On weak Sidon sequences

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Abstract A sequence $\{a_i \mid 1 \leq i \leq k\}$ of integers is a weak Sidon sequence if the sums $a_i + a_j$ are all different for any $i < j$. Let $g(n)$ denote the maximum integer k for which there exists a weak Sidon sequence $\{a_i \mid 1 \leq i \leq k\}$ such that $1 \leq a_1 < \cdots < a_k \leq n$. Let the weak Sidon number $G(k) = \min\{n \mid g(n) = k\}$. In this note, $g(n)$ and $G(k)$ are studied, and $g(n)$ is computed for $n \leq 172$, based on which the weak Sidon number $G(k)$ is determined for up to $k = 17$.

1 Introduction

A sequence $\{a_i|1 \leq i \leq k\}$ of integers is a Sidon sequence if the sums $a_i + a_j$ are all different for any $i \leq j$. Let $f(n)$ denote the maximum integer k for which there exists a Sidon sequence $\{a_i|1 \leq i \leq k\}$ such that $1 \leq a_1 < \cdots < a_k \leq n$. The Sidon sequence was first considered by the Hungarian mathematician Simon Sidon [7] in 1932.

The Sidon sequence, also called Golomb rulers, was well studied by mathematicians and computer scientists. It has many applications, for example, in radio frequency selection, radio antennae placement and error correcting codes [1, 2, 5].

Let the Sidon number $F(k) = \min\{n \mid f(n) = k\}$. Computing values of Sidon numbers is a famous and difficult problem. Shearer [8] proposed one of the first efficient algorithms which was able to compute minimum Golomb ruler up to 16 marks. Later, Dollas et al. [9] introduced the GVANT algorithm and managed to compute the optimal Golomb ruler with 19 marks. With the help of "Distributed.Net" who had taken a liking to Golomb rulers, minimum Golomb rulers with $20 \leq m \leq 23$ were obtained, see [11]. More recently, the OGR project is coordinating the search for optimal Golomb rulers with 24, 25 and 26 marks. It is already difficult to solve for $k \geq 27$, and it is also interesting to build their upper bounds and also some techniques have been proposed in the literature. Projective plane and affine plane constructions for the upper bound on $F(k)$ for more $k \leq 150$ can be found in [12].

Similarly, a sequence $\{a_i | 1 \leq i \leq k\}$ of integers is a weak Sidon sequence if the sums $a_i + a_j$ are all different for any $i < j$. Let $g(n)$ denote the maximum integer k for which there exists a weak Sidon sequence $\{a_i \mid 1 \leq i \leq k\}$ such that $1 \leq a_1 < \cdots < a_k \leq n$. Let the weak Sidon number $G(k) = \min\{n \mid g(n) = k\}$.

In [6], Ruzsa proved that $g(n) < n^{1/2} + O(n^{1/4})$. An alternate proof of this result was presented in [3]. There were little references on values or bounds for small weak Sidon numbers.

In this paper, $g(n)$ and $G(k)$ are studied. After preliminaries in Section 2, some general results are proved in Section 3. In Section 4, $g(n)$ is determined for $n \leq 172$, based on which the weak Sidon number $G(k)$ is determined for up to $k = 17$, with the constructions on the upper bounds given in the Appendix. Section 5 concludes the note.

2 Preliminaries

Given a positive integer n,to search a weak Sidon sequence $S \subseteq \{1, 2, \cdots, n\}$ with the minimum cardinality, we need the following simple lemmas.

Lemma 1 *Let $c(A)$ be the maximum cardinality of a weak Sidon sequence as a subset of A. If t is an integer, then $c(A) = c(A + t)$, where $A + t = \{x + t \mid x \in A\}$.*

If $A = \{1, 2, \cdots, n\}$, then $c(A) = g(n)$.

By reflection symmetry, we have

Lemma 2 *If S is a weak Sidon sequence of order $g(n)$ in $\{1, 2, \cdots, n\}$, α and τ are the prefix and suffix of S of length $\lceil n/2 \rceil$, respectively, then either $\#(\alpha) \geq \lceil g(n)/2 \rceil$ or $\#(\tau) \geq \lceil g(n)/2 \rceil$.*

We also have the following lemma.

Lemma 3 *For any integer $n \geq 1$, $g(n+1) = g(n)$ or $g(n+1) = g(n)+1$, and $g(n+1) = g(n)+1$ if and only if there is a weak Sidon sequence $\{a_i \mid 1 \leq i \leq g(n+1)\}$ that contains $n+1$.*

3 Some general results on weak Sidon numbers

It is not difficult to see that any subsequence of a weak Sidon sequence is also a weak Sidon sequence. This can be used to prove the following theorem, which will be important in the computation in the next section.

Theorem 1 *If $k \geq 4$ and $\{a_i \mid 1 \leq i \leq k\}$ is a weak Sidon sequence, where $1 = a_1 < \cdots < a_k = G(k)$, then $a_i \leq G(k) - G(k-i+1) + 1$ for any $i \in \{2, \cdots, k-1\}$.*

Proof. Since that $\{a_i \mid 1 \leq i \leq k\}$ is a weak Sidon sequencee, $\{a_i, \cdots, a_k\}$ is a weak Sidon sequence too. Thus $a_k - a_i \geq G(k-i+1) - 1$ for any $i \in \{2, \cdots, k-1\}$. So we have $a_i \leq G(k) - G(k-i+1) + 1$ for any $i \in \{2, \cdots, k-1\}$. □

We can see that $G(n) \leq F(n)$. Similar to some known results on $F(n)$, we can prove the following results on $G(n)$ in Theorem 2 and Theorem 3.

For integers $k_1, k_2 \geq 1$, it is not difficult to see that $G(k_1 + k_2) \geq G(k_1) + G(k_2)$. In fact, we can do a little better as in the following theorem.

Theorem 2 *For integers $k_1, k_2 \geq 2$, $G(k_1 + k_2 - 1) \geq G(k_1) + G(k_2) - 1$.*

Proof. Suppose $m = G(k_1 + k_2 - 1)$ and $A = \{a_1, \cdots, a_{k_1+k_2-1}\}$ is a weak Sidon sequence, where $1 = a_1 < \cdots < a_{k_1+k_2-1} = m$. Let $A_1 = \{a_1, \cdots, a_{k_1}\}$, and $A_2 = \{a_{k_1}, \cdots, a_{k_1+k_2-1}\}$. So both A_1 and A_2 are weak Sidon sequences. Thus $a_{k_1} - 1 = a_{k_1} - a_1 \geq G(k_1) - 1$ and $a_{k_1+k_2-1} - a_{k_1} = m - a_{k_1} \geq G(k_1 + k_2 - 1) - 1$, respectively. So $m - 1 \geq G(k_1) + G(k_2) - 2$. Thus $G(k_1 + k_2 - 1) \geq G(k_1) + G(k_2) - 1$. □

For integers $k_1, k_2 \geq 2$, we can prove that $F(k_1 + k_2 - 1) \geq F(k_1) + F(k_2) - 1$ similarly.

We know that $G(k+1) \geq G(k) + 1$. Now we will improve this result.

Theorem 3 *For any integer $k \geq 3$, we have*

(a) $G(k+1) > G(k) + 1$;

(b) if $G(k+1) = G(k) + 2$, then there is a weak Sidon sequence $A = \{a_1, \cdots, a_{k+1}\} \subseteq \{1, \cdots, G(k+1)\}$ such that $a_1 < \cdots < a_{k+1}$, where $a_1 = 1, a_2 = 2, a_k = G(k+1) - 2$ and $a_{k+1} = G(k+1)$.

Proof. (a) We know that $G(k+1) \geq G(k)+1$. Suppose that $m = G(k+1) = G(k)+1$ and $A = \{a_1, \cdots, a_{k+1}\}$ is a weak Sidon sequence, where $1 = a_1 < \cdots < a_{k+1} = m$. Let $A_1 = \{a_1, \cdots, a_k\}$, and $A_2 = \{a_2, \cdots, a_{k+1}\}$. It is not difficult to see that both A_1 and A_2 are weak Sidon sequences, and $a_2 = 2$ and $a_k = G(k) = m-1$. So $\{1, 2, m-1, m\} \subseteq A$, which contradicts with that A is a weak Sidon sequence. Thus $G(k+1) > G(k)+1$.

(b) can be proved similarly. □

From definitions of Sidon sequences and weak Sidon sequences, it is not difficult to see that a Sidon sequence is a weak Sidon sequence without arithmetic progressions of length 3. So we have the following theorem.

Theorem 4 *If $n = G(k) < F(k)$ for an integer $k \geq 2$, and $A = \{a_i \mid 1 \leq i \leq k\}$ is a weak Sidon sequence, where $1 = a_1 < \cdots < a_k = n$, then there is an arithmetic progression of length 3 in A.*

4 Values of some weak Sidon numbers

In this section, we will compute $G(k)$ for $k \in \{2, 3, 4, 5\}$ without the help of computers firstly, and then compute more weak Sidon numbers by computers based on results proved in earlier sections.

4.1 Values of a few small weak Sidon numbers

In this subsection, we obtain the value of $G(n)$ for $n \in \{2, 3, 4, 5\}$ without the help of computers.

It is not difficult to see that $g(2) = 2$ and $g(3) = 3$. So we have $G(2) = 2$, $G(3) = 3$.

Because that $g(4) \geq g(3) = 3$, and $\{1, 2, 3, 4\}$ is not a weak Sidon sequence, we know that $g(4) = 3$. Since $\{1, 3, 4, 5\}$ is a weak Sidon sequence, we can see that $g(5) = 4$. Thus $G(4) = 5$.

By (a) in Theorem 6 we know that $G(5) > G(4)+1 = 5+1 = 6$. Thus $g(6) = 4$.

If $g(7) = 5$, then by (b) in Theorem 6 we know there is a weak Sidon set $A = \{a_i | 1 \leq i \leq 5\}$ such that $a_1 = 1, a_2 = 2, a_4 = 5$ and $a_5 = 7$. So it is not difficult to see that a_3 can not be any integer in $\{3, 4, 6\}$ because that A is a weak Sidon set. Thus $g(7) = 4$ and $g(8) \leq 5$. Since $\{1, 2, 3, 5, 8\}$ is a weak Sidon set, we have $g(8) = 5$ and $G(5) = 8$.

By (a) in Theorem 3 we know that $g(9) = 5$.

4.2 Values of more weak Sidon numbers obtained by computing

In this subsection, we will compute $g(n)$ for $n \in \{10, \cdots, 172\}$ by computers.

In order to speed up the search, we compute $G(k+1)$ based on lemmas and theorems given in Section 2 and Section 3, among which Theorem 4 is an important one.

If we obtain $G(k) = m$, then by Theorem 2 we know that $g(m+1) = k$ and $G(k+1) \geq m+2$.

For $i = m+2$, we will search if there is a weak Sidon sequence $S = \{a_i \,|\, 1 \leq i \leq k+1\}$ such that $1 = a_1 < \cdots < a_{k+1} = i$. If there exists such an S, then $G(k+1) = m+2$. Otherwise, we will do similar computation for $i = m+3$ and so on, until find a weak Sidon sequence S with cardinality i.

Note that (b) in Theorem 3 can be used in the case $G(k+1) = m+2$, where we can suppose that $a_2 = 2$ and $a_k = m$. It seems that $G(k+1) > G(k) + 2$ for any integer $k \geq 4$, but it may be not easy to prove.

Values of some weak Sidon numbers obtained and some known Sidon numbers are listed in Table 1. The constructions are given in the Appendix of this paper.

For $n > 160$, it needs several hours to compute $g(n)$. For instance, it needs about than 25 hours to compute $g(165)$ on our computer (CPU 3.2GHz). So we have not computed $g(n)$ directly one by one for $n \geq 166$. To compute $G(17)$, we compute if $G(17) \geq n$ for $n \in \{175, 174\}$. We find a good construction in $\{1, \cdots, 174\}$, which is in $\{1, \cdots, 172\}$ too. So we obtain $g(17) \leq 172$. Then we obtain that $g(171) < 17$ by computing. So $G(17) = 172$, and $g(n) = 16$ for any $n \in \{152, \cdots, 171\}$.

Table 1: Values of $G(k)$ and known values of $F(k)$

k	2	3	4	5	6	7	8	9	10	11	12
$G(k)$	2	3	5	8	13	19	25	35	46	58	72
$F(k)$	2	4	7	12	18	26	35	45	56	73	86
k	13	14	15	16	17	18	19	20	21	22	23
$G(k)$	87	106	127	151	172						
$F(k)$	107	128	152	178	200						

5 Conclusions and Remarks

Similar to Sidon numbers, weak Sidon numbers are difficult to compute. In this paper, $g(n)$ and weak Sidon number $G(k)$ are studied. By computing $g(n)$ for integer $n \leq 172$, we obtain the exact values of $G(k)$ for $k \leq 17$.

As mentioned above, $G(k) \leq F(k)$. It seems interesting to ask the following question.

Qusetion. Is there a constant C such that $F(k) - G(k) \leq C$ for any integer k?

We know that $g(n) \leq f(n)$. Maybe $g(n)$ is much smaller than $f(n)$ for any integer n that is not very small. For instance, by computing we obtain that $g(172) = 17$, on the other hand, we know that $F(17) = 200$ (see [12]). It is interesting to study upper bounds for more weak Sidon numbers by the projective plane and affine plane constructions similar to those in [12]. This will be our future direction.

Appendix

For any $k \in \{2, \cdots, 17\}$, in the following table, we show the minimum integer n such that $g(n) = k$, together with the construction.

Table 2: Values of $g(n)$

n	$g(n)$	construction
2	2	$\{1,2\}$
3	3	$\{1,2,3\}$
4	5	$\{1,2,3,5\}$
5	8	$\{1,2,3,5,8\}$
6	13	$\{1,2,3,5,8,13\}$
7	19	$\{1,2,3,5,9,14,19\}$
8	25	$\{1,2,3,5,9,15,20,25\}$
9	35	$\{1,2,3,5,9,16,25,30,35\}$
10	46	$\{1,2,8,11,14,22,27,42,44,46\}$
11	58	$\{1,2,6,10,18,32,35,38,45,56,58\}$
12	58	$\{1,2,12,19,22,37,42,56,64,68,70,72\}$
13	87	$\{1,2,12,18,22,35,43,58,61,73,80,85,87\}$
14	106	$\{1,2,7,15,28,45,55,67,70,86,95,102,104,106\}$
15	127	$\{1,2,3,23,27,37,44,51,81,96,108,111,114,119,127\}$
16	151	$\{1,3,5,16,27,37,55,58,75,83,102,116,139,145,146,151\}$
17	172	$\{1,6,10,11,12,44,63,76,89,113,116,130,137,144,152,$ $160,172\}$

Acknowledgements

The authors are very grateful to the anonymous reviewers for their valuable comments. This project is supported by Guangxi Natural Science Foundation (2011GXNSFA018142), Science and Technology Project of Chengdu (11DXYB174JH-027).

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