Domination in the directed zero-divisor graph of ring of matrices

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Abstract

Let R be a noncommutative ring with identity and $Z(R)^*$ be the non-zero zero-divisors of R. The directed zero-divisor graph $\Gamma(R)$ of R is a directed graph with vertex set $Z(R)^*$ and for distinct vertices x and y of $Z(R)^*$, there is a directed edge from x to y if and only if $xy = 0$ in R. S.P. Redmond has proved that for a finite commutative ring R, if $\Gamma(R)$ is not a star graph, then the domination number of the zero-divisor graph $\Gamma(R)$ equals the number of distinct maximal ideals of R. In this paper, we prove that such a result is true for the noncommutative ring $M_2(\mathbb{F})$, where $\mathbb{F}$ is a finite field. Using this we obtain a class of graphs for which all six fundamental domination parameters are equal.

Keywords: directed zero-divisor graph, orbit, regular action, nilpotent, domination number, perfect domination number.

1 Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to rings, groups and semigroups. D. F. Anderson and P. S. Livingston[2] introduced the zero-divisor graph and studied the interplay between the ring-theoretic properties of a commutative ring R and the graph-theory properties of its zero-divisor graph $\Gamma(R)$. The concept of the zero-divisor graph has been extended to non-commutative rings by S.P. Redmond[13]. Throughout this paper R denotes a non-commutative ring and $Z(R)^*$ be its set of non-zero left and right zero-divisors. The directed zero-divisor graph $\Gamma(R)$ of R is a directed graph with vertex set $Z(R)^*$ and two distinct vertices x and y of $Z(R)^*$ are joined by a directed edge from x to y if and only if $xy = 0$ in R. Also, corresponding to R, there is an undirected graph $\overline{\Gamma}(R)$ [13] with the vertex set $Z(R)^*$ in which two distinct vertices x and y are adjacent if and only if either $xy = 0$ or $yx = 0$. For a commutative ring R, S.P. Redmond[14] proved that the domination number of $\overline{\Gamma}(R) = \Gamma(R)$ is equal to the number of distinct maximal ideals in R. In this paper, in Section 2, we prove that such a result is true for the noncommutative ring $M_2(\mathbb{F})$, where $\mathbb{F}$ is a finite field. Using this we obtain a class of graphs for which all the six fundamental domination parameters are equal. In Section 3, we characterize independent and efficient dominating sets in $\Gamma(M_2(\mathbb{F}))$.

For a non-commutative ring R with identity, $Z(R)^*$ denotes the set of all non-zero left and right zero-divisors of R, G denotes the group of all units of R and X is the set of all nonzero non-units of R. The group action on X by G given by $(g, x) \longrightarrow gx$ (resp. $(g, x) \longrightarrow xg^{-1}$) from $G \times X$ to X is called the left (resp. right) regular action. If $\phi : G \times X \longrightarrow X$ is the left(resp. right) regular action, then for each $x \in X$, $o_\ell(x) = \{\phi(g, x) = gx : g \in G\}$ (resp. $o_r(x) = \{\phi(g, x) = xg^{-1} : g \in G\}$) is called left(resp. right) orbit of x. Note that if R is a finite ring, then $Z(R)^* = X$. Recall that, for $x \in Z(R)^*$, the set $ann_\ell(x) = \{y \in Z(R)^* : yx = 0\}$(resp. $ann_r(x) = \{y \in Z(R)^* : xy = 0\}$) is called the left(resp. right) annihilator of x. For basic definitions on rings, one may refer [4].

Let $D = (V, A)$ be a digraph(directed graph) with vertex set V and arc set A. The in-degree and out-degree of a vertex v are respectively denoted by $id(v)$ and $od(v)$. The minimum degree $\delta(D)$ of a digraph D is defined as the minimum of all in-degrees and out-degrees of vertices in D. For a subset S of vertices of D, the the out-neighborhood $N^+(S)$ of S consists of all those vertices w in $D - S$ such that (v, w) is an arc of D for some $v \in S$. The in-neighborhood $N^-(S)$ consists of all those vertices $u \in D - S$ such that (u, v) is an arc of D for some $v \in S$. Also let $N^+[S] = N^+(S) \cup \{S\}$ and $N^-[S] = N^-(S) \cup \{S\}$. For basic definitions on graphs, one may refer [6].

For a digraph $D = (V, A)$, a subset S of V is called an *out-dominating set* of D if for every $v \in V - S$, there exists $u \in S$ such that $(u, v) \in A$. The out-dominating set of a digraph D is commonly called as *dominating set* of D. A subset S of V is called an *in-dominating set* of D if for every $v \in V - S$, there exists $u \in S$ such that $(v, u) \in A$. A dominating set S of V is called an *independent* if the sub digraph induced by S has no arcs. A dominating set S of V is called a *total dominating set* if the induced subdigraph $< S >$ has no isolated vertices. The underlying graph of a digraph D is obtained from D by removing all directions from the arcs of D and replacing any resulting pair of parallel edges by a single edge. A digraph D is *weakly connected* if the underlying graph of D is connected. A dominating set S of V is called a *weakly connected dominating* set of D if the induced sub digraph $< S >$ is weakly connected. A subset S of V is called an *irredundant set* of D if every $v \in S$ has a private out neighbour. The *out-domination number*(resp. *upper out-domination number*) of a digraph D, denoted by $\gamma^+(D)$(resp. $\Gamma^+(D)$), is the minimum (resp. maximum) cardinality of a out-dominating set of D. There are so many domination parameters in the literature and one can refer [9] for more details and undefined terms. Some of the vital results used in this paper are listed below for ready reference. In fact Theorem 1.1 is part of the result due S.P. Redmond[14, Corollary 5.2].

Theorem 1.1. [14, Corollary 5.2] *Let R be a finite commutative ring with identity that is not a domain. If $\Gamma(R)$ is not a star graph, then the domination number of $\Gamma(R)$ equals the number of distinct maximal ideals of R.*

Lemma 1.2. [8, Theorem 2.4 and Lemma 2.7] *Let R be 2×2 matrices over a finite field $\mathbb{F}$. Then G is half-transitive on X by the left (resp. right) regular action and $|o_\ell(a)| = |o_r(a)| = |\mathbb{F}|^2 - 1$ for all $a \in X$.*

Lemma 1.3. [11, Lemma 2.1] *Let $R = M_2(\mathbb{F})$ where $\mathbb{F}$ is a finite field. Then the number of orbits under the left (resp. right) regular action on X by G is $|\mathbb{F}| + 1$.*

Remark 1.4. [11, Remark 1] Let $R = M_2(\mathbb{F})$ where $\mathbb{F}$ is a finite field. Then the number of non-zero nilpotent matrices in R is $|\mathbb{F}|^2 - 1$, $X = \bigcup_{i=1}^{|\mathbb{F}|+1} o_r(x_i) = \bigcup_{i=1}^{|\mathbb{F}|+1} o_\ell(x_i)$ and $o_r(x_i) \cap o_r(x_j) = \emptyset$ (resp. $o_\ell(x_i) \cap o_\ell(x_j) = \emptyset$) for $i \neq j$ where each x_i is a non-zero nilpotent element in R.

Theorem 1.5. [11, Theorem 2.3 and Theorem 2.5] *Let $R = M_2(\mathbb{F})$ where $\mathbb{F}$ is a finite field and N be the set of all non-zero nilpotents in R. Then under the left (resp. right) regular action on X by G, we have the following:*

(i) $|o_\ell(x)) \cap N| = |\mathbb{F}| - 1$

(ii) $o_\ell(x) \cap N = o_r(x) \cap N = o_\ell(x) \cap o_r(x)$

(iii) $|o_\ell(x) \cap o_r(y)| = |\mathbb{F}| - 1$ *for each* $x, y \in X$.

Theorem 1.6. [11, Theorem 3.5] *Let $R = M_2(\mathbb{F})$ where $\mathbb{F}$ is a finite field and $x \in X$. Then $ann_\ell^*(x) = o_\ell(y)$ for all $y \in ann_\ell^*(x)$ (and $ann_r^*(x) = o_r(z)$ for all $z \in ann_r^*(x)$).*

Lemma 1.7. [1, Lemma 14] *Let $\mathbb{F}$ be a finite field and $n \geq 2$. Suppose $M \in M_n(\mathbb{F})$ is a non-zero matrix and $rank(M) = k < n$. Then the in-degree and out-degree of M in $\Gamma(M_n(\mathbb{F}))$ are $|\mathbb{F}|^{n(n-k)} - \epsilon$ and the degree of M in $\overline{\Gamma}(M_n(\mathbb{F}))$ is equal to $2|\mathbb{F}|^{n(n-k)} - |\mathbb{F}|^{(n-k)^2} - \epsilon$, where $\epsilon = 1$, unless $M^2 = 0$ and in this case $\epsilon = 2$.*

Theorem 1.8. [12, Lemma 4.2(i)] *Let $R = M_n(\mathbb{F}_q)$ where $\mathbb{F}_q$ is a finite field with q elements. Then $|Max_\ell(R)| = |Max_r(R)| = \frac{q^n-1}{q-1}$ where $Max_\ell(R)$ (resp. $Max_r(R)$) is the set of all maximal left (resp. right) ideals in R.*

Theorem 1.9. [9] *For any digraph D, $ir(D) \leq \gamma^+(D) \leq \gamma_i(D) \leq \beta(D) \leq \Gamma^+(D) \leq IR(D)$.*

2 Domination in $\Gamma(M_2(\mathbb{F}))$

In this section, we obtain the values of certain domination parameters for the directed zero-divisor graph D on $M_2(\mathbb{F})$, where $|\mathbb{F}| = p^m$ and p is a prime number and $m \geq 1$. In view of this, we obtain a class graphs for which all parameters in the fundamental chain given in Theorem 1.9 are equal.

Proposition 2.1. *Let $\mathbb{F}$ be a finite field, $R = M_2(\mathbb{F})$ and $x \in Z^*(R)$. For each $a \in o_\ell(x)$ (resp. $a' \in o_r(x')$), $o_\ell(a) = o_\ell(x)$ and $ann_r^*(a) = ann_r^*(x)$ (resp. $o_r(a') = o_r(x')$ and $ann_\ell^*(a') = ann_\ell^*(x')$).*

Proof. Let $a \in o_\ell(x)$. Then $a = gx$ for some unit $g \in R$, $x = g^{-1}a$ and so $x \in o_\ell(a)$. Thus $o_\ell(x) \subseteq o_\ell(a)$. By Lemma 1.2, $|o_\ell(x)| = |o_\ell(a)| = |\mathbb{F}|^2 - 1$ and so $o_\ell(a) = o_\ell(x)$.

Let $y \in ann_r^*(x)$. Then $xy = 0$, $ay = gxy = 0$ and so $y \in ann_r^*(a)$. Thus $ann_r^*(x) \subseteq ann_r^*(a)$. If $z \in ann_r^*(a)$, then $az = 0$. Since $x = g^{-1}a$, $xz = 0$ and so $ann_r^*(a) \subseteq ann_r^*(x)$. Hence $ann_r^*(a) = ann_r^*(x)$. □

Recall that a set of vertices S of V is said to be an irredundant set of D if every vertex $v \in S$ has at least one private out neighbour. The minimum cardinality of a maximal irredundant set in D is called the irredundance number and is denoted by $ir(D)$. The maximum cardinality of a maximal irredundant set in D is called the upper irredundance number and is denoted by $IR(D)$[9].

Theorem 2.2. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime and $m \geq 1$. Let $R = M_2(\mathbb{F})$ and $D = \Gamma(M_2(\mathbb{F}))$. Then $ir(D) = IR(D) = p^m + 1$.*

Proof. Note that $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_\ell(x_i)$ where each x_i is a non-zero nilpotent element in R for $1 \leq i \leq p^m + 1 = t$(say). Let $a_i \in o_\ell(x_i)$ for $1 \leq i \leq t$ and $\Omega = \{a_1, a_2, \ldots, a_t\}$. Clearly $ann_r(a_i) \cap ann_r(a_j)$ for some $i \neq j$ is empty by the Remark 1.4 and Theorem 1.6 and so every element in Ω has a private out neighbor. Hence Ω is an irredundant set of D and so $ir(D) \leq t = p^m + 1$.

Suppose Ω' is an irredundant set of D with $|\Omega'| > p^m+1$. By Lemma 1.3, the number of orbits under the left(resp. right) regular action on X by G is $p^m + 1$. From this Ω' contains at least two elements from any one of the orbit of $Z(R)^*$. Let $y_1, y_2 \in \Omega'$ with $y_1, y_2 \in o_\ell(a_i)$ for some i. Then by Proposition 2.1, $ann_r(y_1) = ann_r(y_2) = ann_r(a_i)$ and so both y_1 and y_2 have no private out neighbors. Thus, for every subset of $Z(R)^*$ with cardinality greater than $p^m + 1$ is not an irredundant set and so Ω is both minimum and maximum cardinality of a maximal irredundant set of D. Hence $ir(D) = IR(D) = |\Omega| = p^m + 1$. □

Remark 2.3. Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime and $m \geq 1$. In view of Theorems 1.9 and 2.2, $ir(D) = \gamma^+(D) = \gamma_i(D) = \beta(D) = \Gamma^+(D) = IR(D) = p^m + 1$.

Theorem 2.4. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime and $m \geq 1$. Let $R = M_2(\mathbb{F})$, $D = \Gamma(M_2(\mathbb{F}))$ and $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_\ell(x_i)$ where each x_i is a non-zero nilpotent element in R. Then Ω is an ir-set of D if and only if Ω contains exactly one element in $o_\ell(x_i)$ for each i, $1 \leq i \leq p^m + 1$.*

Proof. Suppose Ω contains exactly one element from $o_\ell(x_i)$ for each i, $1 \leq i \leq p^m+1\}$. Then by Theorem 2.2, Ω is an ir-set of D. Conversely, suppose Ω is an ir-set of D. Then $|\Omega| = p^m + 1$. Suppose $o_\ell(x_i) \cap \Omega = \emptyset$ for some i. Then Ω contains at least two vertices a, b from $o_\ell(x_j)$ for some $j \neq i$. By Proposition 2.1, $ann_r(a) = ann_r(b) = ann_r(x_j)$ and both a and b have no private out neighbors, a contradiction. □

Corollary 2.5. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime and $m \geq 1$. Let $R = M_2(\mathbb{F})$, $D = \Gamma(M_2(\mathbb{F}))$ and $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_\ell(x_i)$ where each x_i is a non-zero nilpotent element in R. Then Ω is a γ^+-set of D if and only if Ω contains exactly one element in $o_\ell(x_i)$ for each i, $1 \leq i \leq p^m + 1$.*

One can prove the following Lemma in analogous to the above.

Lemma 2.6. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime and $m \geq 1$. Let $R = M_2(\mathbb{F})$, $D = \Gamma(M_2(\mathbb{F}))$ and $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_r(x_i)$ where each x_i is a non-zero nilpotent element in R. Then Ω is a γ^--set of D if and only if Ω contains exactly one element in $o_r(x_i)$ for each i, $1 \leq i \leq p^m + 1$.*

Remark 2.7. Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$, where p is a prime number and $m \geq 1$. Let $D = \Gamma(M_2(\mathbb{F}))$. In view of Theorem 1.8 and Remark 2.3, the number of distinct maximal left ideals of $M_2(\mathbb{F})$ is equal to the out-domination number of D and hence D is excellent.

Proposition 2.8. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime and $m \geq 1$. Let $R = M_2(\mathbb{F})$ and $D = \Gamma(M_2(\mathbb{F}))$. Then $\gamma_t(D) = \gamma_{wc}(D) = \gamma_o(D) = p^m + 1$.*

Proof. Take the partition of $Z^*(R)$ as $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_\ell(x_i)$, where each x_i is a non-zero nilpotent element in R. Let $y_i \in o_\ell(x_i)$ for some i. By Theorems 1.5 and 1.6, $|ann_r^*(y_i) \cap o_\ell(x_j)| = |\mathbb{F}| - 1$ for all $j \neq i$ and so there exists $y_j \in o_\ell(x_j)$ such that $(y_i, y_j) \in A$ for all $j \neq i$. By Corollary 2.5, $\Omega = \{y_1, y_2, \ldots, y_{p^m+1}\}$ is a γ^+-set of D and the subdigraph induced by Ω contains no isolated vertices and the underlying graph is connected. Thus Ω is a total as well as weakly connected dominating set of D and so $\gamma_t(D) = \gamma_{wc}(D) = p^m + 1$.

Let $a_i \in o_\ell(x_i) \cap ann_r(x_j)$ and $a_j \in ann_r(x_i) \cap o_\ell(x_j)$ for $i \neq j$. Then (a_i, a_j) and (a_j, a_i) are arcs in $\Gamma(M_2(\mathbb{F}))$. By Theorems 1.5 and 1.6, there exists $a_k \in o_\ell(x_k)$ such that (a_j, a_k) is an arc in $\Gamma(M_2(\mathbb{F}))$ for each k, $k \neq i \neq j$. By Corollary 2.5, $\Omega = \{a_1, \ldots, a_i, \ldots, a_j, \ldots, a_{p^m+1}\}$ is a γ^+-set of D. Also the subdigraph induced by Ω is connected and hence Ω is an open dominating set of D. Thus $\gamma_o(D) = p^m + 1$. □

Proposition 2.9. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime and $m \geq 1$. Let $R = M_2(\mathbb{F})$ and $D = \Gamma(M_2(\mathbb{F}))$. If Ω is an open dominating set of D, then Ω contains no nilpotent elements.*

Proof. Consider the partition $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_\ell(x_i)$, where each x_i is a non-zero nilpotent element in R. Suppose Ω contains a nilpotent element, say, x. Then $x^2 = 0$. Clearly $x \in o_\ell(x_i)$ for some i and by Proposition 2.1, $o_\ell(x) = o_\ell(x_i)$. Let $y \in o_\ell(x)$. Then $y = ux$ for some unit u in R, $yx = 0$ and so $(y, x) \in A$. Thus $y \in ann_\ell^*(x)$ and so $o_\ell(x) \subseteq ann_\ell^*(x)$. Since $|o_\ell(x)| = |ann_\ell^*(x)| = |\mathbb{F}|^2 - 1$ and $ann_\ell^*(x) = o_\ell(x)$, we get that $(a, x) \notin A$ for all $a \in \Omega - \{x\}$. Hence the subdigraph induced by Ω is not connected, a contradiction. □

3 Efficient domination in $\Gamma(M_2(\mathbb{F}))$

In this section, we characterize all independent dominating sets as well as efficient dominating sets of the directed zero-divisor graph $\Gamma(R)$ of $M_2(\mathbb{F})$, where $\mathbb{F}$ is a finite field.

Theorem 3.1. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime and $m \geq 1$. Let $R = M_2(\mathbb{F})$, $D = \Gamma(M_2(\mathbb{F}))$ and Ω is a minimal dominating set of D. Then Ω is independent if and only if $a^2 = 0$ for every $a \in \Omega$.*

Proof. Consider the partition $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_\ell(x_i)$, where each x_i is a non-zero nilpotent element in R. Let Ω be a minimal dominating set in $D = (V, A)$. By Corollary 2.5, $\Omega = \{a_1, \dots, a_{p^m+1} : a_i \in o_\ell(x_i), 1 \leq i \leq p^m+1\}$.

Assume that Ω is an independent dominating set of D. Suppose $a_i^2 \neq 0$ for some i. Without loss of generality one can take that $a_i^2 \neq 0$ for some i and $a_j^2 = 0$ for all $j \neq i$. By Lemma 1.7, $id(a_i) = od(a_i) = p^{2m} - 1$. Since Ω is independent, $a_i \notin ann_r(x_j)$ for all $j \neq i$ and so $a_i \notin \bigcup_{j=1, j\neq i}^{p^m+1} ann_r(x_j)$. Since $id(a_i) > 0$, $a_i \in ann_r(x_i)$, $(x_i, a_i) \in A$ and so $(y, a_i) \in A$ for all $y \in o_\ell(x_i) - \{a_i\}$. Since $|o_\ell(x_i)| = p^{2m} - 1$, $id(a_i) = p^{2m} - 2$, a contradiction.

Conversely, assume that $a^2 = 0$ for all $a \in \Omega$. As in the proof of Lemma 2.9, $ann_\ell(x_i) = o_\ell(x_i)$ for all i and so $x_i \notin ann_r(x_j)$ for all $i \neq j$. Thus the subdigraph induced by Ω has no arcs, and so Ω is an independent dominating set of D. □

Theorem 3.2. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime number and $m \geq 1$. Let $R = M_2(\mathbb{F})$ and $D = \Gamma(M_2(\mathbb{F}))$. Then $d^+(D) = d^-(D) = d_t(D) = p^{2m} - 1$ and hence D is domatically full.*

Proof. Consider the partition $Z(R)^* = \bigcup_{i=0}^{p^m+1} o_\ell(x_i)$, where each x_i is a non-zero nilpotent element in R. Let $o_\ell(x_i) = \{a_{i1}, a_{i2}, \dots, a_{i(p^{2m}-1)}\}$ for all i, $1 \leq i \leq p^m + 1$ and $V_j = \{a_{1j}, a_{2j}, \dots, a_{(p^m+1)j}\}$ for $1 \leq j \leq p^{2m} - 1$. Then $V_t \cap V_k = \emptyset$ for all $k \neq t$. By Corollary 2.5, V_j is a γ^+-set of D for all j and $V(D) = \bigcup_{j=1}^{p^{2m}-1} V_j$. Hence $\{V_1, V_2, \dots, V_{p^{2m}-1}\}$ is a domatic partition for D and so $d^+(D) = p^{2m} - 1$. In a similar way, one can $d^-(D) = d_t(D) = p^{2m} - 1$. □

Theorem 3.3. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime number and $m \geq 1$. Let $R = M_2(\mathbb{F})$ and $D = \Gamma(M_2(\mathbb{F}))$. Then $\gamma^*(D) = \gamma_p(D) = \gamma_e(D) = p^m + 1$.*

Proof. Consider the partition $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_\ell(x_i) = \bigcup_{i=1}^{p^m+1} ann_r^*(x_i) = \bigcup_{i=1}^{p^m+1} ann_\ell^*(x_i)$, where each x_i is non-zero nilpotent element in R. Let $\Omega = \{x_1, x_2, \ldots, x_{p^m+1}\}$. By Corollary 2.5, Ω is a γ^+-set of D. Note that $ann_r^*(x_i) \cap ann_r^*(x_j) = \emptyset$ and $ann_\ell^*(x_i) \cap ann_\ell^*(x_j) = \emptyset$ for all $i \neq j$. By Lemma 1.7, $od(x_i) = p^{2m} - 2$ and $id(x_i) = p^{2m} - 2$ for all i, $1 \leq i \leq p^{2m}+1$. Hence $|N^+[\Omega]| = |N^-[\Omega]| = (p^m+1)(p^{2m}-2)+(p^m+1) = (p^m+1)(p^{2m}-1) = |V(D)|$. From this Ω is both out-dominating and in-dominating set of D. Also Ω is a twin dominating set of D and so $\gamma^*(D) = p^{2m}+1$. By Theorem 3.1, Ω is an independent dominating set of D. Also for each $y \in V(D) - \Omega$, there exists a unique vertex $x_i \in \Omega$ such that $(x_i, y) \in A$, where A is an arc set of D. Hence Ω is perfect and efficient dominating set of D and so $\gamma_p(D) = \gamma_e(D) = p^{2m} + 1$. □

From Theorems 3.1 and 3.3, we state a characterization for an efficient dominating set in D.

Corollary 3.4. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime number and $m \geq 1$. Let $R = M_2(\mathbb{F})$ and $D = \Gamma(M_2(\mathbb{F}))$. If Ω is a minimal dominating set of D, then Ω is perfect (and so efficient) if and only if $a^2 = 0$ for all $a \in \Omega$.*

The following theorem provides the number of efficient dominating sets in D.

Corollary 3.5. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^m$ where p is a prime number and $m \geq 1$. Let $R = M_2(\mathbb{F})$ and $D = \Gamma(M_2(\mathbb{F}))$. Then the number of disjoint efficient dominating sets in D is $p^m - 1$.*

Proof. Consider the partition $Z(R)^* = \bigcup_{i=1}^{p^m+1} o_l(x_i)$, where each x_i is non-zero nilpotent element in R. Let N be the set of all nilpotents in R. By Theorem 1.5, $|o_\ell(x_i) \cap N(p)| = p^m - 1$ for all i, $o_\ell(x_i) \cap N = \{a_{i1}, \ldots, a_{i(p^m-1)}\}$ for all i, $1 \leq i \leq p^m + 1$. Let $V_k = \{a_{1k}, \ldots, a_{(p^m+1)k}\}$ for all k, $1 \leq k \leq p^m - 1$. Then by Corollary 3.4, V_k is an efficient dominating set of D for all k. Hence the number of disjoint efficient dominating sets in D is $p^m - 1$. □

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