2-Color Rado Numbers for $\sum_{i=1}^{m-1} x_i + c = x_m$

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Abstract

Rado numbers are closely related to Ramsey numbers, but pertaining to equations and integers instead of cliques within graphs. For every integer $m \geq 3$ and every integer c, let the 2-color Rado number $r(m, c)$ be the least integer, if it exists, such that for every 2-coloring of the set $\{1, 2, \ldots, r(m, c)\}$ there exists a monochromatic solution to the equation $\sum_{i=1}^{m-1} x_i + c = x_m$. The values of $r(m, c)$ have been determined previously for nonnegative values of c, as well as all values of m and c such that $-m+2 < c < 0$ and $c < -(m-1)(m-2)$. In this paper, we find $r(m, c)$ for the remaining values of m and c.

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1 Introduction and Definitions

For ease of the reader, let us denote the subset $\{a, a+1, a+2, \ldots, b\}$ of the natural numbers $\mathbb{N}$ by $[a, b]$ within this paper. A function $\Delta : A \to [0, t-1]$ is referred to as a *t-coloring* of the set A. Given a linear equation E and a t-coloring Δ of some subset of $\mathbb{N}$, a solution $(x_1, x_2, \ldots, x_m)$ to E is said to be *monochromatic* if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

It was proved by Schur [12] in 1916 that for every positive integer t, there exists a least integer $n = S(t)$, such that for every t-coloring of $[1, n]$, say $\Delta : [1, n] \to [0, t-1]$, there exists a monochromatic solution to the particular linear equation $x_1 + x_2 = x_3$. The values of these Schur numbers $S(t)$ are known exactly for only the first four values of t: $t = 1, 2, 3, 4$.

Seventeen years later, Schur's student R. Rado generalized this notion of Schur numbers to arbitrary systems of linear equations. He found necessary and sufficient conditions to determine if such a given system admits a monochromatic solution under every t-coloring of the natural numbers [7][8][9]. And so, for a given system L of linear equations, we define the t-color *Rado number* to be the least integer n, provided that it exists, such that for every t-coloring of the set $[1, n]$ there exists a monochromatic solution to L. If such an integer n does not exist, then the t-color Rado number for the system L is said to be infinite. The exact Rado numbers for several families of equations have been found in recent years [2][3][4][6][11].

The focus of this paper is to find the 2-color Rado numbers for every integer $m \geq 3$ and every integer c of the linear equation

$$\sum_{i=1}^{m-1} x_i + c = x_m,$$

denoted by $L(m, c)$. We must begin with a definition.

Definition 1.1. *The 2-color Rado number for $L(m, c)$, denoted $r(m, c)$, is the least integer, provided it exists, such that for every 2-coloring $\Delta : [1, r(m, c)] \to [0, 1]$ there exists a monochromatic solution to $L(m, c)$. If such an integer does not exist, we say that $r(m, c)$ is infinite.*

In 1982, Beutelspacher and Brestovansky [1] proved that $r(m, 0) = m^2 - m - 1$ for $m \geq 3$. Schaal found in 1993 [10] that $r(m, c)$ is infinite whenever m is even and c is odd, achieved by the coloring that assigns color 0 to the odd integers and color 1 to the even integers. Additionally, in the case when m is odd or c is even, he showed that $r(m,c) = m^2 + (c-1)(m+1)$ for $c \geq 0$.

In 2001, Schaal and Kosek [5] found $r(m,c)$ for various combinations of even values of m and negative even values of c. This paper finds $r(m,c)$ for the remaining values of m and c. In order to describe these results, it is helpful to represent c as a multiple of $m-2$ plus an appropriate "remainder." Specifically, we let $c = (m-2)\alpha + 2w$, for $w \in \mathbb{N}$ and $0 \leq w < \frac{m-2}{2}$. The results of Schaal and Kosek are listed in rows 1, 5, and 7, while the proofs of rows 2, 3, and 4 can be in Theorems 2.3, 2.1, and 2.2. Finally, row 6 is proved in Theorems 2.5 and 2.6, respectively.

$c = (m-2)\alpha + 2w$		$r(m,c)$
$-\alpha = 1$	$0 \leq w < \frac{m-2}{2}$	$m^2 + (c-1)(m+1)$
$-\alpha = 2$	$1 \leq w < \frac{m-2}{2} - 1$	$\begin{cases} 2w+2+d \\ 2w+2+\left\lceil \frac{d'}{n} \right\rceil \end{cases}$
$-\alpha = 2$	$w = \frac{m-2}{2} - 1$	m
$2 \leq -\alpha \leq m-1$	$w = 0$	$-\alpha$
$3 \leq -\alpha \leq m-1$	$1 \leq w \leq \frac{\alpha+m-1}{2}$	$-\alpha + 2w$
$4 \leq -\alpha \leq m-1$	$\frac{\alpha+m-1}{2} < w < \frac{m-2}{2}$	$\begin{cases} -\alpha+1 \\ -\alpha \end{cases}$
$-\alpha > m-1$	$0 \leq w < \frac{m-2}{2}$	$\left\lceil \frac{1-(m+1)c}{m^2-m-1} \right\rceil$

2 Main Results

For this section, we assume $m \geq 4$ is an even integer, and $c \leq -4$ is also an even number.

Theorem 2.1. *Let $c = -m$. Then $r(m,c) = m$.*

Proof. We first show that $r(m,c) \leq m$. Let $\Delta : [1, m-1] \to [0,1]$ be a 2-coloring and assume, for a contradiction, that it admits no monochromatic solution to $L(m,c)$. Without loss of generality, let $\Delta(1) = 0$. If $m = 4$, then it is straightforward to verify that $r(4,-4) = 2$ with maximal coloring $\Delta(1) = 0$. If $m = 6$, then $r(6,-6) = 5$ with maximal coloring

$$\Delta(x) = \begin{cases} 0, & \text{if } x = 1 \\ 1, & \text{if } 2 \leq x \leq 3 \\ 0, & \text{if } x = 4. \end{cases}$$

So for the remainder of this proof, we can assume that $m \geq 8$. To that end, let $x_i = 1$ for $1 \leq i \leq m-4$ and $x_i = 2$ for $m-3 \leq i \leq m$. Then $\sum_{i=1}^{m-1} x_i + c = 2 = x_m$. Thus, we see that this choice of values for each x_i results in a solution to $L(m,c)$, which implies that $\Delta(2) \neq 0$ else we have a monochromatic solution. We may then assume that $\Delta(2) = 1$.

Similarly, if $x_i = 1$ for $1 \leq i \leq m-2$, $i = m$, and $x_{m-1} = 3$, then this implies that $\Delta(3) \neq 0$ else we have a monochromatic solution. Hence, $\Delta(3) = 1$.

Next we see that if $x_i = 2$ for $1 \leq i \leq m-1$ and $x_m = m-2$, then this implies that $\Delta(m-2) \neq 1$ and so $\Delta(m-2) = 0$.

Now, if $x_i = 2$ for $1 \leq i \leq m-3$, $x_i = 3$ for $m-2 \leq i \leq m-1$, and $x_m = m$, then $\Delta(m) \neq 1$ else we have a monochromatic solution to $L(m,c)$. At the same time, if we let $x_i = 1$ for $1 \leq i \leq m-2$, $x_{m-1} = m$, and $x_m = m-2$, then this implies that $\Delta(m) \neq 0$ else we have a monochromatic solution to $L(m,c)$. And so, we have shown that an arbitrary 2-coloring $\Delta : [1,m] \to [0,1]$ must contain a monochromatic solution, which implies $r(m,c) \leq m$.

Now it will be shown that $r(m,c) \geq m$ by demonstrating a 2-coloring $\Delta : [1,m-1] \to [0,1]$ that avoids a monochromatic solution to $L(m,c)$. Let Δ be defined by

$$\Delta(x) = \begin{cases} 0, & \text{if } x = 1 \\ 1, & \text{if } 2 \leq x \leq m-3 \\ 0, & \text{if } m-2 \leq x \leq m-1. \end{cases}$$

Let us first consider the situation when $\Delta(x_i) = 0$ for every $1 \leq i \leq m$. This is equivalent to the statement $x_i \in \{1, m-2, m-1\}$ for every $i \in [1,m]$. The proofs of these three cases are straightforward and are left to the reader. Thus we cannot create a monochromatic solution in color 0.

Let us now consider the second situation, when $\Delta(x_i) = 1$ for every $1 \leq i \leq m$. This is equivalent to the statement $x_i \in [2, m-3]$ for every $i \in [1,m]$. This implies that $x_m \geq 2 \cdot (m-1) - m = m-2$, implying no solution can be formed in color 1. Thus, Δ avoids a monochromatic solution to $L(m,c)$, which in turn implies that $r(m,c) \geq m$. We finally conclude that $r(m,c) = m$. □

Although the Rado numbers for the following c values of Theorem 2.2 were originally determined in [5], we provide the below proof since the way c is described will be useful in subsequent proofs.

Theorem 2.2. *Suppose that $c \in \mathbb{Z}^-$ such that $-(m-2)(m-1) < c < 0$ and $(m-2)|c$. Let $\alpha \in \mathbb{Z}^-$ be such that $c = (m-2)\alpha$. Then $r(m,c) = -\alpha$.*

Proof. To show $r(m,c) \leq -\alpha$, let Δ be a 2-coloring and assume, for a contradiction, that it admits no monochromatic solution to $L(m,c)$. Indeed, let Δ be such an arbitrary coloring. Notice that if $x_i = -\alpha$ for every $1 \leq i \leq m$, then $\sum_{i=1}^{m-1} x_i + c = -\alpha(m-1)+(m-2)\alpha = -\alpha$. And so, no matter the choice of color assignment for $-\alpha$, we see that the above expression would create a monochromatic solution to $L(m,c)$. Hence, $r(m,c) \leq -\alpha$.

To show that $r(m,c) \geq -\alpha$, we will demonstrate a coloring $\Delta : [1, -\alpha - 1] \to [0,1]$ that avoids a monochromatic solution. Let $\Delta(x) = 0$ for every x in the domain. Notice that since $-(m-2)(m-1) < c < 0$ and $c = (m-2)\alpha$, we see that $\alpha = \frac{c}{m-2}$ and $-\alpha < m-1$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{m-1} x_i + c &\leq [(-\alpha - 1)\cdot(m-1)] + (m-2)\alpha \\
&= -\alpha + 1 - m \\
&< (m-1) + 1 - m \\
&= 0 \\
&< x_m.
\end{aligned}$$

Hence, no solution can be formed from the elements of the domain, which in turn implies that no monochromatic solution is possible. Therefore, $r(m,c) \geq -\alpha$, and we may conclude that $r(m,c) = -\alpha$. □

The next theorem determines the Rado numbers for the c values corresponding to the special case when $-\alpha = 2$.

Theorem 2.3. *Suppose that $c \in \mathbb{Z}^-$ and $c = -2(m-2) + 2w$ for $w \in \mathbb{N}$ and $1 \leq w < \frac{m-2}{2} - 1$. Let $n \in \mathbb{N}$ be the specific natural number for which*

$$n(2w+1) \leq m-1 < (n+1)(2w+1)$$

and let $d = (n+1)(2w+1) - (m-1)$ and $d' = (m-1) - n(2w+1)$. Then

$$r(m,c) = \begin{cases} 2w+2+d & \text{if } d < \left\lceil \frac{d'}{n} \right\rceil \\ 2w+2+\left\lceil \frac{d'}{n} \right\rceil & \text{if } d \geq \left\lceil \frac{d'}{n} \right\rceil \end{cases}.$$

Proof. The case when $w = \frac{m-2}{2} - 1$ corresponds to $c = -m$, which was done in Theorem 2.1. It should next be noted in this theorem that the inequality

$$m - 1 < (n+1)(2w+1)$$

implies that $d \geq 1$. Let us first assume that $d < \left\lceil \frac{d'}{n} \right\rceil$. To show that $r(m,c) \leq 2w+2+d$, let $\Delta : [1, 2w+2+d] \to [0,1]$ be a 2-coloring and assume, for a contradiction, that it admits no monochromatic solution to $L(m,c)$. Without loss of generality, suppose $\Delta(1) = 0$. Then $\Delta(2) = 1$, otherwise a monochromatic solution to $L(m,c)$ can be formed by letting $x_i = 1$ for $1 \leq i \leq 2w$ and $x_i = 2$ for $2w+1 \leq i \leq m$. Similarly, $\Delta(3) = 1$ otherwise a solution can be formed by letting $x_i = 1$ for $1 \leq i \leq \frac{m}{2} + w$ and $i = m$, and $x_i = 3$ for $\frac{m}{2} + w + 1 \leq i \leq m-1$.

Now, $\Delta(2w+2) = 0$ else $x_i = 2$ for $1 \leq i \leq m-1$ and $x_m = 2w+2$ creates a solution. Now notice that if $x_i = 1$ for $1 \leq i \leq m-2-n$ and $x_i = 2w+2$ for $m-1-n \leq i \leq m-1$, then $x_m = 2w+2+d$. This implies that $\Delta(2w+2+d) \neq 0$. But then $x_i = 2$ for $1 \leq i \leq m-1-d$ and $x_i = 3$ for $m-d \leq i \leq m-1$, so that $x_m = 2w+2+d$ and we have a forced monochromatic solution, a contradiction.

To show that $r(m,c) \geq 2w+2+d$, still assuming the case that $d < \left\lceil \frac{d'}{n} \right\rceil$, we will demonstrate a coloring $\Delta : [1, 2w+1+d] \to [0,1]$ that avoids a monochromatic solution to $L(m,c)$. Let such a function Δ be defined by

$$\Delta(x) = \begin{cases} 0, & x = 1 \\ 1, & 2 \leq x \leq 2w+1 \\ 0, & 2w+2 \leq x \leq 2w+1+d. \end{cases}$$

We note that there can be no monochromatic solution in color 1 since if $x_i \in [2, 2w+1]$ for $1 \leq i \leq m-1$, then $x_m \geq (m-2)(2) + c = 2w+2$, which implies $\Delta(x_m) \neq 1$. Now, to see that Δ also avoids a monochromatic solution in color 0, suppose first that there exist precisely n values of i such that $x_i \in [2w+2, 2w+1+d]$ (the same n as stated in the original hypothesis of the theorem). Then $x_m \geq (m-1-n) + n(2w+2) + c = d+1 > 1$ and

$$\begin{aligned} x_m &\leq (m-1-n) + n(2w+1+d) + c \\ &= d+1+n(d-1) \\ &= d+1+nd-n+d'-d' \\ &= (d'+d)+1+nd-n-d' \\ &= (2w+1)+1+nd-n-d' \\ &= 2w+2+n\left(d-1-\frac{d'}{n}\right). \end{aligned}$$

Now, since we have assumed $d < \left\lceil \frac{d'}{n} \right\rceil$, we can say that $d < \frac{d'}{n} + 1$ so that $x_m \leq 2w+2$ meaning $\Delta(x_m) \neq 0$. Now assume that at least $n+1$ values of x_i are elements of the set $[2w+2, 2w+1+d]$. Then an easy calculation gives $x_m \geq 2w+d+2$ so x_m is not in the domain of Δ. Lastly, suppose that at most $n-1$ values of x_i are elements of the set $[2w+2, 2w+1+d]$. Then $x_m \leq n(d-1) - 2w + 1$. By recalling that $d < \lceil \frac{d'}{n} \rceil$, we see that $d < \frac{d'}{n} + 1$, so $n(d-1) < d'$. It is also true that $d' \leq 2w$ since $d' < 2w+1$. Therefore,

$$\begin{aligned} x_m &\leq n(d-1) - 2w + 1 \\ &< d' - 2w + 1 \\ &\leq 2w - 2w + 1 \\ &= 1. \end{aligned}$$

This shows that x_m is not in the domain of Δ, implying no monochromatic solution exists in either color. We conclude that $r(m,c) \geq 2w+2+d$.

Let us now turn to the case when $d \geq \left\lceil \frac{d'}{n} \right\rceil$. To begin, suppose that $\Delta : \left[1, 2w+2+\left\lceil \frac{d'}{n} \right\rceil\right] \to [0,1]$ is a coloring with $\Delta(1) = 0$. Again, $\Delta(2) = \Delta(3) = 1$ and $\Delta(2w+2) = 0$ by the same argument as above. Now, by letting $x_i = 2$ for $1 \leq i \leq m - \left\lceil \frac{d'}{n} \right\rceil$ and $x_i = 3$ for $m - \left\lceil \frac{d'}{n} \right\rceil + 1 \leq i \leq m-1$, we see that $x_m = 2w+1+\left\lceil \frac{d'}{n} \right\rceil$, so $\Delta\left(2w+1+\left\lceil \frac{d'}{n} \right\rceil\right) = 0$ else a monochromatic solution is formed. A solution can also be formed by letting $x_i = 1$ for $1 \leq i \leq m-1-n$, $x_i = 2w+1+\left\lceil \frac{d'}{n} \right\rceil$ for $m-n \leq i \leq n\left(\lceil \frac{m-1}{n} \rceil - 1\right) - \left\lceil \frac{d'}{n} \right\rceil$, $x_i = 2w+2+\left\lceil \frac{d'}{n} \right\rceil$ for $n\left(\lceil \frac{m-1}{n} \rceil - 1\right) - \left\lceil \frac{d'}{n} \right\rceil + 1 \leq i \leq m-1$, and $x_m = 2w+2+\left\lceil \frac{d'}{n} \right\rceil$, implying $\Delta\left(2w+2+\left\lceil \frac{d'}{n} \right\rceil\right) \neq 0$. At the same time, a solution can be formed by letting $x_i = 2$ for $1 \leq i \leq m-1-\left\lceil \frac{d'}{n} \right\rceil$, $x_i = 3$ for $m - \left\lceil \frac{d'}{n} \right\rceil \leq i \leq m-1$, and $x_m = 2w+2+\left\lceil \frac{d'}{n} \right\rceil$, implying $\Delta\left(2w+2+\left\lceil \frac{d'}{n} \right\rceil\right) \neq 1$. Hence $r(m,c) \leq 2w+2+\left\lceil \frac{d'}{n} \right\rceil$.

To prove that $r(m,c) \geq 2w+2+\left\lceil \frac{d'}{n} \right\rceil$ for $d \geq \left\lceil \frac{d'}{n} \right\rceil$, we first assume $\left\lceil \frac{d'}{n} \right\rceil \geq 1$. Consider the coloring $\Delta : \left[1, 2w+1+\left\lceil \frac{d'}{n} \right\rceil\right] \to [0,1]$, defined by

$$\Delta(x) = \begin{cases} 0, & x = 1 \\ 1, & 2 \leq x \leq 2w+1 \\ 0, & 2w+2 \leq x \leq 2w+1+\left\lceil \frac{d'}{n} \right\rceil. \end{cases}$$

If $x_i \in [2, 2w+1]$ for $1 \leq i \leq m-1$, then $x_m \geq (m-1)(2) + c = 2w+2$, implying $\Delta(x_m) \neq 1$. Hence, no monochromatic solution exists in color

1. To see that no monochromatic solution exists in color 0 either, first suppose that there exist precisely n values of $i \in [1, m-1]$ such that $x_i \in \left[2w+2, 2w+1+\left\lceil \frac{d'}{n} \right\rceil\right]$. Then

$$\begin{aligned} x_m &\geq (m-1-n)(1) + n(2w+2) + c \\ &= (n(2w+1) - (m-1)) + 2w + 2 \\ &= -d' + 2w + 2 \\ &> -(2w+1) + 2w + 2 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} x_m &\leq (m-1-n)(1) + n(2w+1+\left\lceil \frac{d'}{n} \right\rceil) + c \\ &< -m + 2nw + 2w + n\left(\frac{d'}{n} + 1\right) + 3 \\ &= -((m-1) - n(2w+1)) + 2w + 2 + d' \\ &= -d' + 2w + 2 + d' \\ &= 2w + 2. \end{aligned}$$

This shows that $\Delta(x_m) \neq 0$. Suppose next that at least $n+1$ values of i exist such that $x_i \in \left[2w+2, 2w+1+\left\lceil \frac{d'}{n} \right\rceil\right]$. Then

$$\begin{aligned} x_m &\geq (m-1-(n+1))(1) + (n+1)(2w+2) - 2m + 4 + 2w \\ &= -m + n + 2nw + 4w + 4 \\ &= -d' + 4w + 3 \\ &= d - (d' + d) + 4w + 3 \\ &= d - (2w+1) + 4w + 3 \\ &= d + 2w + 2 \\ &\geq 2w + 2 + \left\lceil \frac{d'}{n} \right\rceil, \end{aligned}$$

implying that x_m is not in the domain of Δ. Lastly, suppose that at most $n+1$ values of i exist such that $x_i \in \left[2w+2, 2w+1+\left\lceil \frac{d'}{n} \right\rceil\right]$. Then, recalling the fact that $n\left\lceil \frac{d'}{n} \right\rceil \leq d' + n - 1$, we have that

$$
\begin{aligned}
x_m &\leq (m-1-(n-1))(1)+(n-1)(2w+1+\left\lceil \frac{d'}{n} \right\rceil)-2m+4+2w \\
&= -m+2nw+n\left\lceil \frac{d'}{n} \right\rceil - \left\lceil \frac{d'}{n} \right\rceil + 3 \\
&= -d' + \left\lceil \frac{d'}{n} \right\rceil (n-1) - n + 2 \\
&= -d' + \left\lceil \frac{d'}{n} \right\rceil n - \left\lceil \frac{d'}{n} \right\rceil - n + 2 \\
&\leq -d' + (d'+n-1) - \left\lceil \frac{d'}{n} \right\rceil \\
&= -\left\lceil \frac{d'}{n} \right\rceil + 1 \\
&\leq 0.
\end{aligned}
$$

Again, we see that x_m is not in the domain of Δ, implying that no monochromatic solution exists in color 0. Therefore, $r(m,c) \geq 2w+2+\left\lceil \frac{d'}{n} \right\rceil$.

Finally, in the case when $d \geq \left\lceil \frac{d'}{n} \right\rceil$ and $\left\lceil \frac{d'}{n} \right\rceil = 0$, we can easily show that $\Delta : [1, 2w+1] \rightarrow [0,1]$, defined by

$$
\Delta(x) = \begin{cases} 0, & x = 1 \\ 1, & 2 \leq x \leq 2w+1, \end{cases}
$$

avoids a monochromatic solution. By the reasoning in the previous argument, there does not exist a monochromatic solution in color 1. Now, if $x_i = 1$ for $1 \leq i \leq m-1$, then $x_m = m-1+c = -m+2w+3 \leq -m+1 < 0$. Thus, there does not exist a monochromatic in color 0, and we conclude that $r(m,c) \geq 2w+2+\left\lceil \frac{d'}{n} \right\rceil$ in all cases. □

With the case of $-\alpha = 2$ complete, we note that in [5], Schaal and Kosek determined the Rado number for those values when $3 \leq -\alpha \leq m-1$ with $1 \leq w \leq \frac{\alpha+m-1}{2}$. We state their theorem below.

Theorem 2.4. *Let m be an even number and let $c = (m-2)\alpha + 2w$ for $3 \leq -\alpha \leq m-1$ and $1 \leq w \leq \frac{\alpha+m-1}{2}$. Then $r(m,c) = -\alpha + 2w$ with maximal coloring*

$$
\Delta(x) = \begin{cases} 0 & 1 \leq x \leq -\alpha - 1; \\ 1 & -\alpha \leq x \leq -\alpha + 2w - 1. \end{cases}
$$

It is left then to find the appropriate Rado number for $L(m,c)$ when $4 \leq -\alpha \leq m-1$ (Theorem 2.4 covers all cases when $-\alpha = 3$) and $\frac{\alpha+m-1}{2} < w < \frac{m-2}{2}$. These remaining cases will be handled in two theorems, separated by the assumption on w. The proof of the second case follows the proof of the first with only minor modifications.

Theorem 2.5. *Let* $4 \leq -\alpha \leq m-1$. *When* $\frac{\alpha+m-1}{2} < w \leq \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor \left(\frac{-\alpha-1}{2}\right)$, *if*

$$w < \min\left\{ \frac{1+\alpha+\left\lceil \frac{m-1}{-\alpha-1} \right\rceil (m-1)}{2\left\lceil \frac{m-1}{-\alpha-1} \right\rceil + 2}, \left(\frac{m-1}{2}\right)\left(\frac{\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor}{\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor + 1}\right) \right\},$$

then $r(m,c) = -\alpha+1$. *Otherwise,* $r(m,c) = -\alpha$.

Proof. Consider the case when w is less than the above min. To show that $r(m,c) \leq -\alpha+1$, let $\Delta : [1, -\alpha+1] \to [0,1]$ be a 2-coloring and assume, for a contradiction, that it admits no monochromatic solution to $L(m,c)$. We define $\epsilon = -\alpha - m + 2w + 1 \geq 1$ and notice that letting $x_i = -\alpha-1$ for $1 \leq i \leq m-1$ forces $x_m = \epsilon$. This implies $\Delta(\epsilon) \neq \Delta(-\alpha-1)$. Similarly, letting $x_i = -\alpha-1$ for $1 \leq i \leq 2w$ and $x_i = -\alpha$ for $2w+1 \leq i \leq m$ creates a solution, so $\Delta(-\alpha-1) \neq \Delta(-\alpha)$, meaning $\Delta(\epsilon) = \Delta(-\alpha)$. Again, a solution can be created by choosing $x_i = -\alpha-1$ for $1 \leq i \leq \frac{m}{2}+w$ and $x_i = -\alpha+1$ for $\frac{m}{2}+w+1 \leq i \leq m$. This implies that $\Delta(-\alpha+1) \neq \Delta(-\alpha-1)$, meaning $\Delta(-\alpha+1) = \Delta(-\alpha) = \Delta(\epsilon)$. Now, to show that a solution to $L(m,c)$ exists using values of x_i in the set $\{\epsilon, -\alpha, -\alpha+1\}$, and thus forcing $r(m,c) \leq -\alpha+1$, we will establish the existence of a positive integer k with the following two properties: (1) letting $x_i = \epsilon$ for $1 \leq i \leq k$ and $x_i = -\alpha$ for $k+1 \leq i \leq m-1$ forces $x_m \leq -\alpha+1$, and (2) letting $x_i = \epsilon$ for $1 \leq i \leq k$ and $x_i = -\alpha+1$ for $k+1 \leq i \leq m-1$ forces $x_m > -\alpha+1$. Such a value of k satisfying these two statements simultaneously means that the $m-1-k$ values of x_i that are not equal to ϵ can be allocated appropriately (some equalling $-\alpha$ and the rest equalling $-\alpha+1$) to force $x_m = -\alpha+1$. And so, we first claim that

$$\frac{2w-1}{m-2w-1} \leq k < \frac{2w+m-2}{m-2w}.$$

Note that since $w < \frac{m-2}{2}$, we have that $m-2w-2 \geq 0$, which can be used to show that $\frac{2w-1}{m-2w-1} < \frac{2w+m-2}{m-2w}$. To ensure that an integer value exists between these two fractions, it suffices to show that their difference exceeds

1. To that end,

$$\begin{aligned}\frac{2w+m-2}{m-2w}-\frac{2w-1}{m-2w-1} &= \frac{m^2-2wm-m+2}{m^2-4wm-m+4w^2+2w}\\ &> \frac{m^2-2wm-m+2}{m^2-2wm-m+2}\\ &= 1,\end{aligned}$$

using the fact that $4w^2+2w-2wm-2=2w(-m+2w+2)-2$ is negative. Hence, letting k be any such integer allows the creation of a monochromatic solution to $L(m,c)$ using values of x_i in $\{\epsilon,-\alpha,-\alpha+1\}$. Therefore Δ must contain a monochromatic solution to $L(m,c)$, implying $r(m,c)\le -\alpha+1$.

Now it will be shown that $r(m,c)\ge -\alpha+1$ by exhibiting a coloring of $[1,-\alpha]$ that avoids a monochromatic solution to $L(m,c)$. Let $\Delta:[1,-\alpha]\to[0,1]$ be

$$\Delta(x)=\begin{cases}0, & 1\le x\le\epsilon\\ 1, & \epsilon+1\le x\le -\alpha-1\\ 0, & x=-\alpha.\end{cases}$$

There is no monochromatic solution in color 1 since letting $x_i\in[\epsilon+1,-\alpha-1]$ for $1\le i\le m-1$ forces $x_m\le(m-1)(-\alpha-1)+c=-\alpha-m+2w+1=\epsilon$. Now, to show that Δ also avoids a monochromatic solution in color 0, we will assume $\Delta(x_i)=0$ for $1\le i\le m-1$. If at most $\left\lceil\frac{m-1}{-\alpha-1}\right\rceil-2$ values of x_i are in the set $[1,\epsilon]$, then

$$\begin{aligned}x_m &\ge \left(m-1-\left(\left\lceil\frac{m-1}{-\alpha-1}\right\rceil-2\right)\right)(-\alpha)+\left(\left\lceil\frac{m-1}{-\alpha-1}\right\rceil-2\right)(1)+c\\ &= \left(\left\lceil\frac{m-1}{-\alpha-1}\right\rceil\right)(\alpha+1)-3\alpha+2w-2\\ &> \left(\frac{m-1}{-\alpha-1}+1\right)(\alpha+1)-3\alpha+2w-2\\ &= -\alpha+\epsilon-1\\ &\ge -\alpha,\end{aligned}$$

implying x_m is not in the domain of Δ. If exactly $\left\lceil\frac{m-1}{-\alpha-1}\right\rceil-1$ values of i exist such that x_i is in the set $[1,\epsilon]$, then

$$
\begin{aligned}
x_m &\le \left(m-1-\left(\left\lceil\frac{m-1}{-\alpha-1}\right\rceil-1\right)\right)(-\alpha)+\left(\left\lceil\frac{m-1}{-\alpha-1}\right\rceil-1\right)(\epsilon)+c \\
&= 2w\left\lceil\frac{m-1}{-\alpha-1}\right\rceil+\left\lceil\frac{m-1}{-\alpha-1}\right\rceil(-m+1)-\alpha+m-1 \\
&< 2\left(\frac{(m-1)\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor}{2\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor+2}\right)\left\lceil\frac{m-1}{-\alpha-1}\right\rceil(-m+1)-\alpha+m-1 \\
&= \frac{(m-1)\left(1+\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor-\left\lceil\frac{m-1}{-\alpha-1}\right\rceil\right)}{\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor+1}-\alpha \\
&\le -\alpha,
\end{aligned}
$$

and

$$
\begin{aligned}
x_m &\ge \left(m-1-\left(\left\lceil\frac{m-1}{-\alpha-1}\right\rceil-1\right)\right)(-\alpha)+\left(\left\lceil\frac{m-1}{-\alpha-1}\right\rceil-1\right)(1)+c \\
&= \left\lceil\frac{m-1}{-\alpha-1}\right\rceil(\alpha+1)-2\alpha+2w-1 \\
&= \left\lceil\frac{m-\alpha-2}{-\alpha-1}-1\right\rceil(\alpha+1)-2\alpha+2w-1 \\
&> -\left(\frac{m-\alpha-2}{-\alpha-1}\right)(-\alpha-1)-2\alpha+2w-1 \\
&= -m+\alpha+2-2\alpha+2w-1 \\
&= \epsilon.
\end{aligned}
$$

This implies that no solution exists in color 0. Lastly, if there exist at least $\left\lceil\frac{m-1}{-\alpha-1}\right\rceil$ values of x_i in the set $[1,\epsilon]$, then

$$
\begin{aligned}
x_m &\le \left(m-1-\left\lceil\frac{m-1}{-\alpha-1}\right\rceil\right)(-\alpha)+\left\lceil\frac{m-1}{-\alpha-1}\right\rceil(\epsilon)+c \\
&= -\alpha+2w\left(1+\left\lceil\frac{m-1}{-\alpha-1}\right\rceil\right)-(m-1)\left\lceil\frac{m-1}{-\alpha-1}\right\rceil \\
&< -\alpha+2\left(\frac{1+\alpha+\left\lceil\frac{m-1}{-\alpha-1}\right\rceil(m-1)}{2\left\lceil\frac{m-1}{-\alpha-1}\right\rceil+2}\right)\left(1+\left\lceil\frac{m-1}{-\alpha-1}\right\rceil\right) \\
&\quad -(m-1)\left\lceil\frac{m-1}{-\alpha-1}\right\rceil \\
&= 1,
\end{aligned}
$$

which implies x_m is not in the domain of Δ. Therefore, no monochromatic solution exists, which implies $r(m,c) \geq -\alpha+1$. We may now conclude in this case that $r(m,c) = -\alpha+1$.

Now consider the case $w \geq \min\left\{\frac{1+\alpha+\lceil\frac{m-1}{-\alpha-1}\rceil(m-1)}{2\lceil\frac{m-1}{-\alpha-1}\rceil+2}, \frac{(m-1)\lfloor\frac{m-4}{-\alpha-1}\rfloor}{2\lfloor\frac{m-4}{-\alpha-1}\rfloor+2}\right\}$. To show that $r(m,c) \leq -\alpha$, let Δ be a 2-coloring of the set $[1,-\alpha]$ and assume, for a contradiction, that it admits no monochromatic solution. Since letting $x_i = -\alpha-1$ for $1 \leq i \leq m-1$ forces $x_m = \epsilon$, we see that $\Delta(\epsilon) \neq \Delta(-\alpha-1)$. Without loss of generality, we can assume $\Delta(\epsilon) = 0$. Similarly, letting $x_i = -\alpha-1$ for $1 \leq i \leq 2w$ and $x_i = -\alpha$ for $2w+1 \leq i \leq m$ creates a solution, so $\Delta(-\alpha-1) \neq \Delta(-\alpha)$, meaning $\Delta(\epsilon) = \Delta(-\alpha)$. Now, if $\Delta(\epsilon) = \Delta(-\alpha-2)$, then a monochromatic solution can be formed by letting $x_i = -\alpha-2$ for $1 \leq i \leq w+1$ and $i = m$ and $x_i = -\alpha$ for $w+2 \leq i \leq m-1$. Hence, $\Delta(\epsilon) \neq \Delta(-\alpha-2)$, implying $\Delta(-\alpha-2) = \Delta(-\alpha-1)$. Note that using combinations of $x_i \in [-\alpha-2, -\alpha-1]$ for $1 \leq i \leq m-1$ forces $x_m \leq \epsilon$, which implies that

$$\Delta(1) = \cdots = \Delta(\epsilon) = \Delta(-\alpha).$$

It will be shown that this string of equalities, along with our initial assumption that $w \geq \min\left\{\frac{1+\alpha+\lceil\frac{m-1}{-\alpha-1}\rceil(m-1)}{2\lceil\frac{m-1}{-\alpha-1}\rceil+2}, \frac{(m-1)\lfloor\frac{m-4}{-\alpha-1}\rfloor}{2\lfloor\frac{m-4}{-\alpha-1}\rfloor+2}\right\}$, implies a monochromatic solution in color 0 cannot be avoided. If $w \geq \frac{1+\alpha+\lceil\frac{m-1}{-\alpha-1}\rceil(m-1)}{2\lceil\frac{m-1}{-\alpha-1}\rceil+2}$, then letting $x_i = \epsilon$ for $1 \leq i \leq \left\lceil\frac{m-1}{-\alpha-1}\right\rceil$ and $x_i = -\alpha$ for $\left\lceil\frac{m-1}{-\alpha-1}\right\rceil+1 \leq i \leq m-1$ forces

$$1 \leq x_m \leq \epsilon + \left\lceil\frac{m-1}{-\alpha-1}\right\rceil(\epsilon-1),$$

where the upper bound follows from the fact that $\left\lceil\frac{m-1}{-\alpha-1}\right\rceil(\alpha+1) \leq -m+1$. Note that exchanging a particular $x_i = \epsilon$ for $x_i \in [1, \epsilon-1]$ amounts to decreasing x_m by at most $\epsilon-1$. Since there are $\lceil\frac{m-1}{-\alpha-1}\rceil$ such values of x_i, an appropriate allocation of values of x_i for $1 \leq i \leq \left\lceil\frac{m-1}{-\alpha-1}\right\rceil$ from the set $[1,\epsilon]$ will achieve a value of $x_m \in [1,\epsilon]$. Hence, a monochromatic solution in color 0 exists. On the other hand, if

$$\frac{1+\alpha+\left\lceil\frac{m-1}{-\alpha-1}\right\rceil(m-1)}{2\left\lceil\frac{m-1}{-\alpha-1}\right\rceil+2} > w \geq \frac{(m-1)\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor}{2\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor+2},$$

then consider the solution created by letting $x_i = \epsilon$ for $1 \leq i \leq \left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor$

and $x_i = -\alpha$ for $\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor + 1 \leq i \leq m-1$:

$$\begin{aligned} x_m &= \left(m - 1 - \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor\right)(-\alpha) + \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor (\epsilon) + c \\ &= \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor (\alpha+\epsilon) - \alpha + 2w. \end{aligned}$$

Using the upper bound on w yields

$$\begin{aligned} x_m &< \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor (\alpha+\epsilon) - \alpha + 2\left(\frac{1+\alpha+\left\lceil \frac{m-1}{-\alpha-1} \right\rceil (m-1)}{2\left\lceil \frac{m-1}{-\alpha-1} \right\rceil + 2}\right) \\ &= -\alpha + \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor (\epsilon - 1) + \frac{\left\lceil \frac{m-1}{-\alpha-1} \right\rceil \left((\alpha+1)\left\lceil \frac{m-1}{-\alpha-1} \right\rceil + m - 1\right)}{\left\lceil \frac{m-1}{-\alpha-1} \right\rceil + 1} \\ &\leq -\alpha + \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor (\epsilon - 1), \end{aligned}$$

and using the lower bound on w yields $x_m \geq -\alpha$. By the same argument as before, an appropriate allocation of $x_i \in [1, \epsilon]$ for $1 \leq i \leq \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor$ will achieve $x_m = -\alpha$, implying a monochromatic solution in color 0 exists. Hence, $r(m, c) \leq -\alpha$.

It can be shown that $r(m, c) \geq -\alpha$ by using reasoning similar to that of the previous case to verify that the coloring

$$\Delta(x) = \begin{cases} 0, & 1 \leq x \leq \epsilon \\ 1, & \epsilon + 1 \leq x \leq -\alpha - 1 \end{cases}$$

avoids a monochromatic solution in color 1. Letting $\Delta(x_i) = 0$ for $1 \leq i \leq m-1$ forces $x_m \leq 0$ which implies there is no monochromatic solution in color 0 as well. Hence, $r(m, c) = -\alpha$. □

Theorem 2.6. *For* $4 \leq -\alpha \leq m-1$ *and* $\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor \left(\frac{-\alpha-1}{2}\right) < w < \frac{m-2}{2}$, *if*

$$w < \min\left\{\frac{1+\alpha+\left\lceil \frac{m-1}{-\alpha-1} \right\rceil (m-1)}{2\left\lceil \frac{m-1}{-\alpha-1} \right\rceil + 2}, \left(\frac{m-1}{2}\right)\left(\frac{\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor + 1}{\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor + 2}\right)\right\},$$

then $r(m, c) = -\alpha + 1$. *Otherwise,* $r(m, c) = -\alpha$.

Proof. Consider the case when w is less than the above min. To show that $r(m,c) \leq -\alpha+1$, we can appeal to the argument made in Theorem 2.5. To show that $r(m,c) \geq -\alpha+1$, we will also use the same coloring of the previous theorem. The fact that the coloring avoids a monochromatic solution in color 1 follows identically. To show that there is no monochromatic solution in color 0 however, requires us to break this argument into two parts: when $\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor = \left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 1$ and when $\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor = \left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 2$. (These are the only two possible values given the bounds on $-\alpha$).

First note that when $\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor = \left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 2$, we can mimic the proof of Theorem 2.5 by showing that when at most $\left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 2$ values of x_i are from the set $[1,\epsilon]$, then $x_m \geq -\alpha$; when exactly $\left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 1$ values of x_i are from the set $[1,\epsilon]$, then $\epsilon < x_m < -\alpha$; and finally, when at least $\left\lceil \frac{m-1}{-\alpha-1} \right\rceil$ values of x_i are from the set $[1,\epsilon]$, then $x_m < 1$. Therefore, there can be no monochromatic solution in color 0 in this case, and hence $r(m,c) \geq -\alpha+1$.

For the case when $\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor = \left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 1$, we still have that if at least $\left\lceil \frac{m-1}{-\alpha-1} \right\rceil$ values of x_i are from the set $[1,\epsilon]$, then $x_m < 1$. Notice however that in this case, if at most $\left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 1$ values of x_i are from the set $[1,\epsilon]$, then

$$\begin{aligned}
x_m &\geq \left(m - \left\lceil \frac{m-1}{-\alpha-1} \right\rceil\right)(-\alpha) + \left(\left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 1\right)(1) + c \\
&= -\alpha + \left(\left\lceil \frac{m-1}{-\alpha-1} \right\rceil - 1\right)(\alpha+1) + 2w \\
&= -\alpha + \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor (\alpha+1) + 2w \\
&> -\alpha + \left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor (\alpha+1) + 2\left\lfloor \frac{m-4}{-\alpha-1} \right\rfloor \left(\frac{-\alpha-1}{2}\right) \\
&= -\alpha,
\end{aligned}$$

again implying x_m is not in the domain of Δ. Hence no monochromatic solution exists in color 0, and we have thus shown that $r(m,c) \geq -\alpha+1$. We conclude that $r(m,c) = -\alpha+1$.

Now let us turn to the case when w is greater than or equal to the minimum in the theorem. If $w > \frac{1+\alpha+\left\lceil \frac{m-1}{-\alpha-1} \right\rceil (m-1)}{2\left\lceil \frac{m-1}{-\alpha-1} \right\rceil + 2}$, then the proof follows identically to that of the proof of Theorem 2.5. On the other hand, if

$$\frac{1+\alpha+\left\lceil\frac{m-1}{-\alpha-1}\right\rceil(m-1)}{2\left\lceil\frac{m-1}{-\alpha-1}\right\rceil+2} > w \geq \left(\frac{m-1}{2}\right)\left(\frac{\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor+1}{\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor+2}\right),$$

then we must again break this argument up into the cases when $\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor = \left\lceil\frac{m-1}{-\alpha-1}\right\rceil - 2$ and $\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor = \left\lceil\frac{m-1}{-\alpha-1}\right\rceil - 1$. Suppose first that $\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor = \left\lceil\frac{m-1}{-\alpha-1}\right\rceil - 2$. Then letting $x_i = \epsilon$ for $1 \leq i \leq \left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor + 1$ and $x_i = -\alpha$ for $\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor + 2 \leq i \leq m-1$ yields $x_m = -\alpha + \left(\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor + 1\right)(\epsilon - 1) + (\alpha+1)\left(\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor + 1\right) + 2w$. Using the upper and lower bounds on w and simplifying results in:

$$\begin{aligned} x_m &< -\alpha + \left(\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor + 1\right)(\epsilon-1) + \frac{\left\lceil\frac{m-1}{-\alpha-1}\right\rceil(m-1+(m-1)(-1))}{\left\lceil\frac{m-4}{-\alpha-1}\right\rceil+1} \\ &= -\alpha + \left(\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor + 1\right)(\epsilon-1) \end{aligned}$$

and $x_m \geq -\alpha$. Hence, a monochromatic solution in color 0 cannot be avoided. Now we turn to the case when $\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor = \left\lceil\frac{m-1}{-\alpha-1}\right\rceil - 1$. A monochromatic solution in color 0 cannot be avoided by letting $x_i = \epsilon$ for $1 \leq i \leq \left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor$ and $x_i = -\alpha$ for $\left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor + 1 \leq i \leq m-1$:

$$\begin{aligned} x_m &= \left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor(\alpha+\epsilon) - \alpha + 2w \\ &= \left(\left\lceil\frac{m-1}{-\alpha-1}\right\rceil - 1\right)(-m+2w+1) - \alpha + 2w \\ &\geq -\alpha + \frac{m-1}{\left\lceil\frac{m-1}{-\alpha-1}\right\rceil+1} \\ &> -\alpha \end{aligned}$$

and $x_m \leq -\alpha + \left\lfloor\frac{m-4}{-\alpha-1}\right\rfloor(-\epsilon-1)$ by the same reasoning as the previous theorem. Hence, $r(m,c) \leq -\alpha$.

It can be shown that $r(m,c) \geq -\alpha$ by applying the same reasoning as in the previous case that no monochromatic solution exists in color 1, as well as verifying that letting $\Delta(x_i) = 0$ for $1 \leq i \leq m-1$ forces $x_m \leq 0$. Hence, $r(m,c) = -\alpha$.

□

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