A NEW LOOK AT SEARS' ${}_3\phi_2$ TRANSFORMATION FORMULA

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ABSTRACT. In this paper, we give a new look at Sears' ${}_3\phi_2$ transformation formula via a discrete random variable. This interpretation may provide a method to calculate ${}_3\phi_2$ by Monte Carlo experiments.

1. INTRODUCTION

The following is Sears' ${}_3\phi_2$ transformation formula

$$ {}_3\phi_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; q, \frac{b_1 b_2}{a_1 a_2 a_3}\right) = \frac{(b_2/a_3, b_1 b_2/a_1 a_2; q)_\infty}{(b_2, b_1 b_2/a_1 a_2 a_3; q)_\infty} \tag{1.1} $$

$$ \times {}_3\phi_2\left(\begin{matrix} b_1/a_1, b_1/a_2, a_3 \\ b_1, b_1 b_2/a_1 a_2 \end{matrix}; q, \frac{b_2}{a_3}\right), \quad \left|\frac{b_1 b_2}{a_1 a_2 a_3}\right| < 1, \left|\frac{b_2}{a_3}\right| < 1. $$

Sears' ${}_3\phi_2$ transformation is an important formula in basic hypergeometric functions theory. It has been used by Andrews[3, 14] in proving many of Ramanujan's identities for partial theta functions.

The probabilistic method is also a useful tool in the study of basic hypergeometric functions. There are some works available in the literature. For example, K. W. J. Kadell [11] gave a probabilistic proof of Ramanujan's ${}_1\psi_1$ sum based on the order statistics. J. Fulman [8] presented a probabilistic proof of Rogers-Ramanujan identity using Markov chain on the non-negative integers. R. Chapman [5] extended J. Fulman's methods to prove the Andrews-Gordon identity. In particular, the present author [18] established the following new discrete probability distribution $W(x;q)$:

$$ P(\xi = x^n q^k) = \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty}, \tag{1.2} $$

where $x < 0$; $0 < q < 1$; $n = 0, 1$; $k = 0, 1, 2, \cdots$, and gave some applications of this distribution in q-series. q-type distributions play an important

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role in applications. Various q-type distributions have appeared in the physics literature in the recent years [6, 7, 12, 13, 15, 20].

In this paper, we ask a question: is there any probabilistic interpretation for Sears' ${}_3\phi_2$ transformation formula? We try to give an answer.

We first recall some definitions, notation and known results in [4, 9] which will be used in this paper. Throughout the whole paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(1.3)\qquad (a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(1.4)\qquad (a_1, a_2, ..., a_m; q)_n = (a_1; q)_n (a_2; q)_n ... (a_m; q)_n,$$

where n is an integer or ∞. We may extend the definition (1.2) of $(a;q)_n$ to

$$(1.5)\qquad (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},$$

for any complex number α. In particular,

$$(1.6)\qquad (a;q)_{-n} = \frac{(a;q)_\infty}{(aq^{-n};q)_\infty} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n}{(q/a;q)_n} q^{\binom{n}{2}}.$$

Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series, which is defined by

$$(1.7)\qquad {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, ..., a_{r+1} \\ b_1, b_2, ..., b_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_{r+1}; q)_n x^n}{(q, b_1, b_2, ..., b_r; q)_n}.$$

The q-Gauss summation formula

$$(1.8)\qquad {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad |\frac{c}{ab}| < 1.$$

The following is the well known Ramanujan's ${}_1\psi_1$ summation formula

$$(1.9)\qquad \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1.$$

F. H. Jackson defined the q-integral by [10]

$$(1.10)\qquad \int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n) q^n,$$

and

$$(1.11)\qquad \int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t.$$

The q-integral is important in the theory and applications of basic hypergeometric series. The following is the Andrews-Askey integral [2], which can be derived from Ramanujan's ${}_1\psi_1$ summation:

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}, \tag{1.12}$$

provided that no zero factors occur in the denominator of the integrals. Al-Salam and Verma gave an extension of the Andrews-Askey integral, which is called the Al-Salam and Verma [1] q-integral

$$\int_c^d \frac{(qt/c, qt/d, et; q)_\infty}{(at, bt, ft; q)_\infty} d_q t \tag{1.13}$$
$$= \frac{d(1-q)(q, dq/c, c/d, e/a, e/b, e/f; q)_\infty}{(ac, ad, bc, bd, fc, fd; q)_\infty},$$

provided that no zero factors occur in the denominator of the integrals, where $e = abcdf$.

Lebesgue's dominated convergence theorem: Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables, that $X_n \to X$ pointwise almost everywhere as $n \to \infty$, and that $|X_n| \leq Y$ for all n, where random variable Y is integrable. Then X is integrable, and

$$\lim_{n\to\infty} \mathrm{E}X_n = \mathrm{E}X. \tag{1.14}$$

2. A NEW LOOK AT SEARS' ${}_3\phi_2$ TRANSFORMATION

In this section, we use the following lemma to give a new look at Sears' ${}_3\phi_2$ transformation formula via a discrete random variable.

Lemma 1. *Let ξ denote random variable having distribution $W(x;q), -1 < x < 0$, then*

$$E\left\{\frac{(abx\xi; q)_\infty}{(a\xi, b\xi, c\xi; q)_\infty}\right\} = \frac{(abx, bcx; q)_\infty}{(a, b, bx, c, cx; q)_\infty} {}_3\phi_2\left(\begin{matrix} b, bx, c \\ abx, bcx \end{matrix}; q, ax\right), \tag{2.1}$$

provided that $\max\{|a|, |b|, |c|\} < 1$,*where $E(X)$ denotes expected value of the random variable X.*

Proof. Considering the following sequence of random variables (on a probability space):

$$\eta_n = \frac{(abcx\xi; q)_\infty}{(a\xi, b\xi, c\xi; q)_\infty} \sum_{k=0}^{n} \frac{(b\xi, c; q)_k}{(q, abcx\xi; q)_k} (ax)^k,$$

where $n = 0, 1, 2, \cdots$.

Since,

$$\begin{aligned}|\eta_n| &= \left|\frac{(abcx\xi;q)_\infty}{(a\xi,b\xi,c\xi;q)_\infty}\sum_{k=0}^{n}\frac{(b\xi,c;q)_k}{(q,abcx\xi;q)_k}(ax)^k\right| \\ &= \left|\sum_{k=0}^{n}\frac{(abcxq^k\xi;q)_\infty}{(a\xi,bq^k\xi,c\xi;q)_\infty}\cdot\frac{(c;q)_k}{(q;q)_k}(ax)^k\right| \\ &\le \frac{(-|abcx|;q)_\infty}{(|a|,|b|,|c|;q)_\infty}\sum_{k=0}^{n}\left|\frac{(c;q)_k}{(q;q)_k}(ax)^k\right|\end{aligned}$$

and the series

$$\sum_{k=0}^{\infty}\frac{(c;q)_k}{(q;q)_k}(ax)^k$$

converges absolutely. Using Lebesgue's dominated convergence theorem gets:

$$\lim_{n\to\infty}\mathrm{E}\eta_n = \mathrm{E}(\lim_{n\to\infty}\eta_n). \tag{2.2}$$

Using the q-Gauss theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\eta_n &= \frac{(abcx\xi;q)_\infty}{(a\xi,b\xi,c\xi;q)_\infty}\sum_{k=0}^{\infty}\frac{(b\xi,c;q)_k}{(q,abcx\xi;q)_k}(ax)^k \\ &= \frac{(abcx\xi;q)_\infty}{(a\xi,b\xi,c\xi;q)_\infty}\cdot\frac{(abx\xi,acx;q)_\infty}{(abcx\xi,ax;q)_\infty} \\ &= \frac{(acx;q)_\infty}{(ax;q)_\infty}\cdot\frac{(abx\xi;q)_\infty}{(a\xi,b\xi,c\xi;q)_\infty}.\end{aligned} \tag{2.3}$$

Hence, we get the right hand side of (2.2):

$$\mathrm{E}(\lim_{n\to\infty}\eta_n) = \frac{(acx;q)_\infty}{(ax;q)_\infty}\mathrm{E}\left\{\frac{(abx\xi;q)_\infty}{(a\xi,b\xi,c\xi;q)_\infty}\right\}. \tag{2.4}$$

In order to get the left hand side of (2.2), Observe that

$$\begin{aligned}\mathrm{E}\eta_n &= \mathrm{E}\left\{\frac{(abcx\xi;q)_\infty}{(a\xi,b\xi,c\xi;q)_\infty}\sum_{k=0}^{n}\frac{(b\xi,c;q)_k}{(q,abcx\xi;q)_k}(ax)^k\right\} \\ &= \sum_{k=0}^{n}\frac{(c;q)_k(ax)^k}{(q;q)_k}\mathrm{E}\left\{\frac{(abcxq^k\xi;q)_\infty}{(a\xi,bq^k\xi,c\xi;q)_\infty}\right\}.\end{aligned} \tag{2.5}$$

Employing the Al-Salam and Verma q-integral (1.1) gives

$$\text{(2.6)}\quad \mathrm{E}\left\{\frac{(abcxq^k\xi;q)_\infty}{(a\xi,bq^k\xi,c\xi;q)_\infty}\right\}$$

$$= \sum_{n=0}^{1}\sum_{m=0}^{\infty}\frac{(-x)^n(x^{n-1}q^{m+1},x^nq^{m+1},abcx^{n+1}q^{k+m};q)_\infty q^m}{(q,q/x,x,ax^nq^m,bx^nq^{k+m},cq^mx^n;q)_\infty}$$

$$= \frac{1}{(1-q)(q,q/x,x;q)_\infty}[(1-q)\sum_{m=0}^{\infty}\frac{(q^{m+1}/x,q^{m+1},abcxq^{k+m};q)_\infty q^m}{(aq^m,bq^{k+m},cq^m;q)_\infty}$$

$$-x(1-q)\sum_{m=0}^{\infty}\frac{(q^{m+1},xq^{m+1},abcx^2q^{k+m};q)_\infty q^m}{(axq^m,bxq^{k+m},cq^mx;q)_\infty}]$$

$$= \frac{1}{(1-q)(q,q/x,x;q)_\infty}\int_x^1\frac{(qt/x,qt,abcxq^kt;q)_\infty}{(at,bq^kt,ct;q)_\infty}d_qt$$

$$= \frac{(abxq^k,acx,bcxq^k;q)_\infty}{(a,ax,bq^k,bxq^k,c,cx;q)_\infty}.$$

Substituting (2.6) into (2.5) gets

$$\text{(2.7)}\quad E\eta_n = \sum_{k=0}^{n}\frac{(c;q)_k(ax)^k}{(q;q)_k}\mathrm{E}\left\{\frac{(abcxq^k\xi;q)_\infty}{(a\xi,bq^k\xi,c\xi;q)_\infty}\right\}$$

$$= \frac{(abx,acx,bcx;q)_\infty}{(a,ax,b,bx,c,cx;q)_\infty}\sum_{k=0}^{n}\frac{(b,bx,c;q)_k}{(q,abx,bcx;q)_k}(ax)^k.$$

Hence, we get the left hand side of (2.2).

$$\text{(2.8)}\quad \lim_{n\to\infty}E\eta_n = \frac{(abx,acx,bcx;q)_\infty}{(a,ax,b,bx,c,cx;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}b,bx,c\\abx,bcx\end{matrix};q,ax\right).$$

Substituting (2.4) and (2.8) into (2.2) gets (2.1). □

Formula (2.1) gives a new look at Sears' ${}_3\phi_2$ transformation formula. It is obvious that the left-hands of (2.1) is symmetric in a,b, and so is the right-hand sides. Interchanging a,b on the right-hand side of (2.1), we have Sears' ${}_3\phi_2$ transformation formula. In fact, interchanging a and b on the left-hand side of (2.1), we obtain

$$\text{(2.9)}\quad \frac{(abx,bcx;q)_\infty}{(a,b,bx,c,cx;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}b,bx,c\\abx,bcx\end{matrix};q,ax\right)$$

$$= \frac{(abx,acx;q)_\infty}{(a,b,ax,c,cx;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,ax,c\\abx,acx\end{matrix};q,bx\right).$$

Hence,

$$\text{(2.10)}\quad {}_3\phi_2\left(\begin{matrix}b,bx,c\\abx,bcx\end{matrix};q,ax\right) = \frac{(acx,bx;q)_\infty}{(ax,bcx;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,ax,c\\abx,acx\end{matrix};q,bx\right),$$

which is equivalent to Sears' ${}_3\phi_2$ transformation formula. This tells us Sears' ${}_3\phi_2$ transformation formula is nothing but only a symmetry of expectation formula(2.1). So (2.1) gives a probabilistic interpretation for Sears' ${}_3\phi_2$ transformation formula. There is a similar formula in [19]. Unfortunately that formula can not give the above probabilistic interpretation for Sears' ${}_3\phi_2$ transformation formula.

Monte Carlo methods (or Monte Carlo experiments) are a class of computational algorithms that rely on repeated random sampling to compute their results. Monte Carlo methods are often used in computer simulations of physical and mathematical systems. These methods are most suited to calculation by a computer. This method is also used to complement theoretical derivations. We want to point out the interpretation set up a relationship between expected value of a random variable and ${}_3\phi_2$ and it may provide a way to calculate ${}_3\phi_2$ by Monte Carlo method.

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