldots \cup C_{k_0}$ is a chain of boxes.*

Proof. For each $p-q$ geodesic λ in S, there are finitely many subfamilies $\mathcal{C}_\lambda$ of $\mathcal{C}$ for which $\lambda_0 \subseteq \cup\{C : C \text{ in } \mathcal{C}_\lambda\}$. Fix $\mathcal{C}_\lambda$. For each C in $\mathcal{C}_\lambda$, consider the finite collection $\mathcal{K}(C)$ of components of $C \cap \lambda$. Then examine all such components for all C in $\mathcal{C}_\lambda$. That is, examine $\{A_C : A_C \text{ in } \mathcal{K}(C)$ for some C in $\mathcal{C}_\lambda\}$. Each set A_C is a connected subpath of λ of the form $\lambda(a_{A_C}, b_{A_C})$ for a_{A_C}, b_{A_C} on $\lambda(p, q)$, where a_{A_C} precedes (or equals) b_{A_C} relative to our order from p to q. Now select a subfamily of these components A_C whose union contains λ such that the subfamily consists of as few sets as possible.

Observe that the corresponding a_{A_C} points will be distinct, as will the b_{A_C} points. Moreover, if a_{A_C} precedes a_{A_D} on $\lambda(p,q)$, then b_{A_C} precedes b_{A_D} as well (for selected sets A_C and A_D).

For the selected components A_C, list the associated points a_{A_C} in the order established along $\lambda(p,q)$ from p to q. For each a_{A_C} in our ordered list, pass to the corresponding set C. Preserving the order established above, label the selected C sets by $C'_1, \ldots, C'_k$. Of course, at this point, there is no guarantee that the sets $C'_1, \ldots, C'_k$ should be distinct. However, consecutive sets in our ordering will be distinct. Denote the k above by $k(\mathcal{C}_\lambda)$.

For each $p-q$ geodesic $\lambda(p,q)$, consider all associated families C_λ described above and corresponding numbers $k(\mathcal{C}_\lambda)$. Select $k(\mathcal{C}_\lambda)$ as small as possible (for all $\mathcal{C}_\lambda$), and call it $k(\lambda)$. Let k_0 denote the smallest member of $\{k(\lambda) : \lambda \text{ a } p-q \text{ geodesic in } S\}$. Finally, choose a $p-q$ geodesic λ_0 for which $k(\lambda_0) = k_0$, and choose an associated family $\{C_1, \ldots, C_{k_0}\}$ of labeled sets selected according to the scheme described in the previous paragraph (for some $\mathcal{C}_{\lambda_0}$).

We assert that λ_0 and $C_1, \ldots, C_{k_0}$ satisfy the lemma. Certainly $\lambda_0 \subseteq C_1 \cup \ldots \cup C_{k_0}$. We must show that $C_1 \cup \ldots \cup C_{k_0}$ is a chain. That is, we must show that the boxes are distinct and, for $1 \le i < j \le k_0, C_i \cap C_j \ne \emptyset$ if and only if $j = i+1$. If $k_0 = 1$, the result is trivial, so assume that $k_0 \ge 2$. To begin, observe that nonconsecutive sets are disjoint: For $k_0 \ge 3$, consider sets C_i and C_j, where $2 \le i+1 < j \le k_0$. Suppose on the contrary that $C_i \cap C_j \ne \emptyset$. Then by Observation 2 $C_i \cup C_j$ would be staircase convex. The geodesic λ from its first point in C_i to its last point in C_j could be replaced by a staircase in $C_i \cup C_j$ to yield a new $p-q$ path μ. Since every staircase is a geodesic, this replacement would not increase the length of our path. That is, the length of μ would not exceed the length of λ, so μ would be a $p-q$ geodesic in S. Applying our earlier argument to the family $C_1, \ldots, C_i, C_j, \ldots, C_{k_0}$ would yield a family of at most $k_0 - 1$ labeled sets satisfying our requirements. Then $k(\mu) \le k_0 - 1$, contradicting the minimality of k_0. Our supposition must be false, and nonconsecutive sets C_i, C_j in our ordering must be disjoint. Of course, this implies that nonconsecutive C_i, C_j are distinct as well. Since consecutive sets are distinct, the boxes $C_1, \ldots, C_k$ are distinct.

It remains to show that consecutive sets C_i, C_{i+1} have a nonempty intersection for $1 \le i \le k_0 - 1$. Since each of the boxes C_i, C_{i+1} appears exactly once in our list, by earlier comments, the first point a_i of $\lambda_0 \cap C_i$ precedes the first point a_{i+1} of $\lambda_0 \cap C_{i+1}$ and a_i is not in C_{i+1}. Similarly, the last point b_{i+1} of $\lambda_0 \cap C_{i+1}$ follows the last point b_i of $\lambda_0 \cap C_i$, and b_{i+1} is not in C_i. The associated order on λ_0 is $a_i, a_{i+1}, b_i, b_{i+1}$ and $\lambda_0(a_i, b_{i+1}) \subseteq C_i \cup C_{i+1}$. Since $\lambda_0(a_i, b_{i+1})$ is connected, the boxes C_i and C_{i+1} cannot be disjoint. We conclude that, for $1 \le i < j \le k_0, C_i \cap C_j \ne \emptyset$

if and only if $j = i + 1$, and $C_1 \cup \ldots \cup C_{k_0}$ is a chain, finishing the proof of Lemma 1. □

Lemma 2 is a staircase starshaped analogue of Result 2 [2, Theorem 3].

Lemma 2. *Let A be a chain of boxes in $\mathbb{R}^d$, $d \geq 3$, with point p in A. The chain A is staircase starshaped with $p \in$ Ker A if and only if, for each subchain D of A containing p, each projection $\pi(D)$ of D into a coordinate hyperplane is staircase starshaped with $\pi(p) \in$ Ker $\pi(D)$.*

Proof. To establish the necessity, assume that $p \in$ Ker A, and let D be any subchain of A containing p. Let $\pi(D)$ denote the projection of D into a coordinate hyperplane, and for convenience assume that this hyperplane is defined by $\{(x_1, \ldots, x_d) : x_1 = 0\}$. To show that $\pi(p) \in$ Ker $\pi(D)$, select any point $x' \in \pi(D)$. Then $x' = \pi(x)$ for some x in D. Since $p \in$ Ker A, A contains a $p - x$ staircase path λ, and since D is a subchain of chain A, $\lambda \subseteq D$. It is easy to see that $\pi(\lambda)$ defines a staircase from $\pi(p)$ to $\pi(x)$ in $\pi(D)$: Vectors in λ parallel to the x_1-axis map to singleton sets in $\pi(D)$. Each remaining vector $\vec{v}$ in λ maps to a vector in $\pi(A)$ parallel to $\vec{v}$ and having the same direction as $\vec{v}$.

To prove the converse, suppose that, for each subchain D of A containing p, each coordinate projection $\pi(D)$ is staircase starshaped with $\pi(p) \in$ Ker $\pi(D)$, to prove that $p \in$ Ker A. We will use induction on the number of boxes in the chain A. If A is a chain of one or two boxes, then A is staircase convex, and the result is trivial. Inductively, assume that the result holds for chains of $k - 1$ or fewer boxes, $k \geq 3$, to prove for a chain of k boxes. Let A be the chain $B_1 \cup \ldots \cup B_k$ to show that A is staircase starshaped at p. Without loss of generality, assume that one of p or x, say p, belongs to $B_1 \setminus B_2$, while the other point x belongs to $B_k \setminus B_{k-1}$. (Otherwise, p and x would lie in a subchain of A having at most $k - 1$ boxes, and the result would follow from our induction hypothesis.) Let $\lambda = \lambda(p, q)$ be a $p - x$ geodesic in A, and let z, z', respectively, denote the first and last points of $\lambda(p, q)$ in B_{k-1}. By our induction hypothesis, $B_1 \cup \ldots \cup B_{k-1}$ is staircase starshaped at p. Hence $B_1 \cup \ldots \cup B_{k-1} \subseteq A$ contains a $p - z'$ staircase, and by Observation 1, every $p - z'$ geodesic in A is a staircase. Therefore, $\lambda(p, z')$ is a staircase. Since $B_{k-1} \cup B_k$ is staircase convex, $\lambda(z, x)$ is also a staircase. We will show that $\lambda(p, z') \cup \lambda(z', x) = \lambda(p, x)$ is a staircase.

Suppose on the contrary that $\lambda(p, x)$ is not a staircase path. For convenience of notation, assume that the staircase $\lambda(p, z')$ uses vectors in the directions $e_1, \ldots, e_j$ for some $j, 1 \leq j \leq d$. If $\lambda(p, x)$ is not a staircase, then for at least one $e_i, 1 \leq i \leq j$, staircase $\lambda(z', x)$ in B_k must use a vector in direction $-e_i$. Without loss of generality, say $i = 1$. In at least one box $B_m, 1 \leq m \leq k - 1, \lambda$ must use a vector in direction e_1. Observe that $m \neq k - 1$, for if λ used direction e_1 in B_{k-1} and direction $-e_1$ in B_k, then

$\lambda(z,x)$ would not be a staircase. Thus $1 \le m \le k-2$, so by the definition of chain, $B_m \cap B_k = \emptyset$. By Result 1 [2, Lemma 2], for at least $d-1$ of the coordinate projections $\pi_i, 1 \le i \le d, \pi_i(B_m) \cap \pi_i(B_k) = \emptyset$. Since $d \ge 3$, we may choose such a projection π_i with $i \ne 1$. For convenience, assume $i = 2$ and $\pi_2(B_m) \cap \pi_2(B_k) = \emptyset$. But then the corresponding projection $\pi_2(B_1 \cup \ldots \cup B_k)$ of our chain cannot be staircase starshaped at $\pi_2(p)$, since any $\pi_2(p) - \pi_2(x)$ geodesic will require at least one vector in direction e_1 (to travel from $\pi_2(p)$ to $\pi_2(B_{m+1})$) and at least one vector in direction $-e_1$ (to travel from $\pi_2(B_{m+1})$ to $\pi_2(x)$). We have contradicted our hypothesis. Our supposition is false, and $\lambda(p, x)$ is a staircase path, finishing the induction and completing the proof of Lemma 2. □

Corollary 0. *Let n be any fixed integer,* $1 \le n \le d-2$. *Let A be a chain of boxes in* $\mathbb{R}^d, d \ge 3$, *with point p in A. The chain A is staircase starshaped with* $p \in$ *Ker A if and only if, for each subchain D of A containing p, each n-composite projection* $\psi(D)$ *of D into a coordinate* $(d-n)$*-flat is staircase starshaped with* $\psi(p) \in$ *Ker* $\pi(D)$.

Proof. The proof is a direct analogue of the proof of Lemma 2. The main difference occurs in the second part of the proof, and we briefly discuss the needed adjustment: Following the argument and the notation of Lemma 2, we may use Result 1 [2, Lemma 2] to select an appropriate $n \le d-2$ coordinate projections $\pi_i, 1 \le i \le n$, with $\pi_i(B_m) \cap \pi_i(B_k) = \emptyset$ and $i \ne 1$. This allows us to obtain an n-composite projection that violates our hypothesis. □

We are ready for the following result.

Theorem 1. *Let* $\mathcal{C}$ *be a finite family of boxes in* $\mathbb{R}^d, d \ge 3$, *with* $S = \cup\{C : C \text{ in } \mathcal{C}\}$ *connected and* $p \in S$. *Assume that, for every geodesic chain D of* $\mathcal{C}$ *boxes containing p, each coordinate projection* $\pi(D)$ *of D is staircase starshaped with* $\pi(p) \in$ *Ker* $\pi(D)$. *Then S is staircase starshaped and* $p \in$ *Ker S.*

Proof. By Lemma 1, for every x in S, there exist a $p - x$ geodesic $\lambda_0(p,x)$ in S and a corresponding chain A of $\mathcal{C}$-boxes such that $\lambda_0(p,x) \subseteq A$. Certainly we may assume that A is a geodesic chain, as are its subchains. By hypothesis, for A and for each of its subchains D containing p, each coordinate projection $\pi(D)$ of D is staircase starshaped with $\pi(p) \in$ Ker $\pi(D)$. Hence by Lemma 2, $p \in$ Ker A. It follows that S contains a $p - x$ staircase, and by Observation 1, geodesic $\lambda_0(p,x)$ is a staircase path. Therefore, p sees each x in S via staircase paths, and $p \in$ Ker S. □

Corollary 1. *Let n be any fixed integer,* $1 \le n \le d-2$. *Let* $\mathcal{C}$ *be a finite family of boxes in* $\mathbb{R}^d, d \ge 3$, *with* $S = \cup\{C : C \text{ in } \mathcal{C}\}$ *connected and* $p \in S$.

Assume that, for every geodesic chain D of $\mathcal{C}$-boxes containing p and each n- composite projection $\psi(D)$ of D into a coordinate $(d-n)$-flat, $\psi(D)$ is staircase starshaped with $\psi(p) \in Ker\ \psi(D)$. Then S is staircase starshaped and $p \in Ker\ S$.

Proof. The argument parallels the proof of Theorem 1, using Corollary 0 to Lemma 2 instead of Lemma 2 itself. □

Finally, Corollary 2 is an analogue of Result 3 [2, Corollary 3.1].

Corollary 2. *Let n be any fixed integer, $1 \le n \le d-2$. Let $\mathcal{C}$ be a finite family of boxes in $\mathbb{R}^d$, $d \ge 3$, with $S = \cup\{C : C \text{ in } \mathcal{C}\}$ connected. Assume that, for every geodesic chain D of $\mathcal{C}$-boxes and each n-composite projection $\psi(D)$ of D into a coordinate $(d-n)$-flat, $\psi(D)$ is staircase convex. Then S is staircase convex, too.*

Proof. By Corollary 1 above, for every point p in S, $p \in$ Ker S. Hence $S =$ Ker S and S is staircase convex. □

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