Embedding Graphs in Cylinder and Torus Books

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Abstract

In the classical book embedding problem a k-book is defined to be a line L in 3-space (the spine) together with k half-planes (the pages) joined together at L. We introduce two variations on the classical book in which edges are allowed to wrap in either one or two directions. The first is a cylindrical book where the spine is a line L in 3-space and the pages are nested cylindrical shells joined together at L. The second is a torus book where the spine is the inner equator of a torus and the pages are nested torus shells joined together at this equator. We give optimal edge bounds for embeddings of finite simple graphs in cylinder and torus books and give best-possible embeddings of K_n in torus books. We also compare both books with the classical book.

1 Introduction

Throughout this paper, let G denote a finite, simple graph. That is, G consists of finite sets of vertices and edges and does not contain loops or multiple edges. A classical k-page book is a line L in 3-space, called the spine, along with k half-planes, called pages, joined together at L. To embed a graph G in a classical k-book, the vertices of G are placed along the spine and each edge of G is placed on a single page so that no two edges cross each other or the spine. Such an embedding is called a book embedding, or page embedding, of G. The book thickness, or pagenumber, of G, denoted $bt(G)$, is the smallest number k for which G has a k-book embedding.

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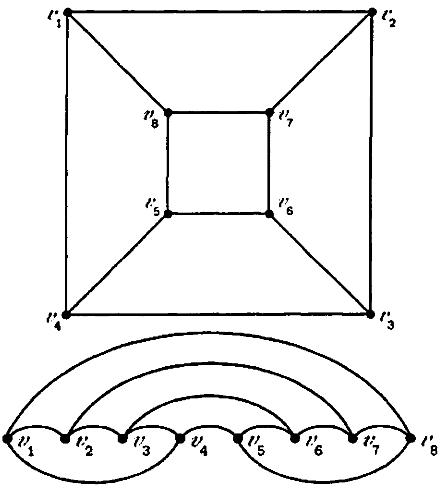

Figure 1 *Two-page book embedding of the 3-cube.*

Figure 1 illustrates a two-page book embedding of the 3-cube. The edges on the half-plane above the spine represent one page of the book and the edges on the half-plane below the spine represent the second page of the book.

Since its introduction by Kainen [10] and Ollmann [13], the classical book embedding problem has been studied by many [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14]. In their 1979 article, Bernhart and Kainen [1] provide many useful foundational results about the book thickness of graphs, including characterizations of one and two-page embeddable graphs. Selected results for classical book embeddings will be discussed in Section 2. In general, determining the book thickness of a graph is a difficult problem since both the relative ordering of the vertices along the spine as well as the assignment of edges to pages must be considered. Even with a fixed vertex ordering, the task of finding the book thickness of an arbitrary graph is NP-Complete [8]. Despite the general complexity of the book embedding problem, edge bounds for n-vertex simple graphs in k-page books as well as the book thickness of many classes of graphs are known [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14].

Books and their properties were originally studied before the recent emergence of numerous practical applications. Book embeddings, also called stack embeddings and stack layout, now have widely documented applications to parallel processing and VLSI chip design [2, 3, 4, 5, 9]. Chung, Leighton, and Rosenberg [4, 5] developed the Diogenes method of designing fault-tolerant VLSI processor arrays. This method involves a linear layout of processing elements (vertices) along the spine. Non-faulty processing elements are connected by wires (edges) and bundles of non-crossing wires correspond to layers or stacks (pages). Book embeddings also have applications to metrics for software complexity and vehicle traffic engineering [11]. In structural biology, book embeddings have been used for modeling RNA folding energy states [7].

In the classical case, each edge may only extend into a single half-plane page of the book before reconnecting to another vertex on the spine. Edges may not cross the spine and they may not wrap around from the front side of a page to the back side of a page before reconnecting to the spine. A natural topological extension of this problem would be to allow edges (wires) to wrap in either one or two directions before connecting back to the spine, while preserving a linear layout of the vertices.

In Section 3, we modify the book to allow edges to wrap in one direction. The resulting book has cylindrical pages, which allow edges to wrap over the surface of a cylinder and reconnect to the spine. Our results include a characterization of one-page cylinder book embeddable graphs. The second modification of the classical book, discussed in Section 4, connects the ends of the cylindrical pages together to form torus pages. This turns the spine into the inner equator of a torus, allowing edges to wrap in two directions on the surface of the torus pages before reconnecting to the spine. For torus books, we prove the optimal torus book thickness for complete graphs and give best bounds for the number of edges that may be embedded in a k-page torus books for a graph with n vertices. The arguments used are nice extensions of the results for classical books, so we include many of the classical proofs in Section 4 as well. We conclude with a comparison of a one-page torus book with a classical three-page book.

2 Results for Classical Book Embeddings

Bernhart and Kainen [1] note that the classical book embedding problem can be viewed as a circular embedding problem. They observe that if the vertices are lined up on the spine in the order $v_1, v_2, \ldots, v_n$ that all edges of the form $\{v_i, v_{i+1}\}, i = 1, 2, \ldots, n-1$ can be placed near the spine on any page without crossing. Finally, the edge $\{v_1, v_n\}$ may be placed outside all other edges on any page, without creating edge crossings. In Figure 1, this cycle appears on the half-plane above the spine. Now it is clear that the maximum number of edges that may be embedded in a one-page book for an n-vertex graph is $2n-3$, corresponding to the n edges of this outer cycle and $n-3$ edges of a complete triangulation of the interior of this cycle.

Stretching this outer cycle into a circle, we see that embedding a graph

G in a k-book is equivalent to placing the vertices of G in a circle and coloring the edges, represented by chords of the circle, with k colors so that no two edges of the same color cross. A graph G is called **outerplanar** if all the vertices of G can be placed in a circle in such a way that all edges of G are non-crossing chords of the circle. This leads to the characterization of one-page embeddable graphs given by Bernhart and Kainen [1].

Theorem 1. *$bt(G) \leq 1$ if and only if G is outerplanar.*

Using this circular depiction of a book, one can see that a two-page book would admit at most $3n - 6$ edges; n for the outer cycle, and up to $2(n-3)$ additional edges for two triangulations of the interior of the cycle. We note that $3n - 6$ also matches the edge bound for all simple planar graphs with n vertices, which is not surprising since a two page book is a planar structure formed by joining two half-planes.

A Hamiltonian cycle in a graph is simple cycle that includes every vertex exactly once. A graph is called **subhamiltonian** if it is the subgraph of a planar graph with a Hamiltonian cycle. Again, appealing to the circular representation of a book, it is clear that a two-page embeddable graph is subhamiltonian. We may form the desired Hamiltonian cycle by proceeding around the circle, adding missing cycle edges, as needed. Conversely, if we begin with a subhamiltonian graph, we embed the graph in the plane and use the Hamiltonian cycle ordering along the spine, again adding any missing edges. Since we began with a planar embedding, the edges inside and on the cycle will form one page of the book and the edges outside the cycle will form the second page. The following theorem of Bernhart and Kainen [1] is now clear.

Theorem 2. *$bt(G) \leq 2$ if and only if G is subhamiltonian.*

In Figure 1, the ordering along the spine corresponds to a Hamiltonian cycle in the 3-cube. The cycle and the edges within the cycle are on one page, while the edges outside the cycle lie on the other page. Determining whether an arbitrary graph has a Hamiltonian cycle is NP-Complete. So, although this characterization is simply stated, determining whether or not a given planar graph is two-page embeddable is not an easy problem. There are, however, large classes of planar graphs known to be subhamiltonian. These include trees, square grids, X-trees, all planar graphs with 10 or fewer vertices, and planar graphs without triangles (see [1, 4, 5, 6, 12]).

Although all two-page embeddable graphs are planar, there are examples of planar graphs that are not subhamiltonian, and hence, not two-page embeddable. One such example is the 11-vertex graph $St^2(K_3)$ (see Figure 2), formed by starting with a triangle, stellating the inner and outer faces of the triangle, and then repeating that process.

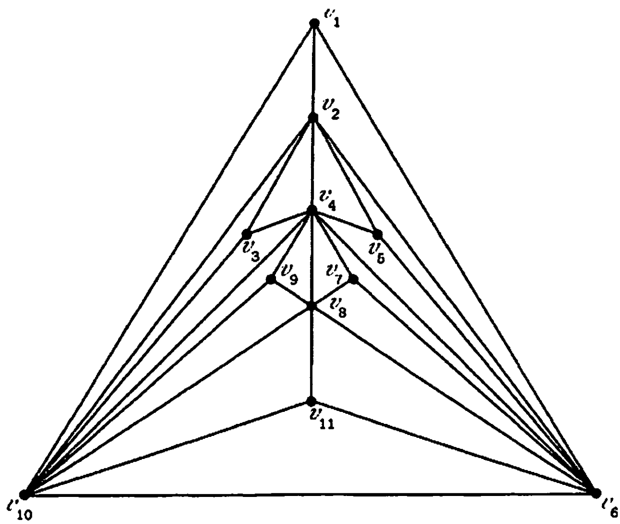

Figure 2 *The stellation graph $St^2(K_3)$.*

Bernhart and Kainen originally conjectured that by repeating this process n times, forming $St^n(K_3)$, that the number of pages required to embed this family of graphs would continue to grow as n increased. This was part of a larger conjecture that the book thickness of planar graphs was unbounded. Heath [8] was later able to embed this family of graphs in three-page books and provide a bound of seven pages for the book thickness of all planar graphs. Yannakakis [14] improved upon this bound in the following theorem.

Theorem 3. *If G is a planar graph, then $bt(G) \leq 4$.*

Proof. See Yannakakis [14]. □

The book thickness of non-planar graphs has also been studied, as well as variations of the book embedding problem involving vertex-ordering restrictions. In the next two sections, we will examine book embeddings in books with modified pages.

3 Cylinder Books

The first modified pages we consider are ones that bend and reconnect at the spine. These **cylinder pages** can be viewed as the outer surfaces of nested cylinders, connected together at the spine, a straight line in 3-space. To embed a graph in a **cylinder book**, all vertices are placed on the spine and each edge is placed on a single cylinder page so that it crosses neither the spine nor any other edge of the graph. We define the **cylinder thickness** $ct(G)$ to be the smallest number of cylinder pages needed to embed the graph G in a cylinder book.

If we cut a one-page cylinder book at the spine and flatten it out, we get a planar structure. This can be realized by making two parallel copies of the spine in the plane. Then all edges that fit on a cylinder page can be drawn in the space between the two copies of the spine. Figure 3 shows a one-page cylinder book embedding of the stellation graph $St^2(K_3)$.

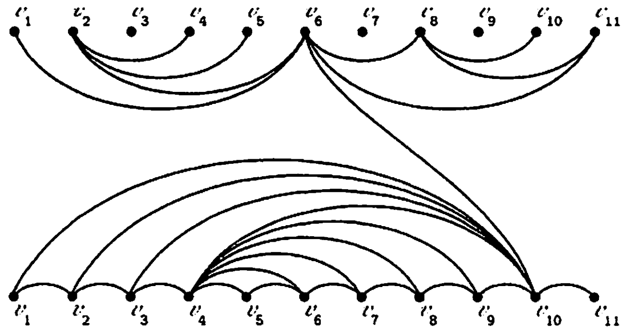

Figure 3 *One-page cylinder book embedding of* $St^2(K_3)$.

If there are no edges between the two copies of the spine, then we have a classical two-page book. Thus, the set of graphs that are embeddable in a one-page cylinder book includes all graphs that admit classical two-page book embeddings. But, we know that $St^2(K_3)$ is not subhamiltonian. Hence, a one-page cylinder book allows embeddings of more planar graphs than the classical two-page book. How much better is a one-page cylinder book? The following theorem helps answer this question.

Theorem 4. *Let G be a graph. Then $ct(G) \leq 1$ if and only if G is a subgraph of a planar graph with a Hamiltonian path.*

Proof. Suppose that $ct(G) \leq 1$. Then G can be embedded in a one-page cylinder book with spine L, a line in 3-space. We realize this embedding in the plane by cutting the cylinder page along the spine. Note that since G is a finite graph, we may assume that the vertices of G appear on a segment of L of finite length, L'. Laying the cylinder flat, the edges of G now lie in the plane between two parallel copies of L'. We next bend the edges of G and re-join the two copies of L' together in the plane (see Figure 4). Now we have a planar embedding of G so that every vertex of G lies along a line segment. Add missing edges along this segment, as needed, to form the desired Hamiltonian path.

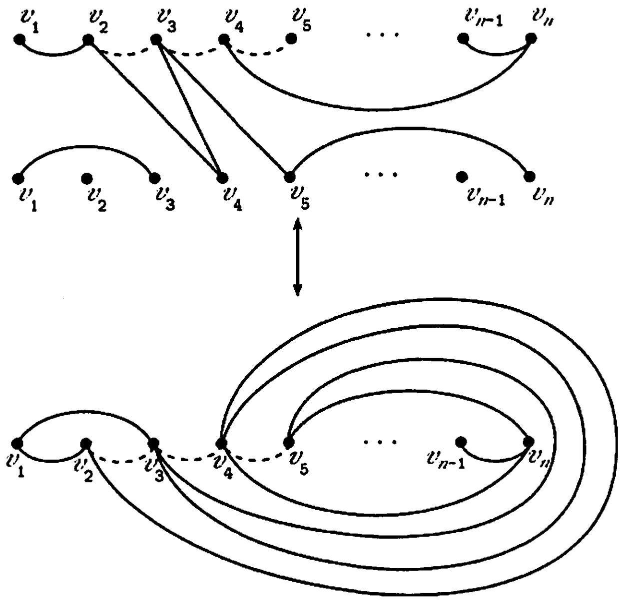

Figure 4 *One-page cylinder book embedding with Hamiltonian path.*

Conversely, suppose that G is a subgraph of a planar graph with a Hamiltonian path. Then the vertices of G can be placed on a straight horizontal line segment P in the plane so that every edge of G lies either above P, below P, or wraps around an endpoint of P. Without violating planarity, the edges that wrap around the endpoints of P can be arranged so that they all wrap around the right endpoint of P. Now we reverse the process and cut along P from left to right to make two copies of P. Rotate the bottom copy of P counter-clockwise about its right endpoint until the two copies of P are parallel in the plane. The edges of G now lie in the plane between the copies of P (see Figure 4). We next form two horizontal lines through the vertices on the two copies of path P and identify these two lines to form the spine of the desired one-page cylinder book. Thus, $ct(G) \leq 1$. □

Any two-page embeddable graph can be embedded in a single-page cylinder book. The graph depicted in Figures 2 and 3 shows that the converse is not true. From Theorem 4, it is now clear that the set of counter-examples consists of all graphs that are subgraphs of planar path-Hamiltonian graphs, but are not subhamiltonian. Since all planar graphs with 10 or fewer vertices are subhamiltonian, then all counter-examples must have at least 11 vertices. The stellation graph, $St^2(K_3)$, in Figures 2 and 3, with 11 vertices is a minimal example of such a graph.

In the proof of Theorem 4, it is clear that a single page of a cylinder book is a planar structure. Hence, any graph that is embeddable on a single cylinder page must satisfy the $3n - 6$ edge bound for simple planar graphs with n vertices. That is the same bound as the number of edges allowed in a classical two-page book. So, while a cylinder page accommodates a slightly larger class of graphs, it does not allow a greater number of edges than a classical two-page book. In either case, the total number of edges allowed is limited by the planarity of the structure. In the next section, we look at a modification of our cylinder page to that of a torus page in hopes that the increase in genus will allow the embedding of more edges for an n-vertex graph.

4 Torus Books

What if we allow edges to wrap in two directions? The second page modification we consider is a torus page, formed by connecting the two ends of a cylinder page. The spine of a **torus book** may now be represented by the inner equator of the torus. Multiple nested **torus pages** are joined together at this common spine. Again, when embedding a graph in a torus book, the vertices are placed on the spine and the edges on the torus pages without crossing each other or the spine. The **torus thickness** $t(G)$ of a graph G is the least number of torus pages needed to embed G in a torus book.

We also have a simple way of realizing a torus page. As we did for the cylinder page, we draw two parallel copies of the spine in the plane forming the top and bottom sides of a rectangle. This cylinder page is then transformed into a torus page by identifying the two vertical sides of this rectangle (see Figure 5). Edges may pass through a vertical side of the rectangle and re-enter at the same point on the opposite side. This horizontal wrapping of edges allows the embedding of many graphs in a one-page torus book that do not admit embeddings in a one-page cylinder book. Figure 5 shows a one-page torus book embedding of the complete graph K_7.

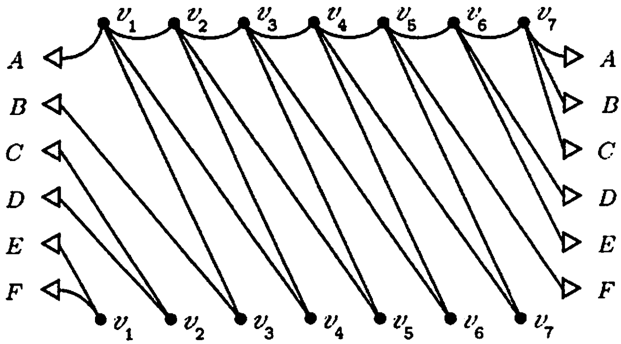

Figure 5 *One-page torus book embedding of K_7.*

The graph K_7 is a non-planar graph, thus, it cannot be embedded either on a one-page cylinder book or on a classical two-page book. By the construction of a torus book it is clear that any graph embeddable on a one-page cylinder book or a classical two-page book is also embeddable in a one-page torus book. How much better is the torus page? To help answer this question, we will examine the number of edges allowed in a one-page torus book embedding of a graph with n vertices.

Our proof of the edge bound for torus books parallels Bernhart and Kainen's proof of the edge bound for the classical k-book given in [1], so we present their argument here. Recall that an n-vertex graph embeddable in a classical two-page book has at most $n + 2(n - 3) = 3n - 6$ edges. Similarly, a k-page embeddable graph with n vertices can have at most $n + k(n - 3)$ distinct edges. We again may have up to n edges on the outer cycle and up to $n - 3$ non-cycle edges on each of the k pages, corresponding to triangulations of the outer n-cycle. Solving for k produces the following bound on the book thickness given by Bernhart and Kainen [1].

Theorem 5. *Let G be a graph with $n \geq 4$ vertices and q edges. Then*

$$bt(G) \geq \frac{q - n}{n - 3}.$$

We can find bounds for the torus thickness of a graph in terms of the numbers of vertices and edges in a similar way.

Theorem 6. *Let G be a graph with n vertices and q edges. Then*

$$t(G) \geq \frac{q - n}{2n}.$$

Proof. Suppose that G is a graph with n vertices and q edges. Consider an embedding of G in a $k = t(G)$-page torus book. The n edges between consecutive vertices of G on the spine can be added on any torus page. So, G can have up to n edges along the spine. Using the representation of a torus page depicted in Figures 5 and 6, we have two possibilities. Either no edge extends from the upper copy of the spine to the lower copy of the spine, or there is at least one edge connecting the upper and lower copies of the spine.

Case 1: Suppose there are no edges extending between the two copies of the spine. Then we have a spine with each edge appearing on only one of the two sides of the spine. This structure is equivalent to a classical two-page book. Hence, if no edge has its ends on opposite sides of the spine, then there are $2(n-3) = 2n-6$ edges that can be added to any torus page in addition to the n for the spine. Thus, there are at most $n + 2k(n-3)$ possible edges in an n-vertex graph that admits a k-page torus book embedding with the restriction that no edges extend from one side of the spine to the other.

Case 2: Suppose that an edge e has ends on opposite sides of the spine on a torus page in a torus book embedding of G. We now use the previous representation of a page of a torus book with two parallel copies of the spine. Label the vertices of G in the order $v_1, v_2, \ldots, v_n$ along the spine. The edge e joins vertex v_i from the top copy of the spine and vertex v_j from the bottom copy of the spine (see Figure 6).

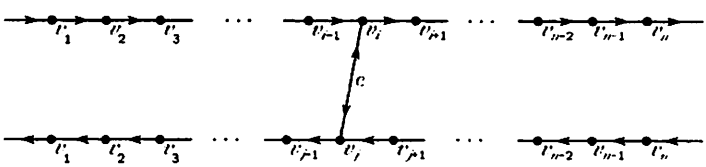

Figure 6 *Edge e joins oppisite sides of the spine.*

Any edge that can be placed on this page is either one of the n cycle edges along the spine, the edge e, or lies within the $2n+2$ closed walk formed by the two copies of the spine and e. Following the arrowed edges of Figure 6, this walk is given by $v_i, v_{i+1}, \ldots, v_n, v_1, v_2, \ldots, v_{i-1}, v_i, v_j, v_{j-1}, \ldots, v_2, v_1, v_n, \ldots, v_{j+1}, v_j, v_i$. There are $(2n+2)-3 = 2n-1$ possible edges that can be drawn in a complete triangulation of the interior of this $2n+2$-walk. Hence, on a single torus page we may embed up to $n+1+(2n-1) = 3n$ edges if there is at least one edge joining opposite sides of the spine. The n edges along the spine may be added to any page, but will only count once toward the edge total. However, the edge extending between the two copies of the spine may be different on different pages. This allows for a possible maximum of $1 + (2n - 1) = 2n$ non-cycle edges on each torus page. Thus, in a k-page book, there are at most $n + 2kn = (2k + 1)n$ edges for an n-vertex graph if edges wrap from one side of the spine to the other.

We see that more edges may be embedded on a torus page if edges wrap around the spine as in Case 2. Hence, if G is a graph with n vertices and q edges embedded in a k-page torus book, it follows that $q \le (2k + 1)n$. Thus, $k = bt(G) \ge \frac{q-n}{2n}$.

□

In the proof of Theorem 6, we noted that a one-page torus book can accommodate up to $3n$ edges. This result is not surprising since it is also the edge bound for any n-vertex graph embedded in a torus, achieved by a complete triangulation of the surface of the torus. Not all such toroidal triangulation graphs have a Hamiltonian cycle. Consequently, the class of one-page torus book embeddable graphs does not include all toroidal graphs since the absence of a Hamiltonian cycle prevents the inclusion of all n cycle edges in the edge count, regardless of the ordering of the vertices along the inner equator. As in the classical book, the restriction of the vertices to the spine does not reduce the total possible number of edges that may be embedded in a one-page book. However, to achieve the maximum, the graph must contain a Hamiltonian cycle and every face must be a triangle on the page.

Theorems 5 and 6 provide theoretical lower bounds for classic book thickness and torus book thickness in terms of the numbers of edges and vertices. In the next two theorems, we will show that for both types of books, these lower bounds for book thickness are the best possible bounds. In particular, we will use the results of Theorems 5 and 6 to prove the optimal book thickness and torus book thickness for the complete graph K_n. Again, the result for torus books extends nicely from the proof for classical books so we include both here.

Theorem 7. *If $n \ge 4$, then $bt(K_n) = \lceil n/2 \rceil$.*

Proof. (Adapted from Bernhart and Kainen [1].) Let $n \ge 4$. First, we show that $bt(K_n) \ge \lceil n/2 \rceil$. The graph K_n has n vertices and $\binom{n}{2}$ edges. Now by Theorem 5, we have that

$$bt(K_n) \ge \frac{\binom{n}{2} - n}{n-3} = \frac{n(n-1)/2 - n}{n-3} = n/2.$$

Since the book thickness of a graph must be an integer, it follows that $bt(K_n) \geq \lceil n/2 \rceil$.

To obtain the other inequality, we will assume that n is even. Suppose that $n = 2m$. We will show that $bt(K_{2m}) \leq 2m/2 = m$. The result for odd n will follow from the fact that K_{2m-1} is a subgraph of K_{2m}. The m pages of the book are formed by rotating the triangulated $2m$-cycle of Figure 7 through m successive positions.

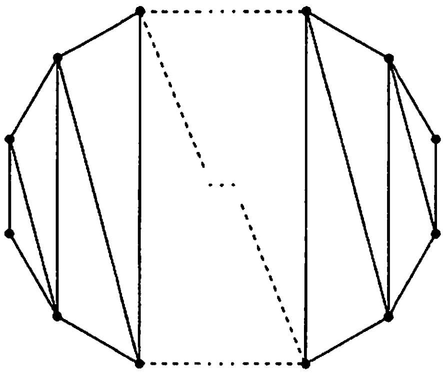

Figure 7 *Triangulation of the 2m-cycle.*

A triangulation of the $2m$-cycle has $2m - 3$ edges. It is easy to see that each inner diagonal of this $2m$-cycle cannot appear in more than one of the m rotations. We get $2m$ edges for the outer cycle and $m(2m - 3)$ edges for the m triangulations for a total of $2m + m(2m - 3) = 2m^2 - m = \binom{2m}{2}$ distinct edges. Hence, all $\binom{2m}{2}$ edges of K_{2m} are accounted for and we have the desired result. □

Theorem 7 gives the optimal book thickness of K_n. Further, when $n \geq 4$ is even, the embedding is as tightly packed as possible since the maximum edge bound of $n-3$ non-cycle edges per page is reached. The next theorem gives the optimal torus book thickness of K_n for all $n \geq 1$. We note that when $n = 4m + 3$, we achieve the maximal torus book edge bound of $2n$ non-cycle edges per page.

Theorem 8. *If $n \geq 1$, then $t(K_n) = \lfloor n/4 \rfloor$.*

Proof. Let $n \geq 1$. We apply Theorem 6 with n vertices and $\binom{n}{2}$ edges. This gives us that

$$t(K_n) \geq \frac{\binom{n}{2} - n}{2n} = (n-3)/4.$$

Since $t(K_n)$ is an integer, it follows that $t(K_n) \geq \lfloor n/4 \rfloor$.

To show that $t(K_n) \leq \lfloor n/4 \rfloor$, we will assume that $n = 4m+3$. Since K_{4m}, K_{4m+1}, and K_{4m+2} are all subgraphs of K_{4m+3}, the result will hold for these graphs as well. Label the vertices $v_1, v_2, \ldots, v_{n-1}, v_n$ along the spine. The n edges $\{v_i, v_{i+1}\}, i = 1, 2, \ldots, n-1$ and $\{v_n, v_1\}$ can all be placed along the spine on any page of the torus book. We will show how to embed the rest of the

$$\binom{4m+3}{2} - (4m+3) = 2m(4m+3)$$

edges of K_{4m+3} on m pages to obtain the desired result.

Consider the cycle $C = \{v_1, v_2, \ldots, v_n\}$. We say that vertices v_i and v_j are **at distance** $k \leq n-1$ **along** C if there is a clockwise simple path of length k in C between v_i and v_j. For example, v_1 and v_2 are at distance one along C and v_1 and v_n are at distance $n-1$ along C. To cover every edge of K_n, we need to ensure that each of the edges joining each vertex v_i, where $1 \leq i \leq n$, to the vertices at distances $1, 2, \ldots, n-1$ from v_i along C appears on a page of the m-page torus book. The edges joining vertices at distances 1 and $n-1$ are included in the cycle along the spine. Next, we embed the remaining $2m(4m+3)$ edges between vertices at distances $2, 3, \ldots, n-2$ onto m pages as follows: edges joining vertices at distance $2k, 2k+1, n-2k-1$, and $n-2k$ are embedded on page k for $k = 1, 2, \ldots, m$. As k ranges over the integers $1, 2, \ldots, m$; $2k$ and $2k+1$ take on the values $2, 3, \ldots, 2m$, and $2m+1$. Similarly, $n-2k-1$ and $n-2k$ take on the values $2m+2, 2m+3, \ldots, 4m$, and $4m+1 = n-2$. Hence, all edges for vertices at distances in the range $2, 3, \ldots, n-2$ appear on the m pages. The edges on page k do not cross as shown by the one-page embedding of K_7 in Figure 5.

Each of the $n = 4m+3$ vertices has degree four on each page (not including the edges of the n-cycle along the spine). Since each edge has two distinct end vertices, there are a total of $4(4m+3)/2 = 2(4m+3)$ edges on each of the m pages. Along with the $n = 4m+3$ vertices along the spine, we have accounted for all

$$n + 2mn = (2m+1)(4m+3) = \binom{4m+3}{2}$$

edges of K_{4m+3}. Thus, $t(K_n) \leq \lfloor n/4 \rfloor$. □

Theorem 8 gives a method for attaining one-page torus book embeddings of K_n for $n \leq 7$. The graph K_7 has $21 = 3(7)$ edges, matching the maximum edge bound for a 7-vertex graph in a one-page torus book. It is the largest complete graph that is embeddable on a torus without any restrictions on the placement of vertices. Through the construction given in Theorem 8, we see that K_7 may be embedded on a torus with all vertices restricted to the inner equator as shown in Figure 5. This graph is of particular interest since Theorem 7 shows that K_7 requires $\lceil 7/2 \rceil = 4$ pages for an embedding in a classical book. Hence, K_7 is a graph that admits an embedding in a one-page torus book but is not embeddable in a classical three-page book. This leads to the following question: Are there graphs that are embeddable in a classical three-page book that are not embeddable in a one-page torus book?

We consider the 10-vertex graph G consisting of an outer 10-cycle and three rotations of the triangulation of Figure 7. Each triangulation fits on a single page, so G is embeddable in a three-page classical book. This graph has 10 edges for the outer cycle and $3(10 - 3) = 21$ edges for the three triangulations for a total of 31 edges and 10 vertices. From Theorem 6, a graph G with 31 edges and 10 vertices must have $t(G) \geq (31-10)/[2(10)] = 21/20 > 1$. Since $t(G)$ is an integer, it follows that $t(G) \geq 2$. So, there are examples of graphs that have classical three-page book embeddings, but are not embeddable on a single-page torus book.

These two graphs demonstrate that a classical three-page book and a single-page torus book are generally not comparable. We can, however, directly compare the number of edges admitted on each of these structures for an n-vertex graph. A classical three-page book admits at most $n + 3(n - 3) = 4n - 9$ edges for an n-vertex graph, while the one-page torus book can accommodate at most $n + 2n = 3n$ edges. So, for graphs with $n \leq 8$ vertices, the one-page torus book allows more edges, for graphs with $n \geq 10$ vertices, the classical three-page book allows more edges, and for graphs with nine vertices, both structures admit the same number of edges.

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