Enumeration of walks in the square lattice according to their areas

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Abstract

We study the area distribution of closed walks of length n, starting and ending at the origin. The concept of algebraic area of a walk in the square lattice is slightly modified and the usefulness of this concept is demonstrated through a simple argument. The idea of using a generating function of the form $(x+x^{-1}+y+y^{-1})^n$ to study these walks is then discussed from a special viewpoint. Based on this, a polynomial time algorithm for calculating the exact distribution of such walks for a given length, is concluded. The presented algorithm takes advantage of the Chinese remainder theorem to overcome the problem of arithmetic with large integers. Finally, the results of the implementation are given for $n = 32, 64, 128$.

Keywords: square lattice; random walk; algebraic area; distribution;

1 Introduction

The problem of finding the area distribution of random walks of a given length is an interesting problem which has many applications, for instance in conformations of polymers and proteins (see [8, 1] and the references there in). When $n \to \infty$ this distribution was computed first by Lévy using Brownian paths. In [3], techniques from non-commutative geometry are applied to the Harper equation and the asymptotic distribution of the area, enclosed by a random walk in the square lattice, is provided. This approach is also studied in [2]. In [9] the authors used a more complicated

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and more informative method to derive the above asymptotic distribution. They have used combinatorial arguments, in which, the enumeration of up-down permutations and the exponential formula for cycles of permutations play fundamental roles. The asymptotic formula in [3] is compared with exact results obtained by computers: For this purpose, the closed walks of lengthes $n = 16, 18$ and 20 are enumerated according to their areas. These computations are then extended in [1] to $n = 28$ by using algorithmic techniques and a DSP processor. However, the algorithm used there, is based on generating walks, and thus the required time grows exponentially with respect to the length of walks.

It is well-known that the number of walks in $\mathbb{Z}^2$ of length n starting at the origin and ending at (p, q) equals the coefficient of $\alpha^p\beta^q$ in $(\alpha + \alpha^{-1} + \beta + \beta^{-1})^n$. We show that how a natural extension of this result leads to a polynomial time algorithm for the previously mentioned problem (i.e. for the enumeration of walks in $\mathbb{Z}^2$ of a given length according to their areas). As examples we present the results of implementation for $n = 32, 64, 128$ in the appendix.

This paper is organized as follows: In Section 2, some preliminaries and notation are discussed. In Section 3 a modification of the concept of area for general walks in the square lattice and some of its properties are discussed; Particularly, it is shown that how the area of composition of two walks cane be found using the areas of these walks and coordinates of the starting and ending points of them. In Section 4, a useful generating function for counting the number of walks in $\mathbb{Z}^2$ is discussed. In Section 5, we see that using modified noncommutative multiplication, the generating function of the previous section, can be applied to count the number of walks according to their areas. Section 6 contains an efficient algorithm to enumerate the number of walks in $\mathbb{Z}^2$. Finally, some results obtained from the implementation are given in tables in the appendix.

2 Preliminaries and Notation

First we mention some formulas about area of a polygon in the plane (The reader is referred to [9]). Let $\mathbf{P}$ be a polygon in in the plane with the vertex sequence $A_0, A_1, \ldots, A_{n-1}, A_0$ and coordinates $A_i = (x_i, y_i), 0 \leq i \leq n-1$ and let $(x_n, y_n) = (x_0, y_0)$. Moreover, for $i = 0, \cdots, n-1$ let $\delta_i = x_{i+1} - x_i$ and $\epsilon_i = y_{i+1} - y_i$. It is known that the algebraic area enclosed by P can be obtained by the following formula

$$S(\mathbf{P}) = \frac{1}{2} \sum_{0 \leq j < i \leq n-1} (\epsilon_i \delta_j - \epsilon_j \delta_i), \tag{1}$$

which is equivalent to

$$S(\mathbf{P}) = \frac{1}{2}\sum_{i=0}^{n-1}(x_i y_{i+1} - x_{i+1} y_i). \tag{2}$$

If moreover, the sides of the polygon **P** are parallel to the axes x or y, i.e. if the equation $\delta_i \epsilon_i = 0$ holds for $i = 0, \cdots, n-1$, then by applying

$$\sum_{0 \leq j < i \leq n-1} \epsilon_i \delta_j + \sum_{0 \leq j < i \leq n-1} \epsilon_j \delta_i = \sum_{j=0}^{n-1} \epsilon_j \sum_{i=0}^{n-1} \delta_i - \sum_{i=0}^{n-1} \delta_i \epsilon_i = 0$$

to (1), one obtains

$$S(\mathbf{P}) = \sum_{0 \leq j < i \leq n-1} \epsilon_i \delta_j \tag{3}$$

and

$$S(\mathbf{P}) = -\sum_{0 \leq j < i \leq n-1} \epsilon_j \delta_i, \tag{4}$$

which respectively lead to

$$S(\mathbf{P}) = \sum_{i=0}^{n-1} x_i (y_{i+1} - y_i) \tag{5}$$

and

$$S(\mathbf{P}) = -\sum_{i=0}^{n-1} y_i (x_{i+1} - x_i). \tag{6}$$

The following notation from generating functions is useful in this paper: For a polynomial (or more generally a formal power series) $f(x)$ the coefficient of x^i in $f(x)$ is denoted by $[x^i]f(x)$. The values $[x^i y^j]f(x,y)$ and $[x^i y^j z^k]f(x,y,z)$ are defined similarly.

For a finite sequence of symbols Σ, called *alphabet*, the free monoid over Σ, denoted as Σ^*, is the set of all finite sequences (together with the empty sequence) constructed using elements of Σ; Any such sequence is called a word over Σ. For an element $w \in \Sigma^*$ the length of this sequence is denoted by $|w|$ and for any $i \in \Sigma$, the value $|w|_i$ is the number of occurrences of i in w. For a nonnegative integer n, Σ^n denotes words of length n over the alphabet Σ.

The square lattice is a graph with vertex set $\mathbb{Z}^2$ in which two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. The square lattice is usually denoted by $\mathbb{Z}^2$. Any walk in $\mathbb{Z}^2$ can be determined by its sequence of vertices. Alternatively, since in each step of such a walk, is just moving to right, left, up or down, a walk of length n in $\mathbb{Z}^2$ is coded by the starting point (x_0, y_0) and a word $w \in \Sigma^n$, where $\Sigma = \{r, \ell, u, d\}$. For this reason, from now on in this manuscript, we fix this four element set as out alphabet Σ. With the mentioned notation, a closed walk of length n in $\mathbb{Z}^2$ corresponds to a polygon in the plane with integer coordinates in which, the identity $\{\delta_i, \epsilon_i\} = \{0, \pm 1\}$ holds for $i = 0, \cdots, n-1$.

We frequently use binomial coefficients and some of the identities satisfied by them, particularly Vandermond's identity. For a real number a and a nonnegative integer m the value of $\binom{a}{m}$ is defined as the ratio $\frac{a(a-1)\cdots(a-m+1)}{m!}$. When a is a nonnegative integer, (as usually holds in this paper), we have $\binom{a}{m} = \frac{a!}{m!(a-m)!}$. When m is not a nonnegative integer, the value of $\binom{a}{m}$ is defined to be 0. If a, b and m are nonnegative integers, the well-known Vandermond's identity states

$$\sum_{i=0}^{m} \binom{a}{i} \binom{b}{m-i} = \binom{a+b}{m}.$$

For elementary concepts of algebra the reader in referred for instance to [6]. Moreover, we need the following definition in Section 5 (See Section 25.3 of the same reference). Let $G = \{g_i\}$ be a multiplicative group and let R be a commutative ring with unity. Then the set $R(G)$ which consists of all formal summations $\sum_i a_i g_i$, with $a_i \in R$ and $g_i \in G$ and with finitely many nonzero a_i's is a ring, called the *group ring of G over R*. If instead of R we have a field F, then $F(G)$ is called the *group algebra of G over F*.

3 Area of walks in the square lattice

As mentioned before, a closed walk in the square lattice corresponds to a polygon with integer coordinates such that for any $0 \leq i \leq n-1$, $\{\delta_i, \epsilon_i\} = \{0, \pm 1\}$. Thus the area of such a walk, is defined as the algebraic area of the corresponding polygon. A natural idea to define the concept of area for a walk in $\mathbb{Z}^2$ which is not necessarily closed, is to close it by a sequence of horizontal and then vertical movements. Before following this idea, we fix our notation by considering W as a given walk in $\mathbb{Z}^2$ with vertex sequence $A_0, A_1, \cdots, A_n$ and coordinates $A_i = (x_i, y_i)$ for $i = 0, \cdots, n$. Moreover, for $i = 0, \cdots, n-1$ let $\delta_i = x_{i+1} - x_i$ and $\epsilon_i = y_{i+1} - y_i$. The following two concepts are associated to W:

Definition 1. The closure of W, denoted as W_c is a closed walk of length $n + |x_0 - x_n| + |y_0 - y_n|$ with vertex sequence

$$W_c = [(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), \cdots, (x_0, y_n), \cdots, (x_0, y_0)],$$

where the path $(x_n, y_n), \cdots, (x_0, y_n)$ consists of all left movements (resp. all right movements) if $x_0 \leq x_n$ (resp. if $x_0 > x_n$) and the path $(x_0, y_n), \cdots,$

(x_0, y_0) consists of all down movements (resp. all up movements) if $y_0 \leq y_n$ (resp. if $y_0 > y_n$).

Definition 2. The area of W is defined to be the area of W_c, that is to say

$$S(\mathbf{W}) = \sum_{i=0}^{n-1} x_i \epsilon_i + x_0(y_0 - y_n). \tag{7}$$

It is easily proved that $S(W)$ is independent from the starting point (x_0, y_0) and depends only on the word w.

Two walks can simply be composed as follows:

Definition 3. Let

$$W = [(x_0, y_0), (x_1, y_1) \ldots, (x_n, y_n)],$$

$$W' = [(x'_0, y'_0), (x'_1, y'_1), \ldots, (x'_m, y'_m)].$$

The composed walk, WW', is defined as

$$WW' = [(x_0, y_0), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1}), \ldots, (x_{n+m}, y_{n+m})],$$

where $x_{n+i} = x_n + x'_i - x'_0$ and $y_{n+i} = y_n + y'_i - y'_0$ for $1 \leq i \leq m$.

It is immediately seen that this composition corresponds to concatenation of words w and w'. Now we have the following result.

Proposition 1. *With the above notation for W and W', for $i = 0, 1, \cdots, n+m-1$ let $\epsilon_i = y_{i+1} - y_i$. The area of WW' then equals*

$$S(WW') = S(W) + S(W') + (x_n - x_0)(y'_m - y'_0). \tag{8}$$

Proof. We have

$$\begin{aligned} S(WW') &= \sum_{i=0}^{n+m-1} x_i \epsilon_i + x_0(y_0 - y_{n+m}) \\ &= \sum_{i=0}^{n-1} x_i \epsilon_i + \sum_{i=n}^{n+m-1} x_i \epsilon_i + x_0(y_0 - y_n) + x_0(y_n - y_{n+m}) \\ &= \sum_{i=0}^{n-1} x_i \epsilon_i + x_0(y_0 - y_n) + \sum_{i=n}^{n+m-1} x_i \epsilon_i \\ &\quad + x_n(y_n - y_{n+m}) + (x_0 - x_n)(y_n - y_{n+m}) \\ &= S(W) + S(W') + (x_n - x_0)(y_{n+m} - y_n), \end{aligned}$$

which yields (8). □

Let $(x_0, y_0) = (x'_0, y'_0) = (0, 0)$, $(x_n, y_n) = (i, j)$, $(x_m, y_m) = (i', j')$, $S(W) = S$ and $S(W') = S'$. Then

$$S(WW') = S + S' + ij'. \tag{9}$$

The geometric interpretation of this fact is demonstrated in Figure 1.

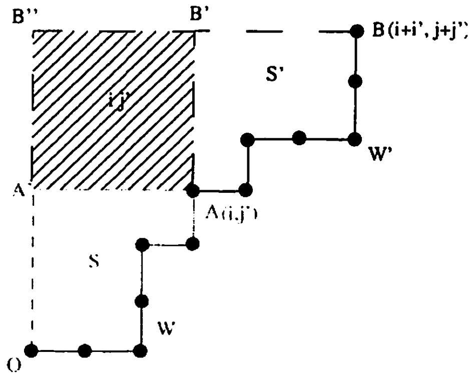

Figure 1

Remark 1. In definition 1 we have closed a walk by horizontal and then vertical movements. One may ask what happens if we close a walk with first vertical and then horizontal movements? In fact if one defines the closure of a walk W by

$$W_{c'} = [(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), \cdots, (x_n, y_0), \cdots, (x_0, y_0)],$$

and let $S(W) = S(W_{c'})$ then it is easily obtained that the right side of (7) is replaced by $\sum_{i=0}^{n-1} x_i\epsilon_i + x_n(y_0 - y_n)$ and the right side of (9) is replaced by $S(W) + S(W') - i'j$. If instead of replacing W by a closed walk in $\mathbb{Z}^2$, we close it as soon as possible, which means if we define $S(W) = S(Q_c)$, where Q_c is the following polygon

$$Q_c = [(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), (x_0, y_0)],$$

then the right side of equation (9) will be replaced by $S(W) + S(W') + \frac{1}{2}(ij' - i'j)$.

4 A useful generating function

In this section we mention a useful generating function of the walks in $\mathbb{Z}^2$ starting at the origin with respect to coordinates of their endpoints (See Section 9.1.1 of [7]) and some of its consequences. We begin by the following known proposition.

Proposition 2. *Let* $u = \alpha + \alpha^{-1} + \beta + \beta^{-1}$ *and denote the number of walks of length* n *in the square lattice which starts at the origin and ends in* (p, q) *by* $a_n(p, q)$. *Then*

(i) *The number* $a_n(p,q) = [\alpha^p\beta^q]\mathrm{u}^n$.

(ii) *We have* $a_n(p,q) = \binom{n}{\frac{n+p+q}{2}}\binom{n}{\frac{n-p+q}{2}}$.

Proof. (i) Any walk of length n, from the origin to (p,q), is coded by a word $w \in \Sigma^n$ with $|w|_r - |w|_\ell = p$ and $|w|_u - |w|_d = q$. Replacing r, ℓ, u and d respectively by α, α^{-1}, β and β^{-1}, we conclude that any such word corresponds to a term $\alpha^p\beta^q$ in the expansion of u^n; Thus the coefficient of $\alpha^p\beta^q$ equals the number of such walks.

(ii) We have

$$\begin{aligned}
a_n(p,q) &= [\alpha^p\beta^q](\alpha+\alpha^{-1}+\beta+\beta^{-1})^n\\
&= [\alpha^p\beta^q]\sum_{i=0}^{n}\binom{n}{i}(\alpha+\alpha^{-1})^i(\beta+\beta^{-1})^{n-i}\\
&= [\alpha^p\beta^q]\sum_{i=0}^{n}\binom{n}{i}\alpha^{-i}\beta^{-n+i}(\alpha^2+1)^i(\beta^2+1)^{n-i}\\
&= \sum_{i=0}^{n}\binom{n}{i}\binom{i}{\frac{p+i}{2}}\binom{n-i}{\frac{q+n-i}{2}}\\
&= \sum_{i=0}^{n}\binom{n}{\frac{n+p+q}{2}}\binom{\frac{n+p+q}{2}}{\frac{n+q-i}{2}}\binom{\frac{n-p-q}{2}}{\frac{-p+i}{2}}\\
&= \binom{n}{\frac{n+p+q}{2}}\binom{n}{\frac{n+q-p}{2}},
\end{aligned}$$

as required. Note that in the last step, Vandermond's identity is used. □

Remark 2. We remark that it is possible to prove Proposition 2(ii) by a direct combinatorial argument, it is enough to project steps of a given walk on the axes $y = x$ and $y = -x$ (This technique is already used, see for instance Proposition 2.3 of [9]). For instance if we project walk steps on the axis $y = x$, the steps to right and up are forward and the steps to left and down are backward, so we obtain $|w|_r + |w|_u - |w|_\ell - |w|_d = p+q$ which together with $|w|_r + |w|_u + |w|_\ell + |w|_d = n$ yields

$$|w|_r + |w|_u = \frac{n+p+q}{2}$$
$$|w|_\ell + |w|_d = \frac{n-p-q}{2}$$

Similarly, by projection on the axis $y = -x$ we obtain

$$|w|_r + |w|_d = \frac{n+p-q}{2}$$
$$|w|_\ell + |w|_u = \frac{n-p+q}{2}$$

Now consider the mapping h, defined by $h(u) = (a,a)$, $h(d) = (b,b)$, $h(\ell) = (b,a)$ and $h(r) = (a,b)$ (See page 451 of [7]). By applying this mapping to the letters of $w = w_0 w_1 \cdots w_{n-1}$ and setting $h(w_i) = (s_i, t_i)$ for $i = 0, 1, \cdots, n-1$, we obtain the words $s = s_0 s_1 \cdots s_{n-1}$ and $t = t_0 t_1 \cdots t_{n-1}$ with $s, t \in \{a,b\}^n$ and $|s|_a = \frac{n+p+q}{2}$ and $|t|_a = \frac{n-p+q}{2}$. Conversely, it is easily observed that given $s, t \in \{a,b\}^n$, the word w is determined uniquely. But there are $\binom{n}{\frac{n+p+q}{2}}$ choices for s and $\binom{n}{\frac{n+p-q}{2}}$ choices for t, which proves Proposition 2(ii).

Although the following identities are concluded immediately from Proposition 2(ii), based on symmetry, we need only part (i) of that proposition to conclude the following.

Proposition 3. *With the notation of the Proposition 2 we have*

(i)
$$a_n(p,q) = a_n(\pm p, \pm q). \tag{10}$$

(ii)
$$a_n(p,q) = a_n(q,p). \tag{11}$$

5 A noncommutative multiplication

In this section, based on a simple combinatorial argument, we present a noncommutative extension of Proposition 2(i). Also we present an extension for Proposition 3. Let $w_n(i,j,s)$ be the number of walks starting at the origin and ending in (i,j) and having algebraic area s. Then by using identity (9) we have the following enumerative identity:

$$w_{n+m}(i,j,s) = \sum_{I_{ijs}} w_n(i_1,j_1,s_1) w_m(i_2,j_2,s_2), \tag{12}$$

where I_{ijs} consists of the set of pairs of integer triples (i_1,j_1,s_1) and (i_2,j_2,s_2) which satisfy the following set of equations:

$$i_1 + i_2 = i, \; j_1 + j_2 = j, \; s_1 + s_2 + i_1 j_2 = s. \tag{13}$$

Equations (12) and (13) lead us to study the multiplication of two monomials $X = x^i y^j z^s$ and $X' = x^{i'} y^{j'} z^{s'}$ defined as

$$x^i y^j z^s . x^{i'} y^{j'} z^{s'} = x^{i+i'} y^{j+j'} z^{s+s'+ij'} \tag{14}$$

It is easy to check that the set of all monomials $x^i y^j z^s$ with this multiplication construct a non-commutative group. Note that the element $x^i y^j z^s$ is just a representation of an element of a group (which may also be represented just by the triple (i, j, s)) and in general does not equal $x^i . y^j . z^s$ (for instance $x.y.z = xyz^2$). This leads to construct a group ring with elements $\sum a(i,j,s) x^i y^j z^s$ with finitely many nonzero coefficients which come from a ring, say $\mathbb{R}$.

Proposition 4. *Let* $\mathfrak{w} = x + x^{-1} + y + y^{-1}$ *and let* $w_n(p,q,s)$ *denote the number of walks in* $\mathbb{Z}^2$ *which start at the origin, end at* (p,q) *and have* s *as their areas. Then we have*

$$w_n(p,q,s) = [x^p y^q z^s]\mathfrak{w}^n, \tag{15}$$

where multiplication is as defined in (14).

Proof. For this note that by setting $m = 1$ in (12) we provide

$$w_{n+1}(p,q,s) = w_n(p-1,q,s) + w_n(p+1,q,s) + w_n(p,q-1,s-p) + w_n(p,q+1,s+p) \tag{16}$$

Now we use induction on n: The basis step, $n = 0$, is trivial. Presuming the validity of (15) for a given integer n and all integers p, q, s, we have

$$\begin{aligned}
\mathfrak{w}^{n+1} &= \mathfrak{w}^n (x + x^{-1} + y + y^{-1}) \\
&= \left(\sum_{p,q,s} w_n(p,q,s) x^p y^q z^s \right) (x + x^{-1} + y + y^{-1}) \\
&= \sum_{p,q,s} w_n(p,q,s) x^{p+1} y^q z^s + \sum_{p,q,s} w_n(p,q,s) x^{p-1} y^q z^s \\
&+ \sum_{p,q,s} w_n(p,q,s) x^p y^{q+1} z^{s+p} + \sum_{p,q,s} w_n(p,q,s) x^p y^{q-1} z^{s-p} \\
&= \sum_{p,q,s} w_n(p-1,q,s) x^p y^q z^s + \sum_{p,q,s} w_n(p+1,q,s) x^p y^q z^s \\
&+ \sum_{p,q,s} w_n(p,q-1,s-p) x^p y^q z^s + \sum_{p,q,s} w_n(p,q,s+p) x^p y^q z^s \\
&= \sum_{p,q,s} w_{n+1}(p,q,s) x^p y^q z^s \quad \text{(by using (16))},
\end{aligned}$$

which concludes the validity of (15) for $n + 1$ and all integers p, q, s and completes the induction. □

Example 1. It is easily checked that

$$\begin{aligned}\mathfrak{w}^2 &= (x + x^{-1} + y + y^{-1}).(x + x^{-1} + y + y^{-1})\\ &= x^2 + 1 + xy + xy^{-1} + 1 + x^{-2} + x^{-1}y + x^{-1}y^{-1}\\ &\quad + xyz + x^{-1}yz^{-1} + y^2 + 1 + xy^{-1}z^{-1} + x^{-1}y^{-1}z + 1 + y^{-2}\\ &= x^2 + x^{-2} + y^2 + y^{-2} + 4 + xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1}\\ &\quad + xyz + x^{-1}yz^{-1} + xy^{-1}z^{-1} + x^{-1}y^{-1}z.\end{aligned}$$

The terms of the form $axyz^i$ are xy and xyz. This means that there are just two walks of length 2 which begin from the origin and end in $(1,1)$: one walk with area 0 and another one with area 1.

Example 2. Again consider the previous example. If we want to answer the same question about walks of length 4, it is enough to calculate the terms $axyz^i$ in $\mathfrak{w}^4$. But using $\mathfrak{w}^4 = (\mathfrak{w}^2)^2$ and applying the above result, these terms are as follows:

$$\begin{aligned}&x^2.x^{-1}y + x^2.x^{-1}yz^{-1} + y^2.xy^{-1} + y^2.xy^{-1}z^{-1} + 4xy + 4xyz +\\ &4xy + xy^{-1}.y^2 + x^{-1}y.x^2 + 4xyz + x^{-1}yz^{-1}.x^2 + xy^{-1}z^{-1}.y^2\\ = {}& xy(2z^2 + 10z + 10 + 2z^{-1}).\end{aligned}$$

Remark 3. Considering Remark 1, Proposition 4 remains still true if we replace the right side of (14) either by $x^{i+i'}y^{j+j'}z^{s+s'-i'j}$ or by $x^{i+i'}y^{j+j'}z^{s+s'+\frac{1}{2}(ij'-i'j)}$. Note that in the second case, our multiplication would be similar to formula (1) in [3].

Proposition 5.

(i)
$$w_n(i,j,s) = w_n(-i,j,-s) = w_n(i,-j,-s) = w_n(-i,-j,s).$$

(ii)
$$w_n(i,j,s) = w_n(j,i,s).$$

(iii)
$$w_n(i,j,s) = w_n(i,j,ij-s).$$

Proof. (i) To prove $w_n(i,j,s) = w_n(-i,j,-s)$ for instance, let W be a walk in $\mathbb{Z}^2$, coded by the word $w \in \Sigma^n$ and let $w' = h(w)$ where h is the following morphism

$$h : \begin{cases} r \to \ell \\ \ell \to r \\ u \to u \\ d \to d \end{cases}$$

and let W' be the walks starting from the origin and coded by w'. Clearly, W' is the mirror image of W with respect to the y axis, so its coordinates satisfy $x'_i = -x_i$ and $y'_i = y_i$ for $i = 0, 1, \cdots, n$, hence, by (7), we have $S(W') = -S(W)$. This correspondence is clearly a bijection so we have $w_n(i, j, s) = w_n(-i, j, -s)$. The other equations are proved similarly. We remark that all the equations of part (i) can be unified as

$$w_n(i, j, s) = w_n(\varepsilon_1 i, \varepsilon_2 j, \varepsilon_1 \varepsilon_2 s), \ (\varepsilon_1, \varepsilon_2) = (\pm 1, \pm 1)$$

and the proofs can be unified by using morphism

$$h_{\epsilon_1,\epsilon_2} : \begin{cases} r \rightarrow \epsilon_1 r \\ \ell \rightarrow \epsilon_1 \ell \\ u \rightarrow \epsilon_2 u \\ d \rightarrow \epsilon_2 d \end{cases}$$

where by definition $-r = \ell$, $-\ell = r$, $-u = d$ and $-d = u$.

(ii) Let the walk W is coded by w and let w' be the word obtained by reversing w and replacing r, ℓ, u, d respectively by u, d, r, ℓ. It is proved that the walk W' coded by w' has the same area as W. It is easily concluded that $w_n(i, j, s) = w_n(j, i, s)$.

(iii) Let W be walk starting at the origin and ending in (i, j) and let W' be its mirror image with respect to the line $y = x$. It is simple to prove that $S(W') = ij - S(W)$. An easy conclusion would be $w_n(i, j, s) = w_n(j, i, ij-s)$, thus using part(ii) we provide $w_n(i, j, s) = w_n(i, j, ij - s)$.

□

Remark 4. If we set $z = 1$ in (14), the relation appears as a usual commutative multiplication; Similarly, considering the result of Proposition 4 as $\mathfrak{w}^n = \sum_{p,q} x^p y^q \sum_s w_n(p, q, s) z^s$ and setting $z = 1$ concludes just Proposition 2(i) as a special case. Hence, Proposition 4 is a noncommutative extension of Proposition 2(i). The absence of a computational result such as part (ii) of Proposition 2 (or a weaker version of it) in the noncommutative version is clear. We wonder whether the following matrix identities [5] are helpful in this respect? Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$A^jB^iZ^s = \begin{bmatrix} 1 & i & s \\ 0 & 1 & j \\ 0 & 0 & 1 \end{bmatrix} \quad (17)$$

hence

$$A^jB^iZ^s\,A^{j'}B^{i'}Z^{s'} = A^{j+j'}B^{i+i'}Z^{s+s'+ij'}. \quad (18)$$

We mention that the last identity is important because it connects the previous complicated multiplication rules to usual matrix multiplication.

6 An algorithm to enumerate walks

The results of the previous section naturally leads to an algorithm for calculation of the coefficients $w_n(p,q,s))$ which is much better than generating the walks themselves (similar to the one used in [1]). In this section, we study and analysis this algorithm. As in examples 1 and 2 one can compute the expression $\mathfrak{w}^n$ for given values of n to obtain values $w_n(i,j,s)$. Of course this can be done for any positive integer n (not only powers of 2,) by calculating expressions of form $\mathfrak{w}^{n_1}.\mathfrak{w}^{n_2}$ at most $\lg(n)$ times. To analyze the provided algorithm, first note that if $|i|+|j|>n$ or if $n+i+j$ is odd or if $|s|>\frac{n^2}{4}$, then $w_n(i,j,s)=0$ (In fact if $|s|>\frac{(n+|i|+|j|)^2}{16}$ then $w_n(i,j,s)=0$). Thus the expression $\mathfrak{w}^n$ has $O(n^4)$ nonzero terms and computation of the expression $\mathfrak{w}^{n_1}.\mathfrak{w}^{n_2}$ needs totally $O({n_1}^4{n_2}^4)$ multiplications. Thus for calculating $\mathfrak{w}^n$, the number of required integer multiplications is $O(n^8\lg(n))$. However, as n grows larger, the coefficients $w_n(i,j,s)$ grow exponentially and should be considered as "large integers" (instead of integers) and the integer multiplication should not be considered as a unitary operation. (For computation with large integers, see for instance [4]). This problem is resolved by using modular arithmetic as follows: Choose k and distinct prime integers $p_1,\cdots,p_k$ such that $w_n(i,j,s)<p_1\cdots p_k$ (It is enough to select these numbers such that $p_1\cdots p_k>4^n$). For any i, $1\le i\le k$, calculate the coefficients of $\mathfrak{w}^n \mod p_i$. Finally for each s with $s\le n^2/16$ a sequence $t_1,\cdots,t_k$ with $w_n(0,0,s)\equiv t_i \pmod{p_i}$ is obtained. Thus the values $w_n(0,0,s)$ can easily be reconstructed using Chinese remainder theorem. Since $k=O(\lg(n))$, the complexity of our algorithm in this case (i.e. when 4^n is a large integer), is computed as $O(n^8\lg^2(n))$.

Let $n=2m$ be an even positive integer; By Proposition 5 (iii), the sequence $\{w_{2m}(0,0,s)\}_s$ is symmetric with respect to the axis $s=0$ and it is easily seen that $w_{2m}(0,0,s)>0$ if and only if $|s|\le\lfloor\frac{m^2}{4}\rfloor$. We have implemented our algorithm for $n=8,16,32,64$; Furthermore, we have obtained the terms $w_{128}(0,0,s)$ in the expression $\mathfrak{w}^{128}$ (As mentioned before, since the coefficients are large for $n=64,128$, we have used some modular arithmetic to simplify the calculations). It is observed that the sequence $\{w_{2m}(0,0,s)\}_s$ is unimodal and takes its maximum at $s = 0$. The results of computations are briefly demonstrated in tables 1.1, 1.2, 2.1, 2.2 (Due to the symmetry, negative values of s are omitted from these tables). Histograms of the number of closed walks with corresponding areas are demonstrated in table 1.1 for $n = 16, 32, 64$, $0 \leq s \leq 50$ and in table 1.2 for $n = 128$, $0 \leq s \leq 50$. We have used sampling of areas to estimate the whole distribution of walks with respect to their areas in table 2.1 for $n = 32, 64$ and in table 2.2 for $n = 128$. Complete tables of values of $w_n(0,0,s)$ for $n = 32, 64$ and 128 can be found at [11]. We mention that it is possible to improve the complexity of this algorithm to $O(n^6 \lg^3(n))$ by some modifications. However, we do not need to implement this modified version for $n \leq 128$.

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Table 1.1. Histogram of the number of closed walks with given algebraic area s for $n = 16, 32, 64,\ 0 \leq s \leq 50$.

Area s	$w_{16}(0,0,s)$	$w_{32}(0,0,s)$	$w_{64}(0,0,s)$
0	33820044	3581690, 9974343308	165545, 3003285874, 5794673311, 4483378060
1	28133728	3444028, 9607452416	163951, 1042472083, 1102818268, 2298389120
2	18569808	3073077, 4567275040	159289, 6955895232, 9652405885, 9706370944
3	10127744	2565083, 0257099008	151905, 3681689474, 6366542956, 7889122560
4	5015108	2025070, 2695492528	142312, 9428052661, 5202704645, 6952968352
5	2289760	1529170, 0875844800	131125, 1853389455, 8748104447, 7490274432
6	1036368	1116254, 1356438464	118976, 8131598077, 3546151378, 9923455744
7	435040	794324, 3235665408	106459, 4734512548, 5432356922, 6277637888
8	184104	554823, 8812436036	94076, 4049114445, 2830982857, 7473179632
9	73056	382197, 6073766784	82219, 0388885868, 0062962560, 1052407296
10	28064	260613, 1312522976	71162, 7986902081, 4394878378, 0740454720
11	10336	176316, 8848622336	61076, 6841799750, 4289859727, 5624957312
12	3760	118576, 4173049648	52040, 7460865277, 2477622381, 7129090240
13	1088	79335, 0438261504	44066, 4566930747, 9914521591, 0107701504
14	352	52859, 9386326560	37116, 4869321275, 0343143165, 4198093312
15	96	35083, 2035517248	31121, 8811662498, 3702411709, 7801894016
16	16	23202, 5420494728	25995, 8037652707, 6596386739, 4450277316
17	0	15290, 8309279936	21643, 8097824910, 3762445371, 2586987264
18	0	10044, 4272588768	17971, 0315275440, 8547703288, 0937254400
19	0	6574, 8927845440	14886, 8517633991, 1765417370, 3945559808
20	0	4289, 6632260736	12307, 6599368221, 5520959165, 4379143840
21	0	2788, 4857169280	10158, 2235010504, 6499042413, 6544733440
22	0	1806, 4537994848	8372, 1122544237, 1269186309, 3162306688
23	0	1165, 8943874752	6891, 5113696382, 9340030299, 8512645376
24	0	749, 7160572048	5666, 6701469563, 7412442711, 0537836464
25	0	480, 1211445312	4655, 1587087675, 5085653450, 8420517888
26	0	306, 2945599680	3821, 0488492849, 7035370375, 8706331264
27	0	194, 5755843520	3134, 0926012667, 5420527449, 4192796416
28	0	123, 0912937696	2568, 9430707074, 7787699462, 7648267040
29	0	77, 5044394624	2104, 4412234929, 2181969190, 3419356416
30	0	48, 5883898144	1722, 9792841968, 7831969876, 7886027968
31	0	30, 3067180160	1409, 9425700569, 3405222424, 1279150592
32	0	18, 8158770672	1153, 2267976836, 0220074562, 9419251848
33	0	11, 6190755520	942, 8248707710, 9783233726, 8937696128
34	0	7, 1372120768	770, 4760703796, 4923852241, 8443915072
35	0	4, 3588560640	629, 3700506407, 3095007510, 8661437568
36	0	2, 6468754368	513, 8983895305, 2165176788, 5718002464
37	0	1, 5971326400	419, 4468635943, 1128763468, 3935688320
38	0	9580227072	342, 2223782721, 8792023636, 1272721856
39	0	5704976448	279, 1091340556, 7526383551, 8177452288
40	0	3374362720	227, 5493796993, 4261990343, 2651390528
41	0	1979897600	185, 4447127237, 3591685480, 3318881920
42	0	1153531776	151, 0745274528, 2500466045, 4542766592
43	0	665930496	123, 0287071199, 9349261843, 0645507456
44	0	381403552	100, 1521446247, 9515175280, 4221836704
45	0	216272192	81, 4990481851, 9987650971, 6280904448
46	0	121397120	66, 2953446385, 4157940925, 1272815296
47	0	67391168	53, 9077623054, 4736632512, 9024635136
48	0	37007392	43, 8184278757, 0688296731, 2469700240
49	0	20046912	35, 6040025857, 9755520652, 8140275072
50	0	10730048	28, 9185589829, 9232653138, 8592603584

Table 1.2. Histogram of the number of closed walks with given algebraic area s for $n = 128$, $0 \leq s \leq 50$.

Area s	$w_{128}(0, 0, s)$
0	1410, 7033892003, 4556275957, 3855536443, 1713372583, 8745556276, 5835946782, 8699656588
1	1407, 3024540489, 7561178225, 1492421016, 5903644384, 0838340709, 2111460482, 4937387520
2	1397, 1649733707, 8547470736, 0499774263, 8920388680, 8359080398, 3163318225, 4162311040
3	1380, 4842678848, 7302789379, 1645965215, 5579137301, 3499661185, 3647061656, 3763726848
4	1357, 5738773060, 9946518605, 0585703091, 4103655038, 8810690309, 8680556206, 4476333696
5	1328, 8551923721, 7677597822, 7209730464, 4139815783, 0161065810, 1704581134, 1916138496
6	1294, 8413214797, 3192840866, 6424244840, 7914453824, 1990913897, 2602303118, 5542646784
7	1256, 1182279806, 0973554574, 6222544230, 8237684539, 6052335327, 6433750005, 5423763456
8	1213, 3242855140, 0028030693, 5376370151, 6590699200, 7160036130, 3615131804, 7547629472
9	1167, 1294078548, 3705936097, 4700188169, 9134665437, 9964875422, 2126644178, 1188844032
10	1118, 2148257931, 4876464340, 5178198327, 6272565882, 5003428615, 7423056334, 4507678592
11	1067, 2544253010, 0491533514, 7894490482, 6439471597, 5277883548, 2780506254, 9148335104
12	1014, 8983533104, 4634867110, 1206819214, 1193382594, 8253734997, 0143386378, 9088034176
13	961, 7593661773, 7189043046, 3337315981, 8652756444, 2697345733, 0322031818, 4291592704
14	908, 4021657996, 3631929848, 5392789504, 0553785827, 1399264505, 9083287686, 8407903488
15	855, 3357592722, 7904570583, 4071833016, 3005561629, 6188755045, 2681666035, 3268214784
16	803, 0087037290, 4609496133, 5353158915, 5592171685, 1608553021, 1235191565, 6504449136
17	751, 8069661057, 2813072696, 6629583064, 1148708036, 7436477079, 4895846268, 6542646784
18	702, 0540396862, 5146881526, 9860430186, 5491459096, 2468812491, 6045345873, 6116161536
19	654, 0129126309, 4424638675, 1129960649, 6515635171, 2836056527, 8268318381, 3914975488
20	607, 8894723270, 7517045571, 1069505624, 9654909825, 9691568380, 4579410665, 3886321856
21	563, 8369457848, 1328923268, 0251406724, 5624510978, 4465364091, 8338626540, 2448514048
22	521, 9610124403, 1971690875, 7285266312, 1361048967, 8464554560, 1550600804, 2947782528
23	482, 3252741185, 5977063374, 9204570737, 0096681055, 7983960514, 9055938353, 6273292544
24	444, 9568211123, 2677528406, 7913521950, 2168637496, 3514310616, 1020629339, 0696154560
25	409, 8516882597, 8820945433, 0086445538, 6875648834, 3075383182, 7515093856, 2496782336
26	376, 9800468596, 7878523294, 6468408515, 6838053818, 2117951574, 8956155126, 1898202880
27	346, 2910248525, 7910212051, 9918713267, 6285219824, 3417736320, 7150365881, 6662952192
28	317, 7170876122, 6649370409, 8769310919, 3372955110, 9885406167, 7375795751, 0850541184
29	291, 1779444806, 8780735863, 2682558559, 3943314167, 7299990195, 1784320302, 7017344000
30	266, 5839720083, 3217144183, 2330534321, 5631455953, 9593549729, 2939952449, 8739436928
31	243, 8391642983, 0152229604, 1800095915, 7149399101, 9378288422, 7765033861, 2981857280
32	222, 8436346973, 8653725051, 1841404543, 0931745709, 4233102150, 3754802180, 2932444612
33	203, 4957022281, 8650274258, 8409511283, 0795000912, 6351287804, 8241811701, 0044212480
34	185, 6936015120, 1442388100, 3756227758, 1997286304, 1641800578, 0316304872, 5931503104
35	169, 3368573323, 1966937758, 8587669558, 1677928523, 5336728042, 7009350227, 8754297344
36	154, 3273651764, 1673709820, 8254120503, 0091312669, 4620532769, 1964334067, 6582675712
37	140, 5702177017, 1811870033, 4744950358, 1126325851, 8097400944, 1897482027, 1138028544
38	127, 9743146199, 7216121242, 6590547944, 5795679103, 3517158052, 5703219655, 5346880768
39	116, 4527903905, 2565093234, 5991490752, 6931680721, 8392377764, 6639015944, 4253611008
40	105, 9232906770, 0813934257, 9389349588, 9826312713, 6299361111, 8150382831, 9241859104
41	96, 3081249848, 4003726572, 9985651294, 9767291595, 5105092725, 2887050944, 2661204480
42	87, 5343194257, 0689809870, 9084153396, 1633523208, 5765141173, 2963363894, 5553767168
43	79, 5335902615, 1001088786, 2229069228, 4944928161, 2471305895, 5160229025, 3471878400
44	72, 2422558339, 9992080512, 9735364612, 5415047397, 5072102242, 8523062539, 9947401984
45	65, 6011017251, 4946098123, 3635319649, 9707716860, 2623541436, 0238379304, 1379316736
46	59, 5552115318, 2629696048, 9600650535, 3649769161, 7005888983, 0456617439, 1624400784
47	54, 0537734745, 4545929513, 5620500429, 2190688647, 1817158178, 7000139728, 0834058496
48	49, 0498711810, 5005814800, 2302396044, 8526299819, 6640254673, 3055286303, 8011851696
49	44, 5002653713, 1010783245, 2854022296, 7683798570, 5477576649, 4078144220, 8843469312
50	40, 3651717975, 4268971940, 8275974742, 9828479525, 6161271917, 9464626860, 0827549440

Table 2.1. Values of $w_{32}(0,0,2k)$ and $w_{64}(0,0,8k)$

Area s	$w_{32}(0,0,s)$	Area s	$w_{64}(0,0,s)$
0	35816909974343308	0	165545, 3003285874, 5794673311, 4483378060
2	30730774567275040	8	94076, 4049114445, 2830982857, 7473179632
4	20250702695492528	16	25995, 8037652707, 6596386739, 4450277316
6	11162541356438464	24	5666, 6701469563, 7412442711, 0537836464
8	5548238812436036	32	1153, 2267976836, 0220074562, 9419251848
10	2606131312522976	40	227, 5493796993, 4261990343, 2651390528
12	1185764173049648	48	43, 8184278757, 0688296731, 2469700240
14	528599386326560	56	8, 2365472231, 4016874589, 9460134624
16	232025420494728	64	1, 5096688485, 6768973162, 6864590576
18	100444272588768	72	2694437402, 9152872362, 8592916864
20	42896632260736	80	467552007, 4041838163, 0604620576
22	18064537994848	88	78744016, 9083977884, 8133019296
24	7497160572048	96	12846824, 7250409523, 5350912480
26	3062945599680	104	2025952, 5183890308, 8567109088
28	1230912937696	112	308080, 5223088698, 5702239872
30	485883898144	120	45051, 1484837085, 3522061280
32	188158770672	128	6315, 2306732457, 4256973920
34	71372120768	136	845, 5556839591, 3528941504
36	26468754368	144	107, 6805280447, 7651259776
38	9580227072	152	12, 9785607608, 7995391104
40	3374362720	160	1, 4718233876, 4286499776
42	1153531776	168	1559345348, 1572357568
44	381403552	176	153004502, 2583865088
46	121397120	184	13753291, 0981167232
48	37007392	192	1116802, 6179713536
50	10730048	200	80423, 8839635904
52	2932896	208	5007, 3152157248
54	743168	216	259, 8435002240
56	172224	224	10, 6187399552
58	35392	232	3102345664
60	5984	240	53445952
62	704	248	337280
64	32	256	64

Table 2.2. Values of $w_{128}(0, 0, 32k)$

Area s	$w_{128}(0, 0, s)$
0	1410, 7033892003, 4556275957, 3855536443, 1713372583, 8745556276, 5835946782, 8699656588
32	222, 8436346973, 8653725051, 1841404543, 0931745709, 4233102150, 3754802180, 2932444612
64	10, 1648073494, 4923419172, 3106847671, 7720107384, 7236429835, 9359307733, 6892311304
96	4138993608, 0697485547, 8360086674, 4923420517, 9089245541, 4042051861, 9613801232
128	161039050, 5304195851, 5419275033, 5083133903, 1254858120, 1415241780, 7554992752
160	5990655, 6605275424, 0970303298, 7170349389, 5743855708, 8743098925, 7949721120
192	212607, 0814985186, 5951934575, 3598563447, 3878472768, 9106908764, 3180434272
224	7180, 6854685731, 4031517786, 4997582279, 9706234195, 6090873096, 6015299584
256	230, 1750942525, 0140859435, 7574600133, 2584273010, 4614639894, 3043516640
288	6, 9816042224, 1138277250, 0582452617, 1692631908, 7011142507, 9215100160
320	1997190758, 6553543306, 5729674881, 2193385181, 4767886062, 4658510144
352	53685272, 9803391380, 6122089350, 6109205977, 4658578703, 0650139264
384	1350455, 5535411518, 0818176830, 9757666811, 6769796180, 3565932544
416	31644, 2527945050, 2410030323, 1585016453, 7594319542, 2645555776
448	687, 1315164172, 5008635859, 1936569312, 0498550867, 2721950208
480	13, 7450595212, 7015217478, 1091307692, 8909430481, 5008168896
512	2515774906, 7368382551, 6728163527, 5448943450, 0831010624
544	41803611, 2117105753, 9735632910, 5539623620, 6960007680
576	624889, 6425097710, 2464041926, 3546679600, 4985488768
608	8312, 9826788959, 9447485149, 7366840822, 7746173056
640	97, 1577615295, 3509138500, 6659985912, 5226822016
672	9821475660, 0178800451, 7922679241, 8657940480
704	84231795, 4751454140, 9292439100, 2628160000
736	598159, 1733186708, 6431903566, 6217660160
768	3408, 4164367454, 1023022248, 2423726336
800	14, 9447433526, 3720269236, 4107405952
832	475760748, 0371846789, 7834797824
864	1010220, 8082355332, 7727527680
896	1253, 0843615061, 8578411264
928	7227374040, 4149925760
960	1233125, 5335552896
992	19, 8894005504
1024	128