

Eternal Domination in Trees

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Abstract

Mobile guards on the vertices of a graph are used to defend the graph against an infinite sequence of attacks on vertices. A guard must move from a neighboring vertex to an attacked vertex (we assume attacks happen only at vertices containing no guard). More than one guard is allowed to move in response to an attack. The m -eternal domination number is the minimum number of guards needed to defend the graph. We characterize the trees achieving several upper and lower bounds on the m -eternal domination number.

Keywords: dominating set, eternal dominating set, connected dominating set, independent set, neo-colonization.

1 Introduction

Let $G = (V, E)$ be a graph with n vertices. Several recent papers have considered problems associated with using mobile guards to defend G against an infinite sequence of attacks; see for instance [1, 2, 6, 7, 9, 10, 11, 12].

Denote the open and closed neighborhoods of a vertex $x \in V$ by $N(x)$ and $N[x]$, respectively. That is, $N(x) = \{v | xv \in E\}$ and $N[x] = N(x) \cup \{x\}$. Further, for $S \subseteq V$, let $N(S) = \bigcup_{x \in S} N(x)$. For any $X \subseteq V$ and $x \in X$, we say that $v \in V - X$ is an *external private neighbor* of x with respect to X if v is adjacent to x but to no other vertex in X ; we sometimes simply

say that v is a private neighbor of x . The set of all such vertices v is the *external private neighborhood of x with respect to X* .

A *dominating set* of graph G is a set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $x \in D$ adjacent to u . A dominating set D is a *connected dominating set* if the subgraph $G[D]$ induced by D is connected. The minimum cardinality amongst all dominating sets of G is the *domination number* $\gamma(G)$, while the minimum cardinality amongst all connected dominating sets is the *connected domination number* $\gamma_c(G)$. Further background on domination can be found in [8].

An *independent set* of vertices in G is a set $I \subseteq V$ with the property that no two vertices in I are adjacent. The maximum cardinality amongst all independent sets is the *independence number*, which we denote as $\beta(G)$.

Let $D_i \subseteq V, 1 \leq i$, be a set of vertices with one guard located on each vertex of D_i . In this paper, we shall allow at most one guard to be located on a vertex at any time. The problems considered in this paper can be modeled as a two-player game between a *defender* and an *attacker*: the defender chooses D_1 as well as each $D_i, i > 1$, while the attacker chooses the locations of the attacks r_1, r_2, \dots . Note that the location of an attack can be chosen by the attacker depending on the location of the guards. Each attack is handled by the defender by choosing the next D_i subject to some constraints that depend on the particular game. The defender wins the game if they can successfully defend any series of attacks, subject to the constraints of the game; the attacker wins otherwise.

In the *eternal dominating set problem*, each $D_i, i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving one guard to r_i from a vertex $v \in D_i, v \in N(r_i)$. The smallest size of an eternal dominating set for G is denoted $\gamma^\infty(G)$. This problem was first studied in [2].

In the *m-eternal dominating set problem*, each $D_i, i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving guards to neighboring vertices. That is, each guard in D_i may move to an adjacent vertex. It is required that $r_i \in D_{i+1}$. The smallest size of an m-eternal dominating set for G is denoted $\gamma_m^\infty(G)$. This “all-guards move” version of the problem was introduced in [6]. It is clear that $\gamma^\infty(G) \geq \gamma_m^\infty(G) \geq \gamma(G)$ for all graphs G .

We say that a vertex is *protected* if there is a guard on the vertex or on an adjacent vertex. We say that an attack at v is *defended* if we send a guard to v .

Our objective in this paper is to describe the trees that achieve some upper and lower bounds on the m -eternal domination number. In Sections 3 through 6 we describe the trees for which equality holds in each of the following:

- (a) $\gamma_m^\infty(T) \leq \gamma_c(T) + 1$;
- (b) $\gamma(T) \leq \gamma_m^\infty(T)$;
- (c) $\gamma_m^\infty(T) \leq 2\gamma(T)$; and
- (d) $\gamma_m^\infty(T) \leq \beta(T)$.

2 Terminology and Background

A *neo-colonization* is a partition $\{V_1, V_2, \dots, V_t\}$ of the vertex set of graph G such that each $G[V_i]$ is a connected graph. A part V_i is assigned a weight $\omega(V_i) = 1$ if it induces a clique and $\omega(V_i) = 1 + \gamma_c(G[V_i])$ otherwise. Then $\theta_c(G)$ is the minimum total weight of any neo-colonization of G , and is called the *clique-connected cover number* of G . Goddard et al. [6] defined this parameter and proved that $\gamma_m^\infty(G) \leq \theta_c(G)$. For $X \subseteq V$ we will write $\theta_c(X)$ as shorthand for $\theta_c(G[X])$. Klostermeyer and MacGillivray then proved the next result, which is key to what follows in this paper.

Theorem 2.1 [10] *For any tree T , $\theta_c(T) = \gamma_m^\infty(T)$.*

It follows from the previous theorem that $\gamma_m^\infty(G) \leq \gamma_c(G) + 1$.

The following property of neo-colonizations will be useful.

Proposition 2.2 *Let T be a tree with at least two vertices. Then there is a neo-colonization of minimum weight in which every part has size at least two.*

Proof. Suppose that $\Pi = \{V_1, V_2, \dots, V_k, \{x\}\}$ is a minimum weight neo-colonization of T . Without loss of generality, $\{x\}$ is adjacent to a vertex of V_k . Since the subgraph of T induced by V_k is connected, x is adjacent to exactly one vertex of V_k . The neo-colonization $\Pi' = \{V_1, V_2, \dots, V_k \cup \{x\}\}$ has minimum weight and fewer parts of size one. Applying this argument repeatedly, one arrives at the desired neo-colonization. \square

A *stem* of a tree T is a vertex of degree at least two that is adjacent to a leaf. A vertex of T that is not a leaf is called an *internal vertex*. A tree is a *star* if it is isomorphic to $K_{1,m}$, $m \geq 1$.

We partition the internal vertices of T into *loners*, *weak stems* and *strong stems* depending on whether they are adjacent to no, exactly one or at least two leaves. Denote the set of leaves of T by $L(T)$ and let $\ell = |L(T)|$. Obviously, $\gamma_c(T) = n - \ell$, the number of *internal vertices*, for any tree T of order $n \geq 3$.

The *eccentricity* of a vertex in a graph is its maximum distance from any other vertex. A vertex of maximum eccentricity in a tree is a leaf which is an end vertex of a longest path. Leaves of maximum eccentricity and the stems to which they are adjacent play an important role in some of our proofs. We use $\text{deg}(v)$ to denote the degree of vertex v .

3 Trees with $\gamma_m^\infty = \gamma_c + 1$

It turns out to be easier to describe the trees for which $\gamma_m^\infty < \gamma_c + 1$. The results below make it possible to look at a tree and determine if $\gamma_m^\infty < 1 + \gamma_c$. They do not, however, give much structural information on the trees for which the inequality holds. Finding such results is an open problem.

Proposition 3.1 *A tree T has $\gamma_m^\infty < 1 + \gamma_c$ if and only if T has a spanning forest consisting of r K_2 's and trees T_1, T_2, \dots, T_k on at least three vertices such that at least k loners of T are leaves of the $k+r$ trees in the collection.*

Proof. Suppose $\gamma_m^\infty < 1 + \gamma_c$. By Lemma 2.2, there is a minimum weight neo-colonization $\Pi = \{V_1, V_2, \dots, V_{k+r}\}$ in which there are no parts of size one. Without loss of generality, V_1, V_2, \dots, V_k each have size at least three, and the remaining parts each have size two. Let $T_i = T[V_i]$, $1 \leq i \leq k+r$. Every internal vertex of some T_i is an internal vertex of T . Since there are no parts of size one, every leaf of T belongs to the same part as the stem to which it is adjacent. In particular, no stem of T is a leaf of any tree T_i that is not isomorphic to K_2 . Each of the trees T_j , $k+1 \leq j \leq k+r$ contains at least one internal vertex of T . The weight of Π is $r+k$ plus the total number of internal vertices of the trees T_i , $1 \leq i \leq k$. All but k units of this quantity are accounted for by internal vertices of T . Since γ_c equals the number of internal vertices of T , it follows that at least k loners of T appear as leaves of the trees in the collection.

On the other hand, suppose that T has a spanning forest consisting of trees T_1, T_2, \dots, T_k on at least three vertices and r trees $T_{k+1}, T_{k+2}, \dots, T_{k+r}$, each isomorphic to K_2 , such that at least k loners of T are leaves of the $k+r$ trees in the collection. Let $V_i = V[T_i]$, $1 \leq i \leq k+r$ and consider the neo-colonization $\Pi = \{V_1, V_2, \dots, V_{k+r}\}$. The weight of Π is $r+k$ plus the total number of internal vertices of the trees T_i , $1 \leq i \leq k$. Since each of the trees which are isomorphic to K_2 contains at least one internal vertex of T , and every leaf of T belongs to the same part as the stem to which it is adjacent, this quantity is at most γ_c , the number of internal vertices of T . \square

Corollary 3.2 *Let T be a tree. Then $\gamma_m^\infty < 1 + \gamma_c$ if and only if there exists a set of edges whose deletion creates a spanning forest consisting of r K_2 's and trees T_1, T_2, \dots, T_k on at least three vertices such that at least k loners of T are leaves of the $k+r$ trees in the collection.*

Proof. The implication that if the condition holds then $\gamma_m^\infty < 1 + \gamma_c$ follows from Proposition 3.1.

Suppose $\gamma_m^\infty < 1 + \gamma_c$. By Proposition 3.1 the tree T has a spanning forest consisting of r K_2 's and trees T_1, T_2, \dots, T_k on at least three vertices such that at least k loners of T are leaves of the $k+r$ trees in the collection. The set of edges to delete consists of the edges of T with ends in different subtrees. \square

We note, in particular, that if every internal vertex is a weak stem then the corollary holds with $k = 0$.

4 Trees with $\gamma_m^\infty = \gamma$

Informally, the corona of a graph G is the graph obtained by joining a new vertex of degree one to each vertex of G . Formally, if G has vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, then $corona(G)$ is the graph with the $2n$ vertices $V(corona(G)) = \{v_1, v_2, \dots, v_n\} \cup \{v'_1, v'_2, \dots, v'_n\}$ and edges $E(corona(G)) = E(G) \cup \{v_i v'_i : 1 \leq i \leq n\}$. A graph H is a corona if it is the corona of some graph G . It is known that a connected graph has domination number $\lfloor \frac{|V|}{2} \rfloor$ if and only if it is either C_4 or a corona [3, 4]. The trees with $\gamma_m^\infty = \gamma$ turn out to be exactly the coronas of trees.

Lemma 4.1 *Let T be a tree for which $\gamma_m^\infty(T) = \gamma(T)$. For each minimum*

weight neo-colonization $\Pi = \{V_1, V_2, \dots, V_k\}$ of T and $i = 1, 2, \dots, k$, we have $\gamma_m^\infty(T[V_i]) = \gamma(T[V_i])$.

Proof. Each induced subgraph $T[V_i]$ is connected. Since $\gamma_m^\infty(T)$ is a minimum taken over all neo-colonizations, it must be that for $i = 1, 2, \dots, k$, the partition $\{V_i\}$ is a neo-colonization of $T[V_i]$ of weight $\gamma_m^\infty(T[V_i])$. Hence,

$$\gamma_m^\infty(T) = \sum_{i=1}^k \omega(V_i) = \sum_{i=1}^k \gamma_m^\infty(T[V_i]) \geq \sum_{i=1}^k \gamma_c(T[V_i]) \geq \sum_{i=1}^k \gamma(T[V_i]) \geq \gamma(T)$$

and the result follows. \square

Theorem 4.2 *A tree T with $n \geq 2$ vertices satisfies $\gamma_m^\infty = \gamma$ if and only if T is a corona.*

Proof. Suppose T is a corona. Then $\gamma(T) = \frac{n}{2}$. It is clear that there is a neo-colonization of weight $n/2$ in which each part consists of a leaf and its unique neighbor. Hence $\gamma_m^\infty = \gamma$.

The proof of the converse is by induction on n . The statement is clearly true if $n = 2, 3, 4$, the trees K_2 and P_4 being the only ones with $\gamma_m^\infty = \gamma$. Suppose the statement holds for all trees on at least two, and at most $n - 1$ vertices, for some $n \geq 5$. Let T be a tree on n vertices for which $\gamma_m^\infty = \gamma$. Then T is not a star.

Let $\Pi = \{V_1, V_2, \dots, V_k\}$ be a minimum weight neo-colonization of T . By Lemma 4.1 we have $\gamma_m^\infty(T[V_i]) = \gamma(T[V_i])$ for $i = 1, 2, \dots, k$. By the induction hypothesis, each tree $T[V_i]$ is a corona. We claim that, in fact, each is isomorphic to K_2 . Otherwise, without loss of generality $T[V_1]$ has at least three vertices and $\omega(V_1) = 1 + \gamma_c(V_1) > \gamma(V_1)$, a contradiction. This proves the claim. It follows that $\gamma_m^\infty(T) = k$ and $n = 2k \geq 6$.

Let u be a leaf of maximum eccentricity in T , and v be the stem to which it is adjacent. By the above argument, without loss of generality $V_k = \{u, v\}$. By Proposition 2.2, the vertex v is not adjacent to another leaf besides u . The choice of u now guarantees that v has degree two in T . Hence $T - u - v$ is a tree. Further, $\{V_1, V_2, \dots, V_{k-1}\}$ must be a minimum weight neo-colonization of $T - u - v$. Thus,

$$\gamma_m^\infty(T - u - v) = k - 1 \leq \gamma(T - u - v) \leq \gamma_m^\infty(T - u - v).$$

It follows that $\gamma(T - u - v) = k - 1 = \gamma_m^\infty(T - u - v)$. By the induction hypothesis, $T - u - v$ is a corona.

Let w be the vertex of $T - u - v$ to which v is adjacent in T . The proof will be complete if we can show that w is a stem. Otherwise, w is a leaf of the corona $T - u - v$, which has at least four vertices. Let s be its unique neighbor. Hence T contains the induced path on four vertices, u, v, w, s . Since the subgraph of T induced by each set V_i is connected and of size two, without loss of generality $V_{k-1} = \{w, s\}$. Thus, $T' = T - \{u, v, w\}$ is a tree. The set X of all internal vertices except s form a dominating set of size $\frac{|V(T')|-1}{2} = \frac{n-3}{2}$. But then $X \cup \{v\}$ is a dominating set of T and

$$\gamma(T) \leq \frac{n-1}{2} < \frac{n}{2} = k = \gamma_m^\infty(T),$$

a contradiction. This completes the proof. \square

An implication of Theorem 4.2 is that, for trees, $\gamma_m^\infty = \gamma$ only if the clique covering number equals γ . This implication holds for all graphs [6]. The converse is not true, even if we restrict the cliques to have size at least two. For example, $\gamma_m^\infty(C_6) = \gamma(C_6) = 2$, but the clique covering number of C_6 equals three. The proof of the theorem shows that trees with $\gamma_m^\infty = \gamma$ have a unique neo-colonization of minimum weight: each part consists of a leaf and the stem to which it is adjacent.

5 Trees with $\gamma_m^\infty = 2\gamma$

The following is from [11] and we include the proof for completeness.

Proposition 5.1 [11] *For any connected graph G , $\gamma_m^\infty(G) \leq 2\gamma(G)$, and the bound is sharp for all values of $\gamma(G)$.*

Proof. Let D be a minimum dominating set. Place a guard at each vertex of D . For each vertex $v \in D$, if v has at least one external private neighbor, pick one of them, say u , and place a guard at u . It is easy to see this configuration is an m -eternal dominating set.

To see that the bound is sharp for $\gamma = 1$, consider any star with at least three vertices. For $\gamma = 2$, consider C_6 and let u and v be two vertices at distance three apart. Add two new internally disjoint $u - v$ paths of length three to form the graph G . Obviously, $\{u, v\}$ is a dominating set of G . Let D be any minimum dominating set of G with $|D| = 3$. Suppose $u \notin D$. Since $N(u)$ is independent with $|N(u)| = 4$, and no two vertices in $N(u)$ have a common neighbor other than u , D does not dominate $N(u)$, a

contradiction. Thus $u \in D$ and similarly $v \in D$. Without loss of generality say $D = \{u, v, w\}$, where $w \in N(u)$. Then D cannot repel an attack at a vertex in $N(v) - N(w)$. It follows that $\gamma_m^\infty(G) = 4 = 2\gamma(G)$.

For $\gamma = k \geq 3$, consider C_{3k} and let $\{u_1, \dots, u_k\}$ be any minimum dominating set of C_{3k} . Note that for each i , $d(u_i, u_{i+1 \pmod{3k}}) = 3$. For each $i = 1, \dots, k$, add a new $u_i - u_{i+1 \pmod{3k}}$ path of length three to form G . Then $\gamma(G) = k$, but it can be shown similar to the previous case that no set of $2k - 1$ vertices eternally protects the vertices of G . \square

We shall now give several characterizations of the trees achieving the bound in Theorem 5.1.

Proposition 5.2 *Let T be a tree. If $\gamma_m^\infty(T) = 2\gamma(T)$ then every minimum dominating set of T is an independent set.*

Proof. We prove the contrapositive. Suppose there is a minimum dominating set D and vertices $w, x \in D$ such that $wx \in E$. Let $\Pi = \{V_1, V_2, \dots, V_\gamma\}$ be a neo-colonization of T in which, for $i = 1, 2, \dots, \gamma$, the subgraph of T induced by V_i is a star containing exactly one vertex of D . If there exists i such that $|V_i| < 3$, then $\gamma_m^\infty(T) < 2\gamma(T)$. Hence assume that $|V_i| \geq 3$ for $i = 1, 2, \dots, \gamma$. Without loss of generality, $w \in V_{\gamma-1}$ and $x \in V_\gamma$. Then, the neo-colonization $\Pi' = \{V_1, V_2, \dots, V_{\gamma-2}, V_{\gamma-1} \cup V_\gamma\}$ has weight at most $2\gamma - 1$ because the connected domination number of $V_{\gamma-1} \cup V_\gamma$ equals two, and hence it contributes $3 < 4$ to the weight of Π' . Therefore $\gamma_m^\infty(T) < 2\gamma(T)$. \square

Proposition 5.3 *Let T be a tree such that $\gamma_m^\infty(T) = 2\gamma(T)$. If D is a minimum dominating set of T , then every $x \in D$ has at least two external private neighbors.*

Proof. We prove the contrapositive. Suppose first D is a minimum dominating set of T such that there exists $w \in D$ with no external private neighbor. Then T admits a neo-colonization $\Pi = \{V_1, V_2, \dots, V_{\gamma-1}, \{w\}\}$ in which the subgraph induced by V_i is a star centered at a vertex in $D - \{w\}$. The weight of Π is at most $2(\gamma - 1) + 1 < 2\gamma$. The argument is similar if w has exactly one private neighbor. \square

Corollary 5.4 *Let T be a tree such that $\gamma_m^\infty(T) = 2\gamma(T)$. Then no leaf of T belongs to a minimum dominating set of T .*

A neo-colonization $\Pi = \{V_1, V_2, \dots, V_k\}$ is called *finest* if it has minimum weight, no parts of size one, and k is maximum over all such neo-colonizations.

Theorem 5.5 *Let D be a minimum dominating set of a tree T such that $\gamma_m^\infty = 2\gamma$. If $\Pi = \{V_1, V_2, \dots, V_k\}$ is a finest neo-colonization of T , then $D \cap V_i \neq \emptyset$ for $i = 1, 2, \dots, k$.*

Proof. The proof is by induction on the number of vertices of T . The statement holds vacuously for all trees with one vertex. Suppose it holds for all trees with $n - 1$ or fewer vertices, for some $n \geq 2$. Let T be a tree with n vertices and $\gamma_m^\infty = 2\gamma$.

Let $\Pi = \{V_1, V_2, \dots, V_k\}$ be a finest neo-colonization of T . The statement holds if $k = 1$, since T must be a star with at least three vertices. Hence, assume $k \geq 2$.

Let x be an end vertex of a longest path in T . Then x is a leaf. By Corollary 5.4, x is adjacent to a vertex $y \in D$. Since Π is finest, the vertex x and every other leaf adjacent to y belong to the same part of Π , say V_k . Let z be the vertex that precedes y on the longest path ending at x . By Proposition 5.2, the vertex z is not in D . We distinguish two cases.

Case 1. Vertex z is a private neighbor of y . Then there is no other stem adjacent to z . By choice of x as an end of a longest path, the vertex z has degree two in T , otherwise there is a longer path.

Suppose $V_k = N[y]$. Let $T' = T - V_k$. Then $\gamma(T') = \gamma(T) - 1$ and the set $D' = D - \{y\}$ is a minimum dominating set of T' . Since V_k can be defended by two guards, $\gamma_m^\infty(T') = 2\gamma(T')$. The sequence $\Pi' = \{V_1, V_2, \dots, V_{k-1}\}$ is a finest neo-colonization of T' . Hence, by the induction hypothesis, $D' \cap V_i \neq \emptyset$ for $i = 1, 2, \dots, k - 1$, and the statement follows.

Suppose $|V_k - N[y]| \geq 2$. Let $T' = T - N[y]$. Then $\gamma(T') = \gamma(T) - 1$ and the set $D' = D - \{y\}$ is a minimum dominating set of T' . Since $N[y]$ can be defended by two guards, $\gamma_m^\infty(T') = 2\gamma(T')$. Since both x and z are internal vertices of the subgraph of T induced by V_k , the weight of the neo-colonization $\Pi' = \{V_1, V_2, \dots, V_{k-1}, V_k - N[y]\}$ of T' is two less than the weight of Π . Hence Π' is a finest neo-colonization of T' , and the statement follows from the induction hypothesis as before.

Finally, suppose $V_k = N[y] \cup \{u\}$. Since $k \geq 2$ and z is not adjacent to a leaf, the vertex u is adjacent to a vertex in some other part of Π , say V_{k-1} . Since the subgraph of T induced by V_{k-1} is connected, u is

adjacent to at most one vertex in V_{k-1} . Again, let $T' = T - N[y]$. As before, $\gamma(T') = \gamma(T) - 1$, the set $D' = D - \{y\}$ is a minimum dominating set of T' , and $\gamma_m^\infty(T') = 2\gamma(T')$. The weight of the neo-colonization $\Pi' = \{V_1, V_2, \dots, V_{k-1} \cup \{u\}\}$ of T' is two less than the weight of Π . Hence Π' is a finest neo-colonization of T' . Since z is a private neighbor of y , we know that the vertex $u \notin D$, and the statement follows from the induction hypothesis as before.

Case 2. Vertex z is not a private neighbor of y . Then, by Proposition 5.2, $z \notin D$, and by Proposition 5.3, there is another leaf $w \neq x$ adjacent to y .

Suppose first that $V_k = N[y]$ or $V_k = N[y] - \{z\}$. As before, the tree $T' = T - V_k$ has $\gamma(T') = \gamma(T) - 1$, the set $D' = D - \{y\}$ is a minimum dominating set of T' , $\gamma_m^\infty(T') = 2\gamma(T')$ and $\Pi' = \{V_1, V_2, \dots, V_{k-1}\}$ is a finest neo-colonization of $T' = T - N[y]$. By the induction hypothesis, $D' \cap V_i \neq \emptyset$ for $i = 1, 2, \dots, k-1$, and the statement follows.

Otherwise, $|V_k - (N[y] - \{z\})| \geq 2$. Then $\Pi' = \{V_1, V_2, \dots, V_{k-1}, V_k - (N[y] - \{z\})\}$ is a finest neo-colonization of $T' = T \setminus (N[y] - \{z\})$. By the induction hypothesis, $D' \cap V_i \neq \emptyset$ for $i = 1, 2, \dots, k-1$, and the statement follows. \square

Corollary 5.6 *If $\Pi = \{V_1, V_2, \dots, V_k\}$ is a finest neo-colonization of a tree T with $\gamma_m^\infty = 2\gamma$, then $k \leq \gamma$.*

Let T be a tree and D be a minimum dominating set of T . A *dominating set partition of T with respect to D* is a neo-colonization $\Pi = \{V_1, V_2, \dots, V_\gamma\}$ such that, for $i = 1, 2, \dots, \gamma$, the subgraph of T induced by V_i is a star centered at a vertex of D . Clearly, every minimum dominating set D gives rise to at least one dominating set partition with respect to D . A dominating set partition will be called *fat* if $|V_i| \geq 3$ for $i = 1, 2, \dots, \gamma$.

Lemma 5.7 *Let T be a tree such that $\gamma_m^\infty(T) = 2\gamma(T)$. Then for any minimum dominating set D , there exists a fat dominating set partition of T .*

Proof. Let $\Pi = \{V_1, V_2, \dots, V_\gamma\}$ be a dominating set partition of T with respect to a minimum dominating set D . Then, by definition, for $i = 1, 2, \dots, \gamma$, the subgraph of T induced by V_i is a star centered at a vertex of D . Hence the weight of each part is at most two. Since the weight of Π equals 2γ , and there are γ parts, each part has weight exactly two. Thus

each star has at least three vertices, and Π is a fat dominating set partition of T . \square

Lemma 5.8 *Let T be a tree such that $\gamma_m^\infty(T) = 2\gamma(T)$. Then $\Pi = \{V_1, V_2, \dots, V_\gamma\}$ is a finest neo-colonization of T in which each part has at least three vertices if and only if Π is a fat dominating set partition with respect to a minimum dominating set D of T .*

Proof. Suppose that $\Pi = \{V_1, V_2, \dots, V_\gamma\}$ is a finest neo-colonization of T in which each part has at least three vertices. Then each part contributes exactly two to the weight of Π . Thus the subgraph induced by each part is a tree with connected domination number one, that is, a star. If D is the set of center vertices of these stars, then clearly D is a dominating set of size γ .

Let Π be a fat dominating set partition with respect to a minimum dominating set D of T . Then the weight of Π equals 2γ , so that Π is a minimum weight neo-colonization of T in which each part has at least three vertices. By Corollary 5.6, the neo-colonization Π has the maximum number of parts among all minimum weight neo-colonizations in which there are no parts of size one, hence Π is finest. \square

Theorem 5.9 *Let T be a tree. Then $\gamma_m^\infty(T) = 2\gamma(T)$ if and only if there is finest neo-colonization of T which is a fat dominating set partition with respect to a minimum dominating set of T .*

Proof. The forward implication is immediate from Lemmas 5.7 and 5.8. For the converse, suppose there is a finest neo-colonization $\Pi = \{V_1, V_2, \dots, V_\gamma\}$ which is a fat dominating set partition with respect to a minimum dominating set D of T . Then, for $i = 1, 2, \dots, \gamma$, $|V_i| \geq 3$, the subgraph of T induced by V_i is a star on at least three vertices. It follows that the weight of each part equals two, so that the weight of Π equals 2γ . Since Π has minimum weight, $\gamma_m^\infty = 2\gamma$. \square

The definition of a neo-colonization can be extended to forests. For each component (tree), the restriction of the neo-colonization to sets consisting of vertices from that tree, forms a neo-colonization.

Theorem 5.10 *Let T be a tree. Then $\gamma_m^\infty = 2\gamma$ if and only if every minimum dominating set D satisfies*

(a) *No vertex is adjacent to more than two vertices of D .*

(b) D is an independent set.

(c) No two vertices adjacent to two vertices of D are adjacent.

(d) Every vertex of D has at least two external private neighbors.

(e) There are no two vertices $x, y \in D$ such that the collection of all private neighbors of x and y induce the P_6 a, x, b, c, y, z .

Proof. Suppose that $\gamma_m^\infty = 2\gamma$. By Proposition 5.2, D is an independent set. Hence (b) holds. Suppose there is a vertex z adjacent to at least three vertices of D . Let $\{u, v, w\} \subseteq N(z) \cap D$. Form a neo-colonization $\Pi = \{V_1, V_2, \dots, V_{\gamma-2}\}$ by letting $V_1 = N[u] \cup N[v] \cup N[w]$ and of $V_2, V_3, \dots, V_{\gamma-2}$ be a partition of $V(T) - V_1$ into stars centered at vertices of $D - \{u, v, w\}$. The weight of V_1 is five, so the weight of Π is at most $5 + 2(\gamma - 3) < 2\gamma$. Since $\gamma_m^\infty = 2\gamma$, it follows that (a) holds.

Suppose that x and y are adjacent vertices that are each adjacent to two vertices of D . Let $N[x] \cap D = \{a, b\}$ and $N[y] \cap D = \{c, d\}$. Then $\{a, b\} \cap \{c, d\} = \emptyset$. Form a neo-colonization $\Pi = \{V_1, V_2, \dots, V_{\gamma-3}\}$ by letting $V_1 = N[a] \cup N[b] \cup N[c] \cup N[d]$ and $V_2, V_3, \dots, V_{\gamma-3}$ be a partition of $V(T) - V_1$ into stars centered at vertices of $D - \{a, b, c, d\}$. The weight of V_1 is seven, so that the weight of Π is at most $7 + 2(\gamma - 4) < 2\gamma$. Since $\gamma_m^\infty = 2\gamma$, it follows that condition (c) holds.

Suppose there is a vertex $x \in D$ with only one external private neighbor, say y , Form a neo-colonization $\Pi = \{V_1, V_2, \dots, V_{\gamma-1}, \{x, y\}\}$ such that, for $i = 1, 2, \dots, \gamma - 1$, the subgraph of T induced by V_i is a star centered at a vertex of $D - \{x\}$. Then the weight of Π is at most $2(\gamma - 1) + 1 < 2\gamma$. Since $\gamma_m^\infty = 2\gamma$, it follows that the component V_1 can not exist. Therefore, condition (d) holds.

Suppose that $x, y \in D$ and the set of all private neighbors of x and y comprise the path x_1, x, x_2, y_1, y, y_2 . Form a neo-colonization $\Pi = \{V_1, V_2, \dots, V_{\gamma-2}, \{x_1, x\}, \{x_2, y_1\}, \{y, y_2\}\}$, where $V_1, V_2, \dots, V_{\gamma-2}$ is a partition of $V(T) - \{x, x_1, x_2, y, y_1, y_2\}$ into stars centered at vertices of $D - \{x, y\}$. Then the weight of Π is at most $2(\gamma - 2) + 3 < 2\gamma$. Since $\gamma_m^\infty = 2\gamma$, it follows that the vertices x and y can not exist. Therefore, condition (e) holds.

The proof of the converse implication is by induction on the number of vertices of T . The statement holds for all trees on one vertex. Suppose it holds for all trees on at most $n - 1$ vertices, for some $n \geq 2$. Let T be a tree on n vertices for which conditions (a) through (e) hold. We need only consider the case when T is not a star. Let D be a minimum dominating set of T .

Let x be an end vertex of a longest path of T , and y the unique neighbor of x . By choice of x , the vertex y has a unique neighbor z which is not a leaf. Since condition (d) holds, no leaf is in D . Hence $y \in D$. Then, by (b), the vertex z can not belong to any minimum dominating set. We claim that the vertex z has degree at most three. Suppose z has four neighbors, y, u, v, w , where y and w lie on a longest path ending at x . By choice of x , the vertices u and v are either leaves or adjacent to a leaf. By condition (d), no leaf is in D . Hence each of u and v is adjacent to a leaf, and $u, v \in D$. Since $y \in D$, the vertex z has three neighbors in D , contrary to (a). Thus, $2 \leq \deg_T(z) \leq 3$. We consider these cases separately.

Case 1. $\deg_T(z) = 3$. Let $N_T(z) = \{a, b, y\}$, where b lies on a longest path starting at x . Then a is not a leaf, but every vertex in $N_T(a) - \{z\}$ is a leaf. Since condition (d) holds, the vertices a and y are each adjacent to at least two leaves.

Let $T' = T - (N[y] - \{z\})$. Then $\gamma(T') = \gamma(T) - 1$, and D' is a minimum dominating set of T' if and only if $D' \cup \{y\}$ is a minimum dominating set of T . Since condition (a) holds, no minimum dominating set of T contains b . Therefore there is no minimum dominating set of T' for which the vertex z belongs to a P_6 as described in condition (e). It follows that conditions (a) through (e) therefore hold for every minimum dominating set of T' . By the induction hypothesis, $\gamma_m^\infty(T') = 2\gamma(T')$.

Let $\Pi = \{V_1, V_2, \dots, V_k\}$ be a minimum weight neo-colonization of T . All leaves adjacent to y belong to the same part as y , say V_k (otherwise Π does not have minimum weight). Since the weight of Π is at most $2\gamma(T)$, it suffices to show that it is also at least $2\gamma(T)$. There are three possibilities, depending on $V_k - (N[y] - \{z\})$.

If $V_k = N[y] - \{z\}$, then $\Pi^0 = \{V_1, V_2, \dots, V_{k-1}\}$ is a neo-colonization of T' , and hence has weight at least $2(\gamma(T) - 1)$. Therefore the weight of Π is at least $2\gamma(T)$.

If $V_k = N[y]$, then the weight of Π is one more than the weight of the neo-colonization $\Pi^1 = \{V_1, V_2, \dots, V_{k-1}, \{z\}\}$ of T' . Since all leaves adjacent to a must belong to the same part as (a), and since the subgraph induced by each part is connected, $N[a] - \{z\}$ is a part of Π^1 , say $V_{k-1} = N[a] - \{z\}$. The neo-colonization $\{V_1, V_2, \dots, V_{k-1} \cup \{z\}\}$ has weight one less than Π^1 , so that the weight of Π^1 is at least $2(\gamma(T) - 1) + 1$. Therefore the weight of Π is at least $2\gamma(T)$.

If $V_k - N[y] \neq \emptyset$, then the connected domination number of the subgraph of T induced by V_k is two more than the connected domination number of

the subgraph induced by $V_k - (N[y] - \{z\})$. Hence, the weight of Π is two more than the neo-colonization $\Pi^2 = \{V_1, V_2, \dots, V_{k-1}, V_k - (N[y] - \{z\})\}$ of T' of T' . Since the weight of Π^2 is at least $2(\gamma(T) - 1)$, the weight of Π is at least $2\gamma(T)$.

Case 2. $\deg_T(z) = 2$. Let $N_T(z) = \{b, y\}$, and $T' = T - N[y]$. Then $\gamma(T') = \gamma(T) - 1$, and D' is a minimum dominating set of T' if and only if $D' \cup \{y\}$ is a minimum dominating set of T . Since conditions (a) through (e) hold for every minimum dominating set of T , they also hold for every minimum dominating set of T' . Therefore, $\gamma_m^\infty(T') = 2\gamma(T')$.

Let D' be a minimum dominating set of T' . Since condition (d) holds for T , the vertex $b \notin D'$.

Let $\Pi = \{V_1, V_2, \dots, V_k\}$ be a minimum weight neo-colonization of T . Then all leaves adjacent to y belong to the same part as y , say V_k (otherwise Π does not have minimum weight). If z also belongs to V_k , the weight of Π is two more than the weight of Π' , and it follows that $\gamma_m^\infty(T) = 2\gamma(T)$.

Hence, suppose that z does not belong to V_k . If there are at least two leaves adjacent to y , or if the part containing z , say V_{k-1} , is not a star centered at b , then the weight of Π is two more than the weight of Π' , and it follows that $\gamma_m^\infty(T) = 2\gamma(T)$.

Assume then, that x is the only leaf adjacent to y and the part V_{k-1} containing z is a star centered at b . The list $\Pi'' = \{V_1, V_2, \dots, V_{k-2}\}$ is a neo-colonization of $T' - V_{k-1}$ which is of weight at most $2(\gamma(T) - 1) - 1$. Hence the domination number of $T' - V_{k-1}$ is at most $\gamma(T) - 2 < \gamma(T')$. Since V_{k-1} is a star centered at b , there is a dominating set D'' of T' that contains b . Therefore $D'' \cup \{y\}$ is a dominating set of T in which y has only one external private neighbor. Hence condition (d) does not hold for T , a contradiction.

The result now follows by induction. \square

Let D be a dominating set of the tree T . The *domination labeling* of T with respect to D is the function $\ell_D : V \rightarrow \{1, 2, \dots, |D|\}$ that assigns to each vertex $x \in V$ the integer $\ell_D(x) = |N[x] \cap D|$.

Given a domination labeling ℓ_D , we use $F_1(T)$, or F_1 when the context is clear, to denote the subgraph of T induced by the set of vertices that are labeled one.

Corollary 5.11 *Let T be a tree. Then $\gamma_m^\infty = 2\gamma$ if and only if every*

minimum dominating set D satisfies

- (a) $1 \leq \ell_D(v) \leq 2$ for every $v \in V$.
- (b) $\ell_D(x) = 1$ for every $x \in D$.
- (c) The set $L_2 = \{x : \ell_D(x) = 2\}$ is an independent set.
- (d) If $x \in D$, then $|N_{F_1}(x)| \geq 2$.
- (e) There are no two vertices $x, y \in D$ such that the subgraph of F_1 induced by $N_{F_1}[x] \cup N_{F_1}[y]$ is isomorphic to P_6 .

Corollary 5.12 *Let T be a tree. Then $\gamma_m^\infty < 2\gamma$ if and only if there exists a minimum dominating set D and a domination partition with respect to D such that at least one of the following statements holds:*

- (a) Some vertex is adjacent to three vertices of D .
- (b) Two vertices of D are adjacent.
- (c) There are adjacent vertices x and y that are each adjacent to two vertices of D .
- (d) The subgraph induced by some part of the partition is isomorphic to K_1 or K_2 .
- (e) There are two parts X and Y such that the subgraph of T induced by $X \cup Y$ is isomorphic to P_6 .

6 Trees with $\gamma_m^\infty = \beta$

The following is a fundamental upper bound on the m -eternal domination number.

Theorem 6.1 [6] *Let G be a graph. Then $\gamma_m^\infty(G) \leq \beta(G)$.*

Let s be a stem in a tree T with at least two vertices. We call s *exposed* if it has at most one neighbor that is an internal vertex of T . Note that a tree with at most two vertices does not have exposed stems by definition.

The next two propositions have simple proofs which we omit.

Proposition 6.2 *Every tree with at least two vertices that is not a star has at least two exposed stems.*

Proposition 6.3 *Iteratively deleting exposed stems and the leaves adjacent to them partitions the vertices of a tree T into subsets that each induce a star, and $\beta(T)$ is the sum of the independence numbers of the stars in the partition.*

We consider the following operation which is a restriction of the procedure in Proposition 6.3.

Operation EWS: *If there is an exposed weak stem, delete it and its leaf.*

Lemma 6.4 [10] *If T' results from one application of operation EWS to T , then $\beta(T') = \beta(T) - 1$ and $\gamma_m^\infty(T') = \gamma_m^\infty(T) - 1$.*

The following lemma can be proved trivially by induction.

Lemma 6.5 *If T has an exposed strong stem that is adjacent to more than two leaves, then $\gamma_m^\infty(T) < \beta(T)$.*

Theorem 6.6 *A tree with at least two vertices has $\beta = \gamma_m^\infty$ if and only if repeated applications of operation EWS reduces T to K_1 or K_2 .*

Proof. Suppose k applications of operation EWS reduces T to K_1 or K_2 . Form a neo-colonization $\{V_1, V_2, \dots, V_{k+1}\}$ of T , where V_i is the set containing the stem and leaf deleted by the i^{th} application of the operation, $1 \leq i \leq k$, and V_{k+1} is the vertex set of the final K_1 or K_2 . By the discussion above, the weight of this neo-colonization is β . We show that there is no neo-colonization of smaller weight by induction on $n = |V|$.

By inspection, the statement is true for trees with two or three vertices. Suppose it holds for trees with between two and $n - 1$ vertices, for some $n \geq 4$. Let T be a tree with n vertices. If operation EWS can be applied to T , then the result follows by induction. Hence suppose EWS cannot be applied to T . Then every exposed stem is strong. Further, by Lemma 6.2, T has at least two strong stems.

By Lemma 6.5, if there is an exposed strong stem which is adjacent to more than two leaves then $\beta(T) > \gamma_m^\infty$ and the statement follows. Hence

assume every exposed strong stem is adjacent to exactly two leaves. Since T has at least four vertices, it follows that every exposed strong stem is adjacent to a unique internal vertex of T .

Let s be an exposed strong stem of T with maximum eccentricity. Let X be the set consisting of s and the two leaves to which it is adjacent. It is clear that $\beta(T) = 2 + \beta(T - X)$. We now consider the outcome of applying operation EWS to $T - X$.

Suppose first that operation EWS does not reduce $T - X$ to K_1 or K_2 . Then, by the induction hypothesis, $\beta(T - X) > \gamma_m^\infty(T - X)$. Let $\{V_1, V_2, \dots, V_p\}$ be a minimum weight neo-colonization of $T - X$. Then $\{V_1, V_2, \dots, V_p, X\}$ is a neo-colonization of T of weight $2 + \gamma_m^\infty(T - X)$. In this case we have

$$\begin{aligned}\beta(T) &= 2 + \beta(T - X) \\ &> 2 + \gamma_m^\infty(T - X) \\ &\geq \gamma_m^\infty(T),\end{aligned}$$

and the result follows.

Now suppose that operation EWS reduces $T - X$ to K_1 or K_2 . We claim that it is reduced to K_1 . Since T has at least two exposed strong stems, $T - X$ has at least one, say w . The reduced graph arising from each application of EWS is a tree. No application of EWS deletes w or a leaf adjacent to w unless the tree under consideration is a path on three vertices. This proves the claim.

Next, we claim that T is a caterpillar with spine $s, x_1, x_2, \dots, x_k, w$, such that

- (a) each of x_1, x_2, \dots, x_k is either a loner or a weak stem.
- (b) there are two leaves adjacent to s and to w ;

To see that T is a caterpillar, first recall from above that EWS cannot be applied to T . Further observe that T can have no vertex v adjacent to three internal vertices because EWS would delete all internal vertices on the path from s to v in $T - X$ and then could not be applied again. (Since w is an exposed stem, $v \neq w$.) Similarly, neither $T - X$ nor any tree derived from it by applying operation EWS has at least two strong stems. Point (a) now follows. Point (b) was established above. This proves the claim.

We complete the proof by showing that $\beta(T) > \gamma_m^\infty(T)$. Partition the spine of T , i.e., the vertices $s, x_1, x_2, \dots, x_k, w$, into maximal paths of stems

and maximal paths of loners. Since none of the reduced trees arising from $T - X$ has two strong stems, and $T - X$ is reduced to K_1 , each maximal path of loners has an even number of vertices.

Suppose first that T has no loners. Then T has connected domination number $k + 2$ and independence number $k + 4$. Hence assume T has loners. Construct a neo-colonization of T by forming a part out of each maximal path of consecutive stems along the spine of T , the leaves adjacent to them, and any loner adjacent to exactly one of these stems. The remaining vertices are all loners belonging to disjoint paths on an even number of vertices. Partition each of these paths into K_2 's.

We argue that there is an independent I set of size greater than the weight of the neo-colonization. The part containing s has independence number two more than connected domination number. Put these vertices into I . The maximum independent set in this part contains the vertex that was a loner in T . Each K_2 part has weight one and contributes one vertex to I . The vertex that it contributes is the one farthest from s along the spine of T . Proceeding away from s along the spine of T , eventually there is a part containing a maximal path of stems. It has independence number two more than connected domination number, but the loner closest to s in this part is cannot be included in I because it is adjacent to the loner from the previous part which belongs to I . Put the remainder of the maximum independent set of the part, i.e. all but this loner, into I . Continuing in this way, eventually the part containing w contributes its connected domination number plus one vertices to I . Thus the weight of the neo-colonization is $|I| - 1$. This completes the proof. \square

An alternate characterization is given as a corollary of the previous result.

Corollary 6.7 *Let T be a tree with at least two vertices. Then $\gamma_m^\infty(T) = \beta(T)$ if and only if there exists a minimum-weight neo-colonization of T containing only parts that are K_2 's or P_3 's and there is at most one P_3 .*

$K_{1,3}$ with each edge subdivided twice is an example showing that the minimum weight condition on neo-colonization is necessary in Proposition 6.7. This graph has a neo-colonization consisting of K_2 's and P_3 's but has $\gamma_m^\infty(T) < \beta(T)$.

7 Concluding Remarks

We begin this section by stating a result of Chambers et al. [5] which also appears in the survey [13].

Theorem 7.1 [5, 13] *For all connected graphs, $\gamma_m^\infty \leq \lceil \frac{|V|}{2} \rceil$.*

The trees for which $\gamma_m^\infty = \gamma$ and those for which $\gamma_m^\infty = \beta$ achieve equality in this bound. There are other trees for which equality holds. One example is the tree obtained from the path v_1, v_2, \dots, v_9 by adding two vertices v_{10}, v_{11} and the edges v_4v_{10}, v_6v_{11} . Characterizing the trees such that $\gamma_m^\infty = \lceil \frac{|V|}{2} \rceil$ remains an open problem.

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