On three families of graphs with constant metric dimension *

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Abstract. A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G), is finite and does not depend upon the choice of G in $\mathcal{G}$. In this paper, we show that the sunlet graphs, the rising sun graphs and the co-rising sun graphs have constant metric dimension.

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1 Notations and preliminary results

For a connected graph G, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. A vertex w of a graph G, is said to resolve two vertices u and v of G if $d(w, u) \neq d(w, v)$. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G, and let v be a vertex of G. The *representation* of a vertex v with respect to W denoted by $r(v|W)$ is the k-tuple $(d(v, w_1), d(v, w_2), ..., d(v, w_k))$. If distinct vertices of G, have distinct representations with respect to W, then W is called a *resolving set* for G, [3]. A resolving set of minimum cardinality is called a *metric basis* for G, and the cardinality of this set is the *metric dimension* of G, denoted by $dim(G)$.
For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the *ith* component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of

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distinct vertices $x, y \in V(G) \backslash W$.
Caceres *et al.* [1] found the metric dimension of the fan graph f_n. Tomescu *et al.* [11] found the metric dimension of *Jahangir graph* J_{2n}.
In [3] Chartrand *et al.* proved that a graph G has metric dimension 1 if and only if it is a *path*, hence path on n vertices constitute a family of graphs with constant metric dimension, and *cycles* with $n \geq 3$ vertices also constitute such a family of graphs as their metric dimension is 2. In [2] J. Caceres *et al.* proved that:

$$dim(p_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

Prisms D_n are the trivalent plane graphs obtained by the cartesian product of the path P_2 with a cycle C_n; they also constitute a family of 3-*regular graphs* with constant metric dimension. In [6], Javaid *et al.* proved that the *antiprism* graph A_n constitutes a family of regular graphs with constant metric dimension as $dim(A_n) = 3$, for every $n \geq 5$.
In this paper, we extend this study by considering the metric dimension of sunlet graphs, the rising sun graphs and the co-rising sun graphs. We show that these graphs constitute families of graphs with constant metric dimension.
The prism D_n, $n \geq 3$, consists of an outer n-cycle $v_1v_2...v_n$, an inner n-cycle $u_1u_2...u_n$, and a set of n spokes u_iv_i, where $n+i$ is taken modulo n.
The sun let graph $S_n^{'}$ is constructed from the graph D_n, by deleting the edges a_ia_{i+1} from $E(D_n)$, for $i = 1, 2, ..., n$, where $n+i$ is taken modulo n.
The antiprism graph A_n, $n \geq 3$, consists of an outer n-cycle $a_1a_2...a_n$, an inner n-cycle $b_1b_2...b_n$, and a set of n spokes b_ia_i and $b_{i+1}a_i$, $i = 1, 2, 3, ..., n$ where $n+i$ is taken modulo n.
The rising sun graph $S_n^{''}$ is obtained from the antiprism graph by deleting the edges a_ia_{i+1} from $E(A_n)$, $i = 1, 2, ..., n$ and the vertex a_n from $V(A_n)$.
The co-rising sun graph S_n^* is the extension of the above graph $S_n^{''}$ as follows: We introduce two new vertices x, y. Introduce two new edges xb_1, yb_n. Relabel the vertices of S_n^* as $\{u_i = b_i | i = 1, 2, ..., n\}$ and $\{x = v_1, a_1 = v_2, ..., y = v_{n+1}\}$.

2 Sun related graphs with constant metric dimension.

In this section we show that the graphs $S_n^{'}$, $S_n^{''}$ and S_n^* defined above have constant metric dimension.

Theorem 1. *For* $n \geq 3$,

$$dim(S_n^{'}) = \begin{cases} 2, & \text{for } 3 \leq n \leq 5; \\ 3, & \text{for } n \geq 6. \end{cases}$$

Proof. By [3] it is easy to show that $W = \{v_1, v_2\}$ is a resolving set for $S_n^{'}$ when $3 \leq n \leq 5$, because it is not a path. For $n \geq 6$ consider the set $W = \{v_1, v_2, v_k\} \subset V(S_n^{'})$. We show that W is a resolving set for $S_n^{'}$. We find the representations of vertices of $V(S_n^{'}) \setminus W$ with respect to W. The representations of $V(S_n^{'}) \setminus W$ vertices are as follows:

$$r(u_i|W) = \begin{cases} (1,2,k), & \text{for } i = 1; \\ (i, i-1, k+1-i), & \text{for } 2 \leq i \leq k; \\ (k+1, k, 2), & \text{for } i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & \text{for } k+2 \leq i \leq n. \end{cases}$$

And

$$r(v_i|W) = \begin{cases} (i+1, i, k+2-i), & \text{for } 3 \leq i \leq k-1; \\ (k+2, k+1, 3), & \text{for } i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & \text{for } k+2 \leq i \leq n. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(S_n^{'}) \leq 3$. We now show that $dim(S_n^{'}) \geq 3$, by proving that there is no resolving set W, with $|W| = 2$ for $S_n^{'}$. Contrarily, suppose that $|W| = 2$, then we have the following possibilities:
(1). Both vertices belong to $\{u_i\} \subset V(S_n^{'})$, $i = 1, 2, ..., n$. Without loss of generality, we suppose that one resolving vertex is u_1, and the other is u_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k$, we have,
$r(u_n|\{u_1, u_t\}) = r(v_1|\{u_1, u_t\}) = (1, t)$.
For $t = k+1$, we have,
$r(u_2|\{u_1, u_t\}) = r(u_n|\{u_1, u_t\}) = (1, k-1)$, a contradiction.
(2). Both vertices belong to $\{v_i\} \subset V(S_n^{'})$, $i = 1, 2, ..., n$. Without loss of generality, we suppose that one resolving vertex is v_1, and the other is v_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k-1$, we have,
$r(v_{t+1}|\{v_1, v_t\}) = r(u_{t+2}|\{v_1, v_t\}) = (t+2, 3)$.
For $t = k$,
$r(u_{t+2}|\{v_1, v_t\}) = r(v_{t-1}|\{v_1, v_t\}) = (k, 3)$, similarly for $t = k+1$, we have,
$r(v_2|\{v_1, v_t\}) = r(u_n|\{v_1, v_t\}) = (3, t)$, a contradiction.
(3). One vertex belong to $\{u_i\}$ and the other vertex belong to $\{v_i\}$, for $i = 1, 2, ..., n$. Without loss of generality consider one resolving vertex is u_1, and the other is v_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k-1$, we have,
$r(v_n|\{u_1, v_t\}) = r(u_{n-1}|\{u_1, v_t\}) = (2, t+2)$.
For $t = k$,
$r(u_{t+2}|\{u_1, v_t\}) = r(v_{t-1}|\{u_1, v_t\}) = (k-1, 3)$, similarly for $t = k+1$, we have,
$r(u_n|\{u_1, v_t\}) = r(u_2|\{u_1, v_t\}) = (1, k)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(S_n^{'})$. Thus, $dim(S_n^{'}) = 3$.

□

Theorem 2. *For $n \geq 3$,*

$$dim(S_n^{''}) = \begin{cases} 2, & \text{for } n = 2k; \\ 3, & \text{for } n = 2k+1. \end{cases}$$

Proof. We distinguish two cases:
Case(1). For $n = 2k$, $k \in \mathbb{Z}^+$. Let $W = \{v_1, v_k\} \subset V(S_n^{''})$, we show that W is resolving set for $S_n^{''}$. Consider the representations of any vertex of $V(S_n^{''}) \backslash W$ with respect to W.
Representations of the vertices are as follows:

$$r(u_i|W) = \begin{cases} (1,k), & i = 1; \\ (i-1, k+1-i), & 2 \leq i \leq k; \\ (k,1), & i = k+1; \\ (2k+2-i, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

And

$$r(v_i|W) = \begin{cases} (i, k-i+1), & 2 \leq i \leq k-1; \\ (2k+2-i, i-k+1), & k+1 \leq i \leq 2k-1. \end{cases}$$

Since these representations are pair-wise distinct, it follows that $dim(S_n^{''}) \leq 2$. By [3] it is clear that $dim(S_n^{''}) \geq 2$. Which implies that $dim(S_n^{''}) = 2$, for even n.

Case(2). For $n = 2k+1$, $k \in \mathbb{Z}^+$. Consider $W = \{v_1, v_2, v_{k+1}\} \subset V(S_n^{''})$, we show that W is resolving set for $S_n^{''}$. Consider the representations of any vertex of $V(S_n^{''}) \backslash W$ with respect to W.
Representations of the vertices are as follows:

$$r(u_i|W) = \begin{cases} (1, 3-i, k+2-i), & \text{for } 1 \leq i \leq 2; \\ (i-1, i-2, k+2-i), & \text{for } 3 \leq i \leq k+1; \\ (k+1, k, 1), & \text{for } i = k+2; \\ (2k-i+3, 2k-i+4, i-k), & \text{for } k+3 \leq i \leq 2k+1. \end{cases}$$

And

$$r(v_i|W) = \begin{cases} (i, i-1, k+2-i), & \text{for } 3 \leq i \leq k; \\ (k+1, k+1, 2), & \text{for } i = k+2; \\ (2k-i+2, 2k-i+3, i-k+1), & \text{for } k+3 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(S_n^{''}) \leq 3$. For the other side of the proof, we show that $dim(S_n^{''}) \geq 3$, by proving that there is no resolving set having two vertices.

Contrarily, suppose that $|W| = 2$, then we have the following possibilities:
(1). Both Vertices belong to $\{u_i\} \subset V(S_n^{''})$, for $i = 1, 2, ..., n$. Without loss of generality, we suppose that one resolving vertex is u_1, and the other is u_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k-1$, we have,
$r(u_{n-1}|\{u_1, u_t\}) = r(v_{n-1}|\{u_1, u_t\}) = (2, t)$.
For $t = k$,
$r(v_t|\{u_1, u_t\}) = r(u_{t+1}|\{u_1, u_t\}) = (t, 1)$, a contradiction. Similarly for $t = k+1$, we have,
$r(v_1|\{u_1, u_t\}) = r(u_n|\{u_1, u_t\}) = (1, t)$, a contradiction.
(2). Both Vertices belong to $\{v_i\} \subset V(S_n^{''})$, for $i = 1, 2, ..., n-1$. Without loss of generality, we suppose that one resolving vertex is v_1, and the other is v_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k-1$, we have,
$r(v_{n-1}|\{v_1, v_t\}) = r(u_{n-1}|\{v_1, v_t\}) = (3, t+2)$.
For $t = k$,
$r(u_{t+2}|\{v_1, v_t\}) = r(v_{t+1}|\{v_1, v_t\}) = (t+1, 1)$, similarly for $t = k+1$, we have,
$r(v_2|\{v_1, v_t\}) = r(u_n|\{v_1, v_t\}) = (2, k)$, a contradiction.
(3). One vertex belong to $\{u_i\} \subset V(S_n^{''})$, $i = 1, 2, ..., n$ and the other vertex belong to $b \in \{v_i\} \subset V(S_n^{''})$, for $i = 1, 2, ..., n-1$. Without loss of generality, we suppose that one resolving vertex is u_1, and the other is v_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k$, we have,
$r(v_{t+1}|\{u_1, v_t\}) = r(u_{t+2}|\{u_1, v_t\}) = (t+1, 2)$.
For $t = k+1$,
$r(u_t|\{u_1, v_t\}) = r(u_{t+2}|\{u_1, v_t\}) = (k, 1)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(S_n^{''})$. Thus, $dim(S_n^{''}) = 3$.

□

Theorem 3. *For $n \geq 3$,*

$$dim(S_n^*) = \begin{cases} 2, & \text{for } n = 2k+1; \\ 3, & \text{for } n = 2k, \textit{ except } n = 4. \end{cases}$$

Proof. We distinguish two cases:
Case(1). For $n = 2k+1$, $k \in \mathbb{Z}^+$. Suppose $W = \{v_1, v_{k+1}\} \subset V(S_n^*)$, we show that W is resolving set for S_n^*. Consider the representations of any vertex of $V(S_n^*) \backslash W$ with respect to W.
Representations of the vertices are as follows:

$$r(u_i|W) = \begin{cases} (i, k+1-i), & 1 \leq i \leq k; \\ (i, 1), & i = k+1; \\ (2k+3-i, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

And

$$r(v_i|W) = \begin{cases} (i, k-i+2), & 2 \leq i \leq k; \\ (2k+4-i, i-k), & k+2 \leq i \leq 2k+2. \end{cases}$$

Since these representations are pair wise distinct it follows that $dim(S_n^*) \leq 2$. By [3] it is clear that $dim(S_n^*) \geq 2$. Which implies that $dim(S_n^*) = 2$, for odd n.

Case(2). For $n = 2k$, $k \in \mathbb{Z}^+$, when $k = 1$ then $dim(S_n^*) = 2$. For $k \geq 2$, suppose $W = \{v_1, v_2, v_{k+1}\} \subset V(S_n^*)$, we show that W is resolving set for S_n^*. Consider the representations of any vertex of $V(S_n^*)\backslash W$ with respect to W.
Representations of the vertices are as follows:

$$r(u_i|W) = \begin{cases} (1, 1, k), & \text{for } i = 1; \\ (i, i-1, k+1-i), & \text{for } 2 \leq i \leq k; \\ (k+1, k, 1), & \text{for } i = k+1; \\ (2k-i+3, 2k-i+3, i-k), & \text{for } k+2 \leq i \leq 2k. \end{cases}$$

And

$$r(v_i|W) = \begin{cases} (i, i-1, k+2-i), & \text{for } 3 \leq i \leq k; \\ (k+2, k+1, 2), & \text{for } i = k+2; \\ (2k+4-i, 2k+4-i, i-k), & \text{for } k+3 \leq i \leq 2k+1; \\ (3, 3, k+2), & \text{for } i = 2k+2. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(S_n^*) \leq 3$. For the other side of the proof, we show that $dim(S_n^*) \geq 3$, by proving that there is no resolving set having two vertices. Contrarily, suppose that $|W| = 2$, then we have the following possibilities:
(1). Both vertices belong to $\{u_i\} \subset V(S_n^*)$, $i = 1, 2, ..., n$. Without loss of generality, we suppose that one resolving vertex is u_1, and the other is u_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k-1$, we have,
$r(v_1|\{u_1, u_t\}) = r(u_n|\{u_1, u_t\}) = (1, t)$.
For $t = k$,
$r(v_t|\{u_1, u_t\}) = r(u_{t+1}|\{u_1, u_t\}) = (t, 1)$, a contradiction. Similarly for $t = k+1$, we have,
$r(v_3|\{u_1, u_t\}) = r(v_n|\{u_1, u_t\}) = (1, k-1)$, a contradiction.
(2). Both vertices belong to $\{v_i\} \subset V(S_n^*)$, $i = 1, 2, ..., n+1$. Without loss of generality, we suppose that one resolving vertex is v_1, and the other is v_t, $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have,
$r(v_{n+1}|\{v_1, v_t\}) = r(v_n|\{v_1, v_t\}) = (3, t+1)$.
For $t = k+1$,
$r(v_2|\{v_1, v_t\}) = r(u_n|\{v_1, v_t\}) = (2, k)$, a contradiction.

(3). One vertex Vertex belong to$\{u_i\} \subset V(S_n^*)$, for $i = 1, 2, ..., n$, and other vertex belong to $\{v_i\} \subset V(S_n^{''})$, for $i = 1, 2, ..., n+1$. Without loss of generality, we suppose that one resolving vertex is u_1, and the other is v_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k$, we have,
$r(v_{n+1}|\{u_1, v_t\}) = r(v_n|\{u_1, v_t\}) = (2, t+1)$.
For $t = k+1$,
$r(v_2|\{u_1, v_t\}) = r(u_n|\{u_1, v_t\}) = (1, k)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(S_n^*)$. Thus, $dim(S_n^*) = 3$.

□

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