GRAPHS SIMULTANEOUSLY ACHIEVING THREE VERTEX COVER NUMBERS

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ABSTRACT. A *vertex cover* of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that every edge is incident with at least one vertex in S, and $\alpha(G)$ is the cardinality of a smallest vertex cover. Let $\mathcal{T}$ be a collection of vertex covers, not necessarily minimum. We say $\mathcal{T}$ is closed if for every $S \in \mathcal{T}$ and every $e \in E$ there is a one-to-one function $f : S \to V$ such that (1) $f(S)$ is a vertex cover, (2) for some s in S, $\{s, f(s)\} = e$, (3) for each s in S, either $s = f(s)$ or s is adjacent to $f(s)$, and (4) $f(S) \in \mathcal{T}$. A set is an eternal vertex cover if and only if it is a member of some closed family of vertex covers. The cardinality of a smallest eternal vertex cover is denoted $\alpha_m^\infty(G)$. Eternal total vertex covers are defined similarly with the restriction that the cover must also be a total dominating set. The cardinality of a smallest eternal total vertex cover is denoted $\alpha_{mt}^\infty(G)$. These three vertex cover parameters satisfy the relation $\alpha(G) \leq \alpha_m^\infty(G) \leq \alpha_{mt}^\infty(G) \leq 2\alpha(G)$. We define a triple (p, q, r) of positive integers such that $p \leq q \leq r \leq 2p$ to be feasible if there is a connected graph G such that $\alpha(G) = p$, $\alpha_m^\infty(G) = q$, and $\alpha_{mt}^\infty(G) = r$. This paper shows all triples with the above restrictions are feasible if $p \neq q$ or $r \leq 3p/2$ and conjectures that there are no feasible triples of the form (p, p, r) with $r > 3p/2$. The graphs with triple $(p, p+1, 2p)$ are characterized and issues related to the conjecture are discussed.

Keywords: vertex cover, total vertex cover, eternal, edge protection, graph characterization, domination, total domination.

1. INTRODUCTION

In this paper, we study graphs $G = (V, E)$ without loops or multiple edges. In some cases, to avoid ambiguity, we will use the notation $V(G)$ for V. A subset $S \subseteq V$ is a *dominating set* if every vertex in $V - S$ is adjacent to a vertex in S; the set is a *total dominating set* if every vertex in V is adjacent to a vertex in S. A *vertex cover* of a graph G is a subset $S \subseteq V$

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such that every edge is incident with at least one vertex in S. A *total vertex cover* is a vertex cover that is also a total dominating set. The *vertex cover* (*total vertex cover*) *number* of G, denoted $\alpha(G)$ ($\alpha_t(G)$), is the cardinality of a smallest vertex cover (total vertex cover). A vertex cover (total vertex cover) of size $\alpha(G)$ ($\alpha_t(G)$) is an α-set (α_t-set).

The concept of "eternal" is relatively new and provides a dynamic aspect to standard graph invariants. Goddard, Hedetniemi, and Hedetniemi [5] introduced this idea by applying it to domination and Klostermeyer [9] applied it to vertex covers and total vertex covers.

A (total) vertex cover S of G can be thought of as a collection of "guards." An *attack* is the selection of an edge in G. A *defense* to the attack is a one-to-one function $f : S \to V$, such that (1) $f(S)$ is a (total) vertex cover, (2) for some s in S, $\{s, f(s)\}$ is the edge that was attacked, and (3) for each s in S, either $s = f(s)$ or s is adjacent to $f(s)$ (informally, we say that the guard on s defends the attacked edge by moving from s to $f(s)$). If there is a collection $\mathcal{T}$ of (total) vertex covers so that, for every $S \in \mathcal{T}$ and for every attack, there is a defense f with $f(S)$ in $\mathcal{T}$, then we say $\mathcal{T}$ is a *closed family of (total) vertex covers* and the (total) vertex covers in $\mathcal{T}$ are *eternal (total) vertex covers.*

The *eternal vertex cover number* and *eternal total vertex cover number* of G, denoted $\alpha_m^\infty(G)$ and $\alpha_{mt}^\infty(G)$, respectively, are the cardinalities of a smallest eternal vertex cover and eternal total vertex cover. An eternal vertex cover (total vertex cover) of size $\alpha_m^\infty(G)$ ($\alpha_{mt}^\infty(G)$) is an α_m^∞-set (α_{mt}^∞-set). Work on eternal sets can be found in [1, 2, 4-15].

A *connected vertex cover* of graph G is a vertex cover that induces a connected subgraph. Let $\alpha_c(G)$ be the cardinality of a smallest connected vertex cover, and let an α_c-set be a connected vertex cover having $\alpha_c(G)$ vertices. The following lemma is an extension of a theorem proved by Klostermeyer [9].

Lemma 1. *For connected graph G, $\alpha_{mt}^\infty(G) \le \alpha_c(G) + 1 \le 2\alpha(G)$.*

Proof. To show the first inequality, let S be an α_c-set of G, $\mathcal{T} = \{S \cup \{v\} : v \in V - S\}$, and $T = S \cup \{v\} \in \mathcal{T}$. Attack any edge $v'w$ where $w \in S$ and $v' \notin T$. The guard at vertex w of S can move along the attacked edge to vertex v'. There is a path from v to w in T and guards can be moved along that path creating a new connected set $S \cup \{v'\} \in \mathcal{T}$. Hence, $\mathcal{T}$ is a closed family of total vertex covers which implies the first inequality.

Let S be an α-set of G that induces a subgraph having $N \le \alpha(G)$ components. S can be transformed into a connected vertex cover by adding at most $N - 1$ vertices so $\alpha_c(G) \le 2\alpha(G) - 1$ and the second inequality is established. □

Since every α_{mt}^{∞}-set is an α_m^{∞}-set and every α_m^{∞}-set is an α-set, Lemma 1 gives us the following theorem.

Theorem 2. *For any graph G,* $\alpha(G) \le \alpha_m^{\infty}(G) \le \alpha_{mt}^{\infty}(G) \le 2\alpha(G)$.

This paper discusses which sets of three positive integers p, q, and r, such that $p \le q \le r \le 2p$, allow a connected graph G such that $\alpha(G) = p$, $\alpha_m^{\infty}(G) = q$, and $\alpha_{mt}^{\infty}(G) = r$. If such a graph does exist for the triple (p,q,r), the triple is termed *feasible.*

Since a total vertex cover must contain at least two vertices, the triple $p = q = r = 1$ is not feasible. Section 2 shows that all other triples (p,q,r) in the range given above such that (1) $q \ge p+1$ or (2) $q = p$ and $r \le 3p/2$ are feasible. Section 3 characterizes the graphs corresponding to the triple $(p, p+1, 2p)$. Section 4 discusses the remaining open case (p,p,r) where $r > 3p/2$ and Section 5 points to a direction for research into this case.

2. Feasible Triples

In this section we show that most triples satisfying the inequalities in Theorem 2 are feasible. The result is expressed in the following theorem.

Theorem 3. *Let p, q, and r be integers. Then*

(1) *(p,q,r) is a feasible triple for $1 \le p < q \le r \le 2p$ and*
(2) *(p,p,r) is a feasible triple for $2 \le p \le r \le 3p/2$.*

We obtain the proof by a series of lemmas. We deal with an extreme case of Theorem 3 Statement 1 first.

Lemma 4. *The triple $(p, p+1, 2p)$ is feasible for $p \ge 1$.*

Proof. For $p, t \ge 1$ and $(p-1)+t \ge 2$, consider the graph $H_{p,t}$ consisting of $p-1$ C_4's joined at a common vertex v, along with t pendant edges incident to v (see Figure 1). It is straightforward to see that the applicable triple for $H_{p,t}$ is $(p, p+1, 2p)$ for $p \ge 1$. □

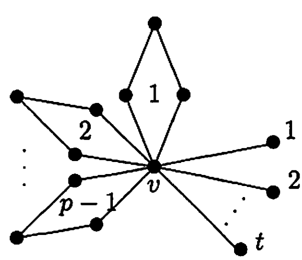

FIGURE 1. Graph $H_{p,t}$

The next lemma finishes the proof of Theorem 3, Statement 1.

Lemma 5. *Let p, q, and r be integers. Then*

(1) $(p, p+1, r)$ *is a feasible triple for* $2 \leq p < r \leq 2p-1$ *and*
(2) (p, q, r) *is a feasible triple for* $2 \leq p$ *and* $p+2 \leq q \leq r \leq 2p$.

Proof. For $p = 2$, the only triple that satisfies the hypotheses of Statement 1 is $(p,q,r) = (2,3,3)$, and its feasibility is shown by a C_3 with one vertex having a pendant edge. The only triple with $p = 2$ that satisfies the hypotheses of Statement 2 is $(p,q,r) = (2,4,4)$, which is shown to be feasible by a C_4 with two pendant edges, one from each of two non-adjacent vertices of the cycle.

For $p > 2$, consider the graph G constructed as follows (see Figure 2). Start with a vertex v. Create m copies of C_4 with vertices labeled in order around the cycle by v, a_i, b_i, and c_i for $1 \leq i \leq m$. On k of these, $0 \leq k \leq m$, add a pendant edge $b_i d_i$. Finally create $s \geq 0$ copies of C_3 with vertices labeled v, e_i, and f_i for $1 \leq i \leq s$. We restrict our attention to the cases for which $m \geq 1$ and $m + s \geq 2$, that is, to graphs with at least one four cycle and at least two cycles in total.

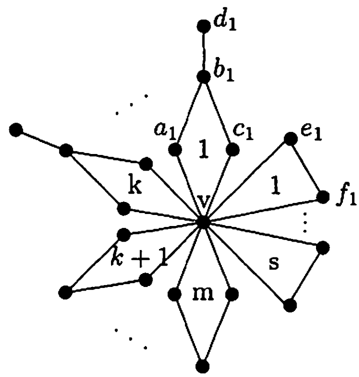

FIGURE 2. Structure of graph G

Since a vertex cover must include guards on at least two vertices in each C_3 and C_4, one of which can be shared by being placed on v, $\alpha(G) \geq m + s + 1$. Furthermore, $S = \{v\} \cup \{b_i : 1 \leq i \leq m\} \cup \{e_i : 1 \leq i \leq s\}$ is a vertex cover, and so $\alpha(G) = m+s+1$. Suppose for an α_m^∞-set B there exists $i \leq k$ such that $|B \cap \{a_i, b_i, c_i, d_i\}| = 1$. Of necessity, $B \cap \{a_i, b_i, c_i, d_i\} = b_i$, so attacking $b_i d_i$ will leave at least one of $a_i b_i$ or $c_i b_i$ unguarded. Hence, for any α_m^∞-set each C_4 with a pendant edge must have at least two guards in addition to v.

Thus, $\alpha_m^\infty(G) \geq m + s + k + 1$. However, $m + s + k + 1$ guards are insufficient under the condition $m \geq 1$ and $m+s \geq 2$. If $k < m$ we consider a C_4 without a pendant edge and with vertices $\{v, a_i, b_i, c_i\}$. Either b_i has a guard or an attack on a_ib_i or c_ib_i can force one there. If there is no guard on either a_i or c_i, then v must contain a guard and an attack on edge va_i forces two guards onto the C_4, neither of which is on v. The remaining $m + s + k - 1$ guards are fewer than the $m - 1 + k + s + 1 = m + k + s$ guards required to be in the other $m - 1 + s$ structures. If $k = m$, then $\{v, a_1, b_1, c_1, d_1\}$ induces a C_4 with a pendant edge. By attacking edges b_1d_1, a_1b_1 and/or va_1, if necessary, at least three guards can be forced onto $\{a_1, b_1, c_1, d_1\}$. Again, the remaining $m + s + k - 2$ guards are fewer than the $m-1+k-1+s+1 = m+k+s-1$ guards required to be in the other $m-1+s$ structures. On the other hand, one more guard makes it possible to respond to any attack by moving a guard to v from the structure with the extra guard. Hence, $\alpha_m^\infty(G) = m + s + k + 2$.

Similarly, an eternal total vertex cover requires two guards in each C_3 and three in each C_4, where a guard on v can be shared by all the structures. Thus, $\alpha_{mt}^\infty(G) \geq 2m + s + 1$. If $k = 0$, $2m + s + 1$ guards are sufficient since any attack can be handled entirely by the guards in the structure containing the attacked edge, and a guard can always be returned to v. On the other hand, an analysis similar to the above shows three guards can be forced to vertices of a C_4 with a pendant edge, none of which is on v. The $2m+s-2$ remaining guards are too few to guard the rest of the structures, but one more guard is sufficient.

Summarizing, we have

$$p = \alpha(G) = m + s + 1$$

$$q = \alpha_m^\infty(G) = m + s + 2 + k$$

$$r = \alpha_{mt}^\infty(G) = \begin{cases} 2m + s + 1 & \text{if } k = 0 \\ 2m + s + 2 & \text{if } 1 \leq k \leq m \end{cases}$$

We solve for m, s, and k (if $k \neq 0$) in order to determine the specific graph for the triple (p, q, r). We deal with two cases separately.

(1) If $k = 0$ we see that $q = p+1$, $m = r-p$, and $s = 2p-r-1$. Here, q is non-negative and $p < q \leq r$ implies $m \geq 1$. Also, $q = p + 1$ restricts us to Statement 1 of the lemma; hence, $r \leq 2p - 1$ and $s = 2p - r - 1 \geq 0$. Furthermore, $m + s = p - 1 \geq 2$ since $p > 2$. Thus, these values for m, k, and s also correspond to graphs that satisfy the constraints of our construction, and Statement 1 holds.

(2) If $k \geq 1$ we find $m = r - (p+1)$, $k = q - (p+1)$, and $s = 2p - r$. Since $p < q \leq r \leq 2p$, m, k, and s are nonnegative. Furthermore, $k \geq 1$ and $k = q-(p+1)$ implies that $q \geq p+2$. Hence, $r \geq p+2$, so

$m = r-(p+1)$ implies $m \geq 1$. Also, $m+s = 2p-(p+1) \geq 2$ since $p > 2$. Hence, these values for m, k, and s correspond to graphs that satisfy the constraints of our construction and Statement 2 holds.

□

The proofs to the next two lemmas employ the following proposition of Klostermeyer and Mynhardt [12].

Proposition 6. *If G has two disjoint minimum vertex covers and each edge of G is contained in a maximum matching, then $\alpha_m^\infty(G) = \alpha(G)$.*

The first hypothesis of Proposition 6 implies that G is bipartite and, if G is connected, that $\alpha(G) = n/2$, where n is the number of vertices in G. The next lemma deals with an extreme case of Theorem 3, Statement 2.

Lemma 7. *The triple $(p, p, 3p/2)$ is feasible for any positive even integer p.*

Proof. The triple $(2, 2, 3)$ is demonstrated by C_4. Let $p = 2t \geq 4$. Construct a graph G_p as follows. Start with a C_p with the vertices labeled in order by $0, 1, \ldots, p-1$. For each edge $\{2i, 2i+1\}$, $0 \leq i \leq t-1$, create a C_4 using that edge and two new vertices. The graph G_8 is shown in Figure 3.

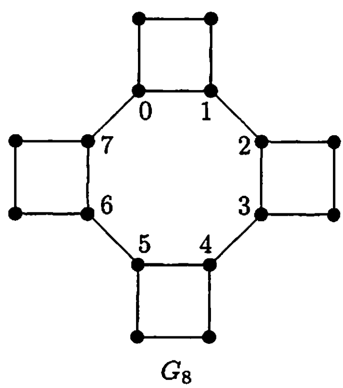

FIGURE 3. A graph showing the triple $(8, 8, 12)$ is feasible

The graph G_p is a bipartite graph with each partite set being a minimum vertex cover and every edge in a maximum matching. Thus, by Proposition 6 and the comment following it, $\alpha_m^\infty(G_p) = \alpha(G_p) = 2t$. Since each C_4 requires three guards in any eternal total vertex cover, $\alpha_{mt}^\infty \geq 3t$ and it is easy to see that this is sufficient. □

Note that Statement 2 of Theorem 3 cannot be extended to $p = 1$, since in this case, $r \leq 3p/2$ implies $\alpha_{mt}^{\infty}(G) = 1$, which is impossible. Lemma 8 completes the proof of Theorem 3.

Lemma 8. *The triple (p, p, r) is feasible for $2 \leq p \leq r < 3p/2$.*

Proof. The triple (p, p, p) is shown to be feasible by K_{p+1} for $p \geq 2$. Hence, we may assume $r > p$. Furthermore, since $p = 2$ and $r < 3p/2$ implies $p = q = r = 2$, we also assume $p \geq 3$.

Construct the graph G (illustrated in Figure 4) by adding edges to the disjoint union of a $K_{s,s}$ for $s \geq 1$ and $m \geq 1$ copies of C_4. Let $\{a_1, a_2, \ldots, a_s\}$ and $\{b_1, b_2, \ldots, b_s\}$ be the two partite sets of $K_{s,s}$ (only s edges of the $K_{s,s}$ are shown in Figure 4). Label the vertices of each C_4 in order $c_{i,1}$, $d_{i,1}$, $c_{i,2}$, and $d_{i,2}$ for $1 \leq i \leq m$. Now add the edges $a_1 d_{i,1}$ and $b_1 c_{i,1}$ for $1 \leq i \leq m$ and the edges $a_i d_{1,1}$ and $b_i c_{1,1}$ for $2 \leq i \leq s$.

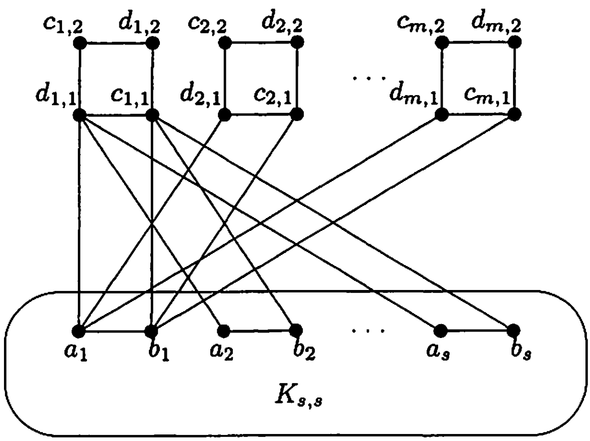

FIGURE 4. Graph G demonstrating the feasible triple (p, p, r) with $p \geq 3$ and $r < 3p/2$

Observe that the set of vertices with an "a" or "c" label and the set of vertices with a "b" or "d" label are each minimum vertex covers, the two sets are disjoint, and every edge is in a perfect matching. By Proposition 6 and the comment following it, $\alpha_m^{\infty}(G) = \alpha(G) = 2m + s$. Any eternal total vertex cover must have three guards on vertices of each C_4 and s guards in the $K_{s,s}$. Hence, $\alpha_{mt}^{\infty} \geq 3m + s$. This number is sufficient since any attack can still leave three guards in each C_4 and in particular always maintain guards on both c_{i1} and d_{i1} for $1 \leq i \leq m$. Only s guards are needed to cover the edges in the $K_{s,s}$, and since every guard in the $K_{s,s}$ is adjacent to c_{11} or d_{11}, the cover is total. Thus, $\alpha_{mt}^{\infty} = 3m + s$.

Setting $p = \alpha(G) = \alpha_m^\infty(G) = 2m + s$ and $r = \alpha_{mt}^\infty(G) = 3m + s$, we can solve for $m = r - p$ and $s = 3p - 2r$. Since $r > p$, $m \geq 1$ and, since $r < 3p/2$, $s \geq 1$. Hence these values for m and s correspond to graphs that satisfy the lemma for $p \geq 3$.

□

3. The Triple $(p, p+1, 2p)$

Although completely characterizing feasible graphs for a particular triple (p, q, r) appears difficult, it is possible for the triple $(p, p+1, 2p)$. In this section, we show every graph yielding a triple of this form is contained in the collection illustrated in Figure 1.

Let $\mathcal{H} = \{H_{p,t} : p, t \geq 1, p + t \geq 3\}$, the set of graphs defined in the proof of Lemma 4, and let $\mathcal{Q}$ be the set of all graphs with the triple $(p, p+1, 2p)$ for some p. Note that $\mathcal{H} \subseteq \mathcal{Q}$ by the proof of Lemma 4. In Theorem 16 we will show that G corresponds to a triple of the form $(p, p+1, 2p)$ if and only if $G \in \mathcal{H}$, that is, we show $\mathcal{H} = \mathcal{Q}$.

Suppose $G \in \mathcal{Q}$. Let A be an α-set of G and B an α_m^∞-set of G. Using the notation, $\overline{X} = V - X$ to represent the complement of X in V, we note that the sets $A \cap B, A \cap \overline{B}, \overline{A} \cap B$, and $\overline{A} \cap \overline{B}$ form a partition of the vertex set V. The following lemma shows A is an independent set.

Lemma 9. *For any graph $G = (V, E)$, if $\alpha_{mt}^\infty(G) = 2\alpha(G)$, then every α-set of G is independent.*

Proof. Suppose $A \subseteq V$ is an α-set and N_A is the number of components in the subgraph induced by A. A connected vertex cover can be obtained by adding $N_A - 1$ vertices to A so, by Lemma 1, $2\alpha(G) = \alpha_{mt}^\infty(G) \leq |A| + N_A - 1 + 1 = \alpha(G) + N_A \leq 2\alpha(G)$ which implies $N_A = \alpha(G)$, that is, A is an independent set of vertices. □

Since all vertices not in a vertex cover form an independent set, we see that A, $\overline{A}$, and $\overline{B}$ are all independent sets of vertices. This means each edge in G is between (1) $A \cap \overline{B}$ and $\overline{A} \cap B$, (2) $\overline{A} \cap B$ and $A \cap B$, or (3) $A \cap B$ and $\overline{A} \cap \overline{B}$. We determine several structural properties of G.

Lemma 10. *If $G \in \mathcal{Q} - \{P_3\}$, then G has no cut vertex of degree two.*

Proof. Suppose G has a cut vertex v of degree two with neighbors u and w which are, respectively, vertices in C_u and C_w, the connected components of $G - \{v\}$. Since $G \neq P_3$ we may assume without loss of generality that $deg_G(u) \geq 2$, so there exists $x \in N(u) - \{v\}$. Let B be an α_m^∞-set chosen to minimize $|B \cap (V(C_u) \cup \{v, w\})|$. By attacking the edges vw, uv, and xu, if necessary, we may assume $\{u, v, w\} \subseteq B$. If $deg(w) = 1$ then $B - \{w\}$ is an α-set containing two adjacent vertices, contradicting Lemma 9. If

$deg(w) \geq 2$ there exists $y \in N(w) - \{v\}$. Since $B - \{v\}$ is an α-set, Lemma 9 implies $(N(w) - \{v\}) \cap B = \emptyset$. Hence, attacking wy produces an α_m^∞-set B' such that $|B' \cap (V(C_u) \cup \{v, w\})| = |B \cap (V(C_u) \cup \{v, w\})| - 1$, contradicting the choice of B. □

Observation 11. *Using the above notation, if $G \in Q$ then*

(1) $|B| = |A| + 1$,
(2) *if $x \in \overline{A}$, then $1 \leq deg_G(x) \leq 2$,*
(3) *if $x \in A$, then $deg(x) \geq 2$, and*
(4) *B induces either an independent set, a K_2 and $\alpha(G)-1$ independent vertices, two K_2's and $\alpha(G) - 3$ independent vertices, or a P_3 and $\alpha(G) - 2$ independent vertices.*

Proof. We treat each item separately.

(1) Immediate since $\alpha_m^\infty = \alpha + 1$.
(2) If $deg_G(x) \geq 3$, then $A \cup \{x\}$ induces at most $\alpha(G) - 2$ components and we get the contradiction $2\alpha(G) = \alpha_{mt}^\infty(G) \leq \alpha(G) + 1 + (\alpha(G) - 2) = 2\alpha(G) - 1$.
(3) If $G = P_3$, then this holds. If not, then by Lemma 10, the neighbor x of a degree-one vertex in G must have degree at least three. However, by Lemma 9, x must be in $\overline{A}$, and so the degree of x is less than 3, by Statement 2. Thus, A can have no degree-one vertices.
(4) Let N_B be the number of components in the subgraph induced by B. By computations similar to the above, $2\alpha(G) = \alpha_{mt}^\infty(G) \leq \alpha_m^\infty(G) + N_B \leq \alpha(G) + 1 + \alpha_m^\infty(G) = 2\alpha(G) + 2$, implying $\alpha(G) - 1 \leq N_B \leq \alpha(G) + 1$. The only possibilities are those listed.

□

Lemma 12. *If G is a graph consisting of $k \geq 2$ internally disjoint paths of even length with distinct common endpoints x and y, then $G \notin Q$.*

Proof. For $1 \leq i \leq k$ let P_i be the path $x, v_{i,1}, v_{i,2} \ldots, v_{i,m_i}, y$ where m_i is a positive odd integer. Let $A = \{x, y\} \cup \{v_{i,j} :$ for which j is even$\}$. It is easy to see that A is an independent vertex cover. The independence of A implies $|E| = deg(x) + deg(y) + 2(|A| - 2)$. Since all the vertices in G except x and y have degree two, no smaller set is a vertex cover. Therefore, A is an α-set. Since the m_i's are odd, $\alpha(G) = |A| = 2 + \sum_{i=1}^k (\frac{m_i - 1}{2})$. Hence, $2\alpha(G) = 4 + \sum_{i=1}^k (m_i - 1) = 4 - k + \sum_{i=1}^k m_i = 4 - k + (n - 2) = n - k + 2$. For $1 \leq i \leq k$ let $S_i = \{v_{i,1}\} \cup (V(G) - N(x))$. Each of the S_i's is a total vertex cover. An attack on $xv_{j,1}$ where $j \neq i$ is met by moving the guard from x to $v_{j,1}$ and moving the guard from $v_{i,1}$ to x. An attack on $v_{j,1}v_{j,2}$ where $j \neq i$ is met by moving the guards on the path from $v_{i,1}$ to $v_{j,1}$ which goes through y. Both responses result in S_j. It follows that the collection of S_i's is a closed family of total vertex covers, so each S_i is an eternal total vertex cover. Since $|S_i| = 1 + |V(G)| - |N(x)| = 1 + n - k$, we have $\alpha_{mt}^{\infty}(G) \leq n - k + 1 < n - k + 2 = 2\alpha(G)$, so $G \notin \mathcal{Q}$. □

Much of the structure of $G \in \mathcal{Q} - \mathcal{H}$ can be derived by considering the subgraph $\widehat{G}$ induced by $(\overline{A} \cap B) \cup (A \cap \overline{B})$. Note that if $v \in A \cap \overline{B}$ then $deg_G(v) = deg_{\widehat{G}}(v)$.

Lemma 13. *For any $G \in \mathcal{Q} - \mathcal{H}$ and vertex x of $\widehat{G}$, $deg_G(x) \leq 2$.*

Proof. By Observation 11, Statement 2, vertices of $\overline{A} \cap B$ have degree at most two in G; hence we need only show that vertices in $A \cap \overline{B}$ have degree at most two in G. Let n_i be the number of vertices of degree i in $\widehat{G}$, $0 \leq i \leq 4$, and $n_{\geq 5}$ be the number of degree at least five. Since A has no degree one vertices and $deg_G(v) = deg_{\widehat{G}}(v)$ for $v \in A \cap \overline{B}$, vertices in $A \cap \overline{B}$ have degree at least two in $\widehat{G}$. The number of edges m in $\widehat{G}$ is equal to the sum of the degrees in $\widehat{G}$ of the vertices in $\overline{A} \cap B$ and also the sum of the degrees of those in $A \cap \overline{B}$. Thus, $m = 0n_0 + 1n_1 + 2(|\overline{A} \cap B| - n_0 - n_1) = 2|\overline{A} \cap B| - 2n_0 - n_1$ and $m \geq 5n_{\geq 5} + 4n_4 + 3n_3 + 2(|A \cap \overline{B}| - n_3 - n_4 - n_{\geq 5}) = 2|A \cap \overline{B}| + n_3 + 2n_4 + 3n_{\geq 5}$. Simplifying, using $|B| = |A| + 1$ so $|\overline{A} \cap B| = |A \cap \overline{B}| + 1$, yields $2 \geq 2n_0 + n_1 + n_3 + 2n_4 + 3n_{\geq 5}$, implying $n_{\geq 5} = 0$ and equality holds, that is, $2 = 2n_0 + n_1 + n_3 + 2n_4$.

If $n_0 = 1$ or $n_1 = 2$, the degree-zero vertex or the two degree-one vertices must be in $\overline{A} \cap B$. Also, the equality implies $n_3 = n_4 = 0$ and, since $deg_G(v) = deg_{\widehat{G}}(v)$ for all v in $A \cap \overline{B}$, the lemma holds.

The only remaining possibilities are $n_3 = 2$, $n_4 = 1$, and $n_3 = n_1 = 1$. We examine each of these cases. In the first two, all vertices in $\overline{A} \cap B$ are degree two in both G and $\widehat{G}$, so there are no edges between $A \cap B$ and $\overline{A} \cap B$. By connectivity, $G = \widehat{G}$ so $B = \overline{A}$ is an independent set of vertices and $|V(G)| = 2\alpha(G) + 1$.

(1) $n_3 = 2$. Here $G = \widehat{G}$ consists of either two degree three vertices joined by three vertex disjoint paths of even length or two even cycles joined by an even length path. The first possibility is excluded by Lemma 12. For the second possibility, note that $V(G)$ minus one degree-two vertex in each of the two cycles is an eternal total vertex cover. Hence, $\alpha_{mt}^{\infty}(G) \leq |V(G)| - 2 < 2\alpha(G)$ implying $G \notin \mathcal{Q}$.

(2) $n_4 = 1$. In this case, $G = \widehat{G}$ is composed of two even cycles with a common degree four vertex. The argument given in the previous case for two even cycles joined by an even length path is valid for this case as well.

(3) $n_3 = n_1 = 1$. Let u and v be the vertices with degrees 1 and 3, respectively, in $\widehat{G}$. By Lemma 10, u (which has degree at most 2), is not adjacent to any vertex in $A \cap B$, since it would be a cut vertex of degree 2. Hence, the degree of u in G is one and so, by Lemma 10, u is not adjacent to any vertex of degree two in G. It follows that u is adjacent to v. Since every other vertex, besides u and v, has degree two, $G = \widehat{G}$. Hence, G consists of the edge uv and an even cycle C_{2k} containing v. We know $\alpha(G) = k$ and $\alpha^{\infty}_{mt}(G) \leq \lceil 4k/3 \rceil + 1$ since $\alpha^{\infty}_{mt}(C_{2k}) = \lceil 4k/3 \rceil$. Therefore, $k = 2$ (since $2k \leq \lceil 4k/3 \rceil + 1$ if and only if $k \leq 2$) and $G = H_{2,1} \in \mathcal{H}$. Thus, $G \notin \mathcal{Q} - \mathcal{H}$.

Since the above cases have been eliminated, every vertex of $\widehat{G}$ must have degree at most two in G. □

Lemma 14. *If $G \in \mathcal{Q} - \mathcal{H}$ and $deg_G(v) > 2$, then for every α-set A and every α^{∞}_{m}-set B, $v \in A \cap B$ and $N(v) \cap B \neq \emptyset$.*

Proof. By Observation 11 Statement 2 and Lemma 13, $v \in A \cap B$. If $N(v) \cap B = \emptyset$, then attacking an edge incident on v creates an α^{∞}_{m}-set that does not contain v. □

We now show that at most one vertex of $G \in \mathcal{Q} - \mathcal{H}$ has degree greater than two.

Lemma 15. *If $G \in \mathcal{Q} - \mathcal{H}$ then G has at most one vertex with degree larger than two.*

Proof. Suppose vertices $v_1, ..., v_k$ have degree at least three and $k \geq 2$. Hence, $G - \{v_1, ..., v_k\}$ is the disjoint union of even length paths. By Lemma 14 each of the v_i's must be in every α-set and in every α^{∞}_{m}-set. Thus, for an α-set A and α_m-set B, there exists exactly one path, R, of $G - \{v_1, ..., v_k\}$ such that $|V(R) \cap B| = |V(R) \cap A| + 1$. Also, since the paths all have even length, the vertices from A and B in the other paths will be identical and none of these vertices will be adjacent to a vertex in $\{v_1, ..., v_k\}$. By Lemma 14, each v_i is adjacent to a vertex in B. This implies $k = 2$ and one end vertex of R is adjacent to v_1 while the other end vertex of R is adjacent to v_2.

Suppose P is a path in $G - \{v_1, v_2\}$ that has no endpoint adjacent to v_2. Since G is connected some endpoint of P must be adjacent to v_1. Obtain a new α^{∞}_{m}-set B' by attacking an edge in P incident with v_1. Since the attack causes a guard to move from v_1 onto P, and since v_1 must be in B' a guard must move from R to v_1. This implies none of the paths which connect to v_2 have an extra vertex. Hence, $N(v_2) \cap B' = \emptyset$, contradicting

Lemma 14. Therefore, $G - \{v_1, v_2\}$ consists of at least three paths, each of which has one endpoint adjacent to v_1 and one endpoint adjacent to v_2.

By Lemma 12, $G \notin \mathcal{Q}$ contradicting $G \in \mathcal{Q} - \mathcal{H}$. □

The characterization of graphs with triple $(p, p+1, 2p)$ is given in the next theorem.

Theorem 16. *A graph G corresponds to a triple $(p, p+1, 2p)$ if and only if $G \in \mathcal{H}$, that is, $\mathcal{H} = \mathcal{Q}$.*

Proof. The first paragraph of this section shows $\mathcal{H} \subseteq \mathcal{Q}$. Suppose there is a graph G in $\mathcal{Q} - \mathcal{H}$. By Lemma 15, G has at most one vertex of degree greater than two.

If G has no vertex of degree greater than two then, by connectivity, G is either an even cycle C_{2k} with $k \geq 2$ or a path P_n with $n \geq 3$. The former is impossible since $\alpha_{mt}^{\infty}(C_{2k}) = \lceil 4k/3 \rceil < 2k = 2\alpha(C_{2k})$ for all $k \geq 2$. By Lemma 10, the only path satisfying the triple requirements is $P_3 = H_{1,2} \in \mathcal{H}$.

Therefore, we may assume G has exactly one vertex v of degree at least three. By Lemma 10, every vertex of degree one must be adjacent to v. Since G is connected and bipartite, G must be composed of even cycles all sharing v and pendent vertices all adjacent to v. Thus the components of $G - v$ consist of r isolated vertices and s even length paths containing c_i vertices for $1 \leq i \leq s$ (if $s > 0$). A minimum vertex cover can be formed from v and $(c_i - 1)/2$ vertices from the s paths; hence $\alpha(G) = 1 + \sum_{i=1}^{s}(c_i - 1)/2$.

Since $\alpha_{mt}^{\infty}(C_n) = \lceil 2n/3 \rceil$, a not necessarily minimum eternal total vertex cover can be formed using v, $\lceil 2(c_i + 1)/3 \rceil - 1$ vertices from the i^{th} cycle, and, if $r \geq 1$, an additional vertex to handle the case when the guard at v moves to a degree one vertex.

When $r \geq 1$, we have

$$2\alpha(G) = 2 + \sum_{i=1}^{s}(c_i - 1) = \alpha_{mt}^{\infty}(G) \leq 2 + \sum_{i=1}^{s}(\lceil 2(c_i + 1)/3 \rceil - 1)$$

which implies

$$\sum_{i=1}^{s}(c_i - \lceil 2(c_i + 1)/3 \rceil) = \sum_{i=1}^{s}\lfloor (c_i - 2)/3 \rfloor \leq 0.$$

For each i, c_i is odd and $c_i \neq 1$ since G has no multiple edges, so $c_i \geq 3$. Therefore, the last inequality implies $c_i = 3$ for all i, that is, all cycles have four vertices.

When $r = 0$, the extra vertex is not required in a total eternal vertex cover, so

$$2\alpha(G) = 2 + \sum_{i=1}^{s}(c_i - 1) = \alpha_{mt}^{\infty}(G) \leq 1 + \sum_{i=1}^{s}(\lceil 2(c_i + 1)/3 \rceil - 1)$$

implying

$$\sum_{i=1}^{s}(c_i - \lceil (2(c_i + 1)/3 \rceil) \leq -1,$$

an impossibility that shows $r \geq 1$.

When $s = 0$ and $r = 1$, the graph is K_2 which does not have the required triple assignment. Thus $r + s \geq 2$ and $G = H_{s+1,r} \in \mathcal{H}$. We conclude that $\mathcal{Q} - \mathcal{H} = \emptyset$, that is, $\mathcal{Q} \subseteq \mathcal{H}$ implying $\mathcal{Q} = \mathcal{H}$. □

4. Triples (p, p, r) with $r > 3p/2$

The triple for K_2 is $(1, 1, 2)$. No other graphs have been found to show the feasibility of triples (p, p, r) with $r > 3p/2$. This suggests the following conjecture.

Conjecture 17. *If $\alpha(G) = \alpha_m^{\infty}(G) \geq 2$, then $\alpha_{mt}^{\infty}(G) \leq 3\alpha(G)/2$.*

There is some evidence in support of the conjecture. Klostermeyer [9] has shown that the triple $(p, p, 2p)$ is not feasible if $p \geq 2$. Also, the largest r to p ratio for cycles occurs with C_4's and is 3/2. Interestingly, no graphs have been uncovered which contradict the following related conjecture.

Conjecture 18. *If $\alpha(G) = \alpha_m^{\infty}(G)$, then $\alpha_t(G) = \alpha_{mt}^{\infty}(G)$.*

In the next section, we show that $\alpha_t(G) \leq 3\alpha(G)/2$, when G is connected, $n \geq 3$, and $\alpha(G) = \alpha_m^{\infty}(G)$.

5. An Upper Bound on $\alpha_t(G)$ When $\alpha(G) = \alpha_m^{\infty}(G)$

One of the problems encountered in dealing with (p, p, r), where $r > 3p/2$, is that there is no known characterization for graphs G satisfying $\alpha(G) = \alpha_m^{\infty}(G)$. This condition does imply that every vertex appears in at least one α-set of G. We call graphs with this property *α-complete.* These concepts are not the same. For example, a path P_n on an even number of vertices is α-complete, but $\alpha(P_n) \neq \alpha_m^{\infty}(P_n)$ if $n \geq 3$. The property of being α-complete is all that is needed to show the results of this section.

Definition 19. *Let $G = (V, E)$ be a graph.*

(1) *$\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is the collection of all α-sets of G.*
(2) *$\mathcal{A}^* \subseteq \mathcal{A}$ is* exhaustive *if every vertex of V appears in at least one α-set of $\mathcal{A}^*$.*

(3) *For any vertex* $v \in V$, $\mathcal{A}_v = \{A_i \in \mathcal{A} : v \in A_i\}$.

With this definition, a graph is α-complete if and only if the collection of all its α-sets is exhaustive. The next lemma establishes facts needed for the induction argument in Theorem 22.

Lemma 20. *Let* $G = (V, E)$ *be an* α*-complete graph.*

(1) *If there is a vertex* $v \in V$ *such that* $\mathcal{A}_v$ *is exhaustive, then* $\alpha(G - v) = \alpha(G) - 1$ *and* $G - v$ *is* α*-complete.*
(2) *If there are vertices* $v, w \in V$ *such that* $\mathcal{A}_v \cap \mathcal{A}_w = \emptyset$ *and* $\mathcal{A}_v \cup \mathcal{A}_w$ *is exhaustive, then* $\alpha(G - \{v, w\}) = \alpha(G) - 1$ *and* $G - \{v, w\}$ *is* α*-complete.*

Proof. We treat each statement separately.

(1) If A is an α-set of $G - v$, then $A \cup \{v\}$ is a vertex cover of G so $\alpha(G-v) \geq \alpha(G)-1$. Also, if $A_i \in \mathcal{A}_v$ then $A_i - \{v\}$ is a vertex cover of $G-v$ implying $\alpha(G-v) \leq \alpha(G)-1$. Hence, $\alpha(G-v) = \alpha(G)-1$ and $\mathcal{A}_v^* = \{A_i - \{v\} : A_i \in \mathcal{A}_v\}$ is a collection of α-sets of $G - v$. Since $\mathcal{A}_v$ is exhaustive in G, $\mathcal{A}_v^*$ is exhaustive in $G - v$.
(2) If A is an α-set of $G - \{v, w\}$, then $A \cup \{v, w\}$ is a vertex cover of G. Since $\mathcal{A}_v \cap \mathcal{A}_w = \emptyset$, no α-set of G contains both v and w, so $|A \cup \{v, w\}|$ is not a minimum vertex cover of G. Therefore, $|A \cup \{v, w\}| \geq \alpha(G) + 1$ which implies $\alpha(G - \{v, w\}) = |A| \geq \alpha(G) - 1$. Also, if $A_i \in \mathcal{A}_v \cup \mathcal{A}_w$ then $A_i - \{v, w\}$ is a vertex cover of $G - \{v, w\}$ of size $\alpha(G) - 1$. Hence, $\alpha(G - \{v, w\}) = \alpha(G) - 1$ and $\mathcal{A}_{vw}^* = \{A_i - \{v, w\} : A_i \in \mathcal{A}_v \cup \mathcal{A}_w\}$ is a collection of α-sets of $G - \{v, w\}$. Here again, since $\mathcal{A}_v \cup \mathcal{A}_w$ is exhaustive, $\mathcal{A}_{vw}^*$ is exhaustive.

□

The next lemma is a useful structural one and it is followed by the establishment of a lower bound for $\alpha(G)$.

Lemma 21. *For an arbitrary collection of* α*-sets of a graph, let* X *be the vertices appearing in every set of the collection and* Y *the vertices appearing in no set of the collection. Then, for every* $v \in Y$, $N(v) \subseteq X$.

Proof. Suppose it is not true for some vertex $v \in Y$, so v has a neighbor w that is not a member of at least one α-set, say A_i, of the collection. Then v and w are adjacent vertices, neither of which is in A_i, so A_i is not a vertex cover, a contradiction. □

Theorem 22. *If* $G = (V, E)$ *is an* α*-complete graph, then* $\alpha(G) \geq n/2$.

Proof. We induct on the number of vertices of G. The only α-complete graph on at most two vertices is P_2 and the result holds for it. Let $n \geq 3$

and assume the result is true for α-complete graphs having $n-1$ vertices. If G has a vertex v such that $\mathcal{A}_v$ is exhaustive then, by Lemma 20 Statement 1, $G-v$ is α-complete and $\alpha(G-v) = \alpha(G) - 1$. Employing the inductive hypothesis yields $\alpha(G) = \alpha(G-v) + 1 \geq (n-1)/2 + 1 > n/2$.

Now assume $\mathcal{A}_v$ is not exhaustive for any $v \in V$. Select a vertex v such that $|\mathcal{A}_v| \geq |\mathcal{A}_w|$ for any $w \in V$. Let X be the intersection of all the α-sets of $\mathcal{A}_v$ and Y be the set of vertices not appearing in any set in $\mathcal{A}_v$. The set X is nonempty since it contains v, and Y is nonempty since $\mathcal{A}_v$ is not exhaustive. Let $y \in Y$. By Lemma 21, y is adjacent to a vertex $x \in X$. The definition of X implies every α-set in $\mathcal{A}_v$ is in $\mathcal{A}_x$, so the choice of v implies $\mathcal{A}_v = \mathcal{A}_x$. Since the edge xy must have at least one end vertex in every α-set, y is in every set of $\mathcal{A}$ - $\mathcal{A}_x$.

Thus, $\mathcal{A}_x \cap \mathcal{A}_y = \emptyset$ and $\mathcal{A}_x \cup \mathcal{A}_y = \mathcal{A}$ is exhaustive.

By Lemma 20 Statement 2, $G-\{x,y\}$ is α-complete and $\alpha(G-\{x,y\}) = \alpha(G)-1$. Again using the inductive hypothesis, $\alpha(G) = \alpha(G-\{x,y\})+1 \geq (n-2)/2 + 1 = n/2$. □

Dutton [3] has established the following upper bound.

Theorem 23. *For connected graphs G with $n \geq 3$, $\alpha_t(G) \leq (n + \alpha(G))/2$.*

Our final theorem follows immediately from Theorems 22 and 23.

Theorem 24. *If G is connected, $n \geq 3$, and $\alpha(G) = \alpha_m^{\infty}(G)$, then $\alpha_t(G) \leq 3\alpha(G)/2$.*

Notice that Theorem 24 along with Conjecture 18 would imply Conjecture 17.

6. Open Questions

The following open questions are of interest:

(1) Is the triple (p,p,r) feasible when $r > 3p/2$?
(2) Is Conjecture 18 true?
(3) Characterize graphs which correspond to any particular feasible triple (p,q,r). For example, are the graphs of the type illustrated in Figure 3 the only ones for $(p,p,3p/2)$?
(4) Characterize graphs G, where $\alpha(G) = \alpha_m^{\infty}(G)$.

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