Isolated toughness condition for a graph to be a fractional (g, f, n)-critical graph *

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Abstract: Let $i(G)$ be the number of isolated vertices in graph G. The isolated toughness of G is defined as $I(G) = +\infty$ if G is complete; $I(G) = \min\{|S|/i(G-S) : S \subseteq V(G), i(G-S) \geq 2\}$ otherwise. In this paper, we determine that G is a fractional (g, f, n)-critical graph if $I(G) \geq \frac{b^2+bn-1}{a}$ if $b > a$; $I(G) \geq b + n$ if $a = b$.

Key words: isolated toughness, fractional (g, f)-factor, fractional (g, f, n)-critical graph

1 Introduction

The graphs considered here are finite and simple. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $x \in V(G)$, we denote by $d_G(x)$ and $N_G(x)$ the degree and the neighborhood of x in G, respectively. Let $\delta(G)$ denote the minimum degree of G. For any $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. Let $i(G-S)$ be the number of isolated vertices in $G - S$. The readers can refer to [1] for standard graph theoretic concepts and terms used but undefined in this paper.

Let g and f be two integer-valued functions on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a

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(g,f)-*factor* if $g(x) \le d_F(x) \le f(x)$ for evert vertex $x \in V(G)$. A *fractional* (g,f)-*factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \le \sum_{e\in E(x)} h(e) \le f(x)$. If $g(x) = a$, $f(x) = b$ for all $x \in V(G)$, then a fractional (g,f)-factor is a fractional $[a,b]$-factor. Moreover, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer throughout this paper, and we will not reiterate it again) for all $x \in V(G)$, then a fractional (g,f)-factor is just a fractional k-factor.

Liu and Zhang [2] gave the result for the existence of a fractional (g,f)-factor. Several characterization of fractional (g,f)-factors due to Liu and Zhang [2, 3]. A graph G is called a *fractional* (g,f,n)-*critical graph* if after deleting any n vertices from G, the resulting graph still has a fractional (g,f)-factor. Similarly, a graph G is called a (g,f,n)-*critical graph* if after removing any n vertices from G, the resulting graph admits a (g,f)-factor. Some sufficient conditions for (a,b,n)-critical graphs can refer [9] and [10].

A milestone result on fractional (g,f,n)-critical graph was obtained by Liu [5], and its equal version can be stated as follows.

Lemma 1 (Liu [5]) *Let G be a graph and let g,f be two non-negative integer-valued functions defined on $V(G)$ satisfying $g(x) \le f(x)$ for all $x \in V(G)$. Let n be a non-negative integer. Then G is a fractional (g,f,n)-critical graph if and only if*

$$f(S) - g(T) + d_{G-S}(T) \ge \max\{f(U) : U \subseteq S, |U| = n\} \qquad (1)$$

for any disjoint subsets S and T of $V(G)$ with $|S| \ge n$.

Yang et al. [8] introduced the concept of isolated toughness $I(G)$ of graph G as follows. If G is not complete,

$$I(G) = \min\{\frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \ge 2\}.$$

Otherwise, $I(G) = +\infty$.

Ma and Liu [7] confirmed G is fractional k-factor if $I(G) \ge k$ and $\delta(G) \ge k$, and showed that the result is sharp. Recently, in [6], Liu studied a isolated toughness condition for graphs to be fractional (g,f,n)-critical. It is determined that G is a fractional (g,f,n)-critical graph if $a \equiv b \pmod 2$, $\delta(G) \ge \frac{[(a+b)^2+2(b-a)](n+1)}{4a}$ and $I(G) \ge \frac{[(a+b)^2+2(b-a)](n+1)}{4a}$; or $a \not\equiv b \pmod 2$, $\delta(G) \ge \frac{[(a+b)^2+2(b-a)+1](n+1)}{4a}$ and $I(G) \ge \frac{[(a+b)^2+2(b-a)+1](n+1)}{4a}$. However, the author of [6] didn't know whether the condition is best or not. The question which tight isolated toughness condition for fractional (g,f,n)-critical graphs is subject of a open problem. It motivates considering the better $I(G)$ for fractional (g,f,n)-critical graphs, and we are interested in deriving such isolated toughness bound. Our main result to be proved in next section can be stated as follows.

Theorem 2 *Let G be a graph, n be a non-negative integer, g, f be two non-negative integer-valued functions on $V(G)$, and $a \le g(x) \le f(x) \le b$ for all $x \in V(G)$, where a, b are two integers with $1 \le a \le b$ and $b \ge 2$. $\delta(G) \ge \frac{bn}{a} + \frac{(b+1)^2}{4a} + b - 1$. If G satisfies isolated toughness*

$$I(G) \ge \begin{cases} \frac{b^2+bn-1}{a}, & \text{if } b > a \\ b+n, & \text{if } a = b. \end{cases}$$

Then, G is a fractional (g, f, n)-critical graph.

The proof strategy is similar to the one in Liu and Zhang [4] but we need to cope with the more detail case now and hence new methods are necessary. Begin on the way to the proof of Theorem 2, we would like to show some useful lemmas.

Lemma 3 (Liu and Zhang [4]) *Let G be a graph and let $H = G[T]$ such that $\delta(H) \ge 1$ and $1 \le d_G(x) \le k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \ge 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1}(k-j)c_j \le \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for every $j = 1, \ldots, k-1$.

The lemma below can be deduced from Lemma 2.2 in [4].

Lemma 4 (Liu and Zhang [4]) *Let G be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \ge 2$. Then there exists a maximal independent set I and the covering set $C = V(H) - I$ of H satisfying*

$$|V(H)| \le \sum_{i=1}^{k}(k-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

and

$$|C| \le \sum_{i=1}^{k}(k-i)|I^{(i)}| - \frac{|I^{(1)}|}{2},$$

where $I^{(i)} = \{x \in I, d_H(x) = k-i\}$, $1 \le i \le k$ and $\sum_{i=1}^{k} |I^{(i)}| = |I|$.

2 Proof of Theorem 2

The aim of this Section is to prove our main result. We only verify that Theorem 2 holds when $b > a$, because the case analysis for $a = b$ is similar to that of the case $b > a$. We always assume that G is not complete since the result for complete graph immediately follows from $\delta(G) \geq \frac{bn}{a} + \frac{(b+1)^2}{4a} + b - 1$.

Suppose that G is a counter-example of Theorem 2, then G satisfies the conditions of Theorem 2, but exists subsets S and T of $V(G)$ such that

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \leq f(S) - g(T) + d_{G-S}(T) < bn. \qquad (2)$$

We choose subsets S and T such that $|T|$ is minimum. Obviously, $T \neq \emptyset$. If $d_{G-S}(x) \geq g(x)$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2), which contradicts the choice of S and T. Which led to $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for any $x \in T$.

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{x \in V(H') | d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_b.

If $|V(H)| = 0$, then we deduce $|S| < \frac{b(|T_0|+l)+bn}{a}$ by (2). Let S' be set of vertices such that it contains exactly $b-1$ vertices in each component of K_b in H'. Clearly, $i(G - S \cup S') \geq |T_0| + l \geq 1$. If $i(G - S \cup S') > 1$, then $I(G) \leq \frac{|S \cup S'|}{i(G-S-S')} < \frac{b(|T_0|+l)+bn+al(b-1)}{a(|T_0|+l)} \leq \frac{b}{a} + \frac{bn}{2a} + b - 1$, which contradicts $I(G) \geq \frac{b^2+bn-1}{a}$. If $i(G - S \cup S') = 1$, then $|T_0| + l = 1$. Hence $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq \frac{bn}{a} + \frac{b}{a} + b - 1$. We have $d_{G-S}(x) \geq \frac{bn}{a} + \frac{(b+1)^2}{4a} + b - 1 - |S| > \frac{bn}{a} + \frac{b}{a} + b - 1 - \frac{b(n+1)}{a}$, which contradicts $d_{G-S}(x) \leq b - 1$ for any $x \in T$.

Now, we consider $|V(H)| \geq 1$. Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = b - 1$ for every vertex $x \in V(H_1)$ and $H_2 = H - H_1$. According to Lemma 4, there exist a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ of H_1 such that

$$|V(H_1)| \leq \sum_{i=1}^{b} (b - i + 1)|I^{(i)}| - \frac{|I^{(1)}|}{2} \qquad (3)$$

and

$$|C_1| \leq \sum_{i=1}^{b} (b - i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \qquad (4)$$

where $I^{(i)} = \{x \in I_1, d_{H_1}(x) = b - i\}$, $1 \leq i \leq b$ and $\sum_{i=1}^{b} |I^{(i)}| = |I_1|$. Let $T_j = \{x \in V(H_2) | d_{G-S}(x) = j\}$ for $1 \leq j \leq b-1$. Each component of

H_2 has a vertex of degree at most $b-2$ in $G-S$ by the definitions of H and H_2. From Lemma 3, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{b-1}(b-j)c_j \le \sum_{j=1}^{b-1}(b-2)(b-j)i_j, \tag{5}$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \ldots, b-1$. Set $W = V(G)-S-T$ and $U = S \cup S' \cup C_1 \cup (N_G(I_1) \cap W)) \cup C_2 \cup (N_G(I_2) \cap W)$. We infer

$$|U| \le |S| + l(b-1) + |C_1| + \sum_{j=1}^{b-1} j i_j + \sum_{i=1}^{b}(i-1)|I^{(i)}| \tag{6}$$

and

$$i(G-U) \ge t_0 + l + |I_1| + \sum_{j=1}^{b-1} i_j, \tag{7}$$

where $t_0 = |T_0|$. Then when $i(G-U) > 1$, we yield

$$|U| \ge I(G)i(G-U). \tag{8}$$

If $i(G-U) = 1$ then $G[T]$ is a clique with vertices number less than b. By (2), we get

$$\begin{aligned} |S| &< \frac{bn + b|T| - d_{G-S}(T)}{a} \\ &\le \frac{bn + b|T| - |T|(|T|-1)}{a} \\ &\le \frac{bn + b\frac{b+1}{2} - (\frac{b+1}{2})(\frac{b+1}{2} - 1)}{a} \\ &= \frac{bn}{a} + \frac{(b+1)^2}{4a}, \end{aligned}$$

and $d_{G-S}(x) \ge \frac{bn}{a} + \frac{(b+1)^2}{4a} + b - 1 - |S| > \frac{bn}{a} + \frac{(b+1)^2}{4a} + b - 1 - (\frac{bn}{a} + \frac{(b+1)^2}{4a})$, which contradicts $d_{G-S}(x) \le b-1$ for any $x \in T$.

In view of (6), (7) and (8), we get

$$|S| + |C_1| \ge \sum_{j=1}^{b-1}(I(G)-j)i_j + I(G)(t_0+l) + I(G)|I_1| - \sum_{i=1}^{b}(i-1)|I^{(i)}| - l(b-1). \tag{9}$$

In terms of $b|T| - d_{G-S}(T) > a|S| - bn$, we obtain

$$bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j > a|S| - bn.$$

Combining with (9), we deduce

$$\begin{aligned} & |V(H_1)| + \sum_{j=1}^{b-1}(b-j)c_j + a|C_1| \\ > & \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + (aI(G) - b)(t_0 + l) + aI(G)|I_1| \\ & -a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn - la(b-1). \end{aligned} \tag{10}$$

By (3) and (4), we have

$$|V(H_1)| + a|C_1| \le \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| - \frac{(a+1)|I^{(1)}|}{2}. \tag{11}$$

Using (5), (10) and (11), we get

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| \\ > & \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} \\ & -a\sum_{i=1}^{b}(i-1)|I^{(i)}| + (aI(G) - b)(t_0 + l) - bn - la(b-1). \end{aligned} \tag{12}$$

The following proof splits into two cases according to the value of $t_0 + l$.

Case 1. $t_0 + l \ge 1$. By $aI(G) \ge b^2 + bn - 1$, we have $(aI(G) - b)(t_0 + l) - bn - la(b-1) \ge 0$ by $b \ge a+1$ and $b \ge 2$. Thus, (12) becomes

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| \\ > & \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} \\ & -a\sum_{i=1}^{b}(i-1)|I^{(i)}|. \end{aligned}$$

Therefore, at least one of the following two cases must hold.

Subcase 1. There is at least one j such that

$$(b-2)(b-j) > aI(G) - aj - b + j,$$

which implies

$$\begin{aligned} aI(G) &< b(b-2) + (a-b+1)j + b \\ &\leq b(b-2) + (a-b+1) + b \\ &= (b^2-1) + (a-b) + (2-b) \\ &< b^2 - 1, \end{aligned}$$

which contradicts $I(G) \geq \frac{b^2-1+bn}{a}$.

Subcase 2.

$$\begin{aligned} &\sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| \\ > \;& aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\ \geq \;& (b^2 + bn - 1)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\ \geq \;& (b^2 - 1)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|. \end{aligned}$$

This implies,

$$\sum_{i=2}^{b}(ab + b - a - i + 2 - b^2)|I^{(i)}| + (ab + b - \frac{3}{2}a - b^2 + \frac{1}{2})|I^{(1)}| > 0.$$

Let

$$h_1(b) = -b^2 + (a+1)b - \frac{3}{2}a + \frac{1}{2}.$$

From $b \geq a+1$, we get

$$\max\{h_1(b)\} = h_1(a+1) = -\frac{3a}{2} + \frac{1}{2} < 0.$$

On the other hand, $ab + b - a - i + 2 - b^2 \leq -b^2 + (a+1)b - a$ due to $i \geq 2$.
Let

$$h_2(b) = -b^2 + (a+1)b - a.$$

We infer

$$\max\{h_2(b)\} = h_2(a+1) = -a < 0$$

by $b \geq a+1$. This is a contradiction.

Case 2. $t_0 + l = 0$. In this case, (12) becomes

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| \\ > & \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} \\ & -a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn. \end{aligned}$$

From what we have discussed in Subcase 1, we get $\sum_{j=1}^{b-1}(b-2)(b-j)i_j \leq \sum_{j=1}^{b-1}(at - aj - b + j)i_j$. If $|I_1| > 0$, we deduce

$$\begin{aligned} & \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| \\ > & aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn \\ \geq & (b^2 + bn - 1)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn \\ \geq & (b^2 - 1)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|. \end{aligned}$$

The result follows from what we discussed in Subcase 2 above.

The last situation is $|I_1| = 0$ and $\sum_{j=1}^{b-1}(b-2)(b-j)i_j > \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j - bn$. Let $h_3 = (b-2)(b-j) - (aI(G) - aj - b + j) + bn$. By $b \geq a+1$, we infer

$$\begin{aligned} h_3 &= b(b-2) + (a - b + 1)j + b - aI(G) + bn \\ &\leq b(b-2) + (a - b + 1) + b - (b^2 + bn - 1) + bn \\ &= -2(b-1) + a < 0, \end{aligned}$$

a contradiction.

Therefore, we can conclude that Theorem 2 holds. □

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