Decompositions of complete graphs with holes of the same size into the graph-pair of order 4

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Abstract

We give necessary and sufficient conditions for the decomposition of the complete graphs with multiple holes, $K_n \setminus hK_v$, into the graph-pair of order 4.

1 Introduction

A graph-pair of order t consists of two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$. In [4], Abueida and Daven showed that there exists a $\{K_m, K_{1,m}\}$-decomposition of λK_n for all $m \geq 3$, $\lambda \geq 1$, and $n \equiv 0, 1 \pmod{m}$. For graph-pairs of order 4 and 5, G and H, Abueida, Daven, and Roblee (in [3, 5]) determined the the values of n for which there exists $\{G, H\}$-decomposition of λK_n for $\lambda \geq 1$. In [6], Abueida and O'Neil showed that there exists a $\{C_m, K_{1,m-1}\}$-decomposition of λK_n for $m = 3$, 4, and 5 and $n \geq m + 1$.

For positive integers h, n and $v \geq 2$ where $hv < n$, the complete graph of order n with h holes of size v is formed by removing all the edges of h complete subgraphs K_v from K_n, while retaining all vertices (i.e. $K_n \setminus hK_v$). The non-isomorphic graphs with no isolated vertices, G and H on t vertices form a *graph-pair* of order t if $G \cup H \cong K_t$. The graphs $G \cong C_4$ and

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$H \cong 2K_2$ are the only graph-pair of order 4 . A (G, H)-decomposition of K_n is a partitioning of the edges of K_n into copies of G and H with at least one copy of G and at least one copy of H. If a decomposition does not exist, we then attempt the decomposition of the graph as closely as possible. With a maximum *packing*, one can obtain a leave L (the set of unused edges) with as few edges as possible. For a minimum *covering*, one can obtain a padding P (the set of edges used more than once) that has as few edges as possible.

Recently, Shyu [9] gave decompositions of the complete graph K_n into p copies of P_{k+1} and q copies of S_{k+1} when $n \geq 4k$, $k(p+q) = \binom{n}{2}$, and either k is even and $p \geq \frac{k}{2}$, or k is odd and $p \geq k$. In [10], Shyu investigated the decomposition of K_n into paths and cycles. He obtained necessary and sufficient condition for decomposing K_n into p copies of P_5 and q copies of C_4 for all possible values of $p \geq 0$ and $q \geq 0$.

Let $V(K_n) = \mathbb{Z}_n$ and $V(K_{s,t}) = \mathbb{Z}_{s+t}$. If $S \subseteq \mathbb{Z}_n$, then $K_n[S]$ is the subgraph of K_n induced by the vertices in S. For a disjoint sets S and T, if $S \cup T \subseteq \mathbb{Z}_n$, then $K_n[S;T]$ is the bipartite subgraph of K_n on the vertices $S \cup T$. When $s = |S|$ and $t = |T|$, it is clear that $K_n[S] \cong K_s$ and $K_n[S;T] \cong K_{s,t}$. Define $[a,b] = \{t \in \mathbb{Z}_n \mid a \leq t \leq b\}$. If $S = [a,b]$ and $T = [c,d]$, then we write $K_n[a,b]$ and $K_n[a,b;c,d]$ rather than $K_n[S]$ and $K_n[S;T]$. We say that $C_4 \cong \{a,b,c,d\}$ to denote the cycle on 4 vertices a, b, c and d in order.

2 Preliminaries

One of the nicest results in cycle decomposition of graphs is due to Alspach and Gavlas in [8]. The results are summarized in the following theorem.

Theorem 2.1. *[8] For odd integers m and n with $3 \leq m \leq n$, there exists a C_m-decomposition of K_n if and only if $m | n(n-1)/2$.*

Another nice result is due to Sotteau in [11] which is summarized in the following theorem.

Theorem 2.2. *[11] $K_{2a,2b}$ is C_m-decomposable if and only if $m \leq 4a$, $m \leq 4b$, and $m | 4ab$.*

One can deduce the following corollary directly from Sotteau's theorem.

Corollary 2.3. *For nonzero even integers a and b, $K_{a,b}$ is C_4-decomposable.*

The authors in [3] settled the problem of multidesigns of the complete graph K_n into the graph-pair of order 4. The following theorem summarizes their results.

Theorem 2.4. *[3]*

1. *There is a $(C_4, 2K_2)$-decomposition of K_n if and only if $n \equiv 0, 1$ mod 4 ($n \geq 4$, $n \neq 5$).*
2. *There is a $(C_4, 2K_2)$-maximum packing (minimum covering) of K_n with L (and P) $\cong K_2$ if and only if $n \equiv 2, 3$ mod 4.*

In [1], Abueida solved the proposed problem when there is exactly one hole. The results are summarized in the following theorem.

Theorem 2.5. *Suppose n, p, v are integers with $p = n - v$ and $v \geq 2$. The following are true:*

1. *There is a $(C_4, 2K_2)$-decomposition of $K_n \setminus K_v$ if and only if:*
   (a) $p \equiv 0$ mod 4;
   (b) $p \equiv 1$ mod 4 and $v \equiv 0$ mod 2; or
   (c) $p \equiv 3$ mod 4 and $v \equiv 1$ mod 2.
2. *There is a $(C_4, 2K_2)$-maximum packing (minimum covering) of K_n with a leave L (padding P) $\cong K_2$ if and only if:*
   (a) $p \equiv 1$ mod 4 and $v \equiv 1$ mod 2;
   (b) $p \equiv 2$ mod 4; or
   (c) $p \equiv 3$ mod 4 and $v \equiv 0$ mod 2.

Hence, throughout this paper, we can assume that $h \geq 2$. We make use of the following lemma in the proof of the main result.

Lemma 2.6. 1. *There is a $2K_2$-decomposition of $K_{2,3}$.*

2. *There is a $(C_4, 2K_2)$-decomposition of $K_{3,4}$.*

Proof. Let $K_{2,3}$ have parts A labeled $0, 1$ and B labeled $2, 3, 4$. Then $\{0, 2 : 1, 3\}, \{0, 3; 1, 4\}, \{0, 4; 1, 2\}$ is a $2K_2$-decomposition of $K_{2,3}$.

Let $K_{3,4}$ have parts A labeled $0, 1, 2$ and B labeled $3, 4, 5, 6$. Then the following is a $(C_4, 2K_2)$-decomposition of $K_{3,4}$:

$C_4 \cong \{0, 3, 1, 4\}, \{1, 5, 2, 6\}$; and

$2K_2 \cong \{0, 5; 2, 3\}, \{0, 6; 2, 4\}$. □

Lemma 2.7 is a result of combining Corollary 2.3 and Lemma 2.6.

Lemma 2.7. 1. *There is a* $(C_4, 2K_2)$*-decomposition of* $K_{even,even}$ *and* $K_{even,odd}$.

2. *There is a* $(C_4, 2K_2)$*-packing of* $K_{odd,odd}$ *with a leave* $L \cong K_2$.

3 The main result

Now, we now present our main result.

Theorem 3.1. *For integer* $v \geq 2$*, there is a* $(C_4, 2K_2)$ *decomposition of the* $K_n \setminus hK_v$ *if and only if either:*

1. $n \equiv 0$ *or 1 (mod 4), and either* h *is even or* $v \equiv 0$ *or 1 (mod 4); or*

2. $n \equiv 2$ *or 3 (mod 4),* h *is odd, and* $v \equiv 2$ *or 3 (mod 4).*

Proof. Note that $|E(K_n \setminus hK_v)| = \binom{n}{2} - h\binom{v}{2}$. Since both of C_4 and $2K_2$ have even number of edges then the existence of $(C_4, 2K_2)$ decomposition of $K_n \setminus hK_v$ requires that either both of the terms $\binom{n}{2}$ and $h\binom{v}{2}$ to be even or both odd. Cases 1 and 2 show the $(C_4, 2K_2)$-decomposition when both $\binom{n}{2}$ and $h\binom{v}{2}$ are even, and Cases 3 and 4 show the $(C_4, 2K_2)$-decomposition when both $\binom{n}{2}$ and $h\binom{v}{2}$ are odd.

Next, we will exhibit the decomposition by splitting the problem into cases based on the parity of n modulo 4. We note that there are three type of edges that are involved in the decomposition. For simplicity, we call the edges between any two vertices in $K_n \setminus hK_v$ type 1 edges, the ones between vertices in different holes type 2 edges, and the ones between vertices in the holes and vertices in $K_n \setminus hK_v$ type 3 edges. (see figure 1)

Case 1: $n \equiv 0 \pmod 4$. If h is even, and v is even (or h is odd and $v \equiv 0 \pmod 4$), then $n - hv \equiv 0 \pmod 4$. Applying Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$, we get a $(C_4, 2K_2)$-decomposition that uses all the edges of type 1. For $1 \leq i \neq j \leq h$, we apply Lemma 2.7 to $K_{v,v}[b_1^i, b_v^i; b_1^j, b_v^j]$ and $K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$, to get a $(C_4, 2K_2)$-decomposition that uses all the type 2 and type 3 edges, respectively. If h is even and v is odd, then $n - hv \equiv 0$ or $2 \pmod 4$. For $1 \leq i \neq j \leq h$, apply Lemma 2.7 to the type 2 edges in $K_{v,v}[b_1^i, b_v^i; b_1^j, b_v^j]$. For each choice of $i \neq j$, the packing will have a leave of a single edge. Without loss of generality, we can assume that the leave is the edge $b_1^i\ b_1^j$. The collection of those leave edges is the complete graph $K_h[b_1^1, \cdots b_1^h]$. Apply Lemma 2.7 to the type 3 edges in

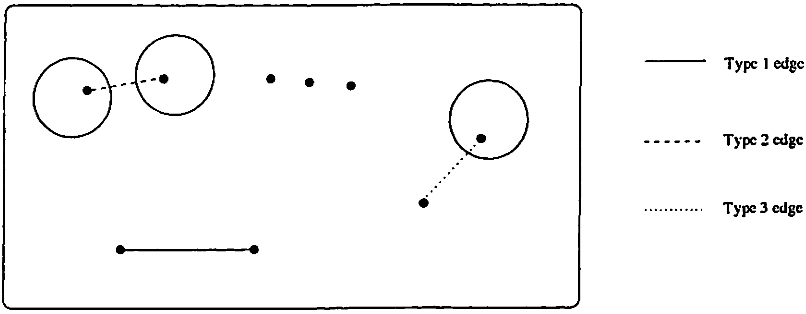

Figure 1: The three types of edges in $K_n \setminus hK_v$.

$K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$ to get a $(C_4, 2K_2)$-decomposition that uses all of the type 3 edges.

If $h \equiv 0 \pmod 4$, then $n - hv \equiv 0 \pmod 4$, then apply Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$ and $K_h[b_1^1, b_1^h]$ to get a $(C_4, 2K_2)$-decomposition that uses all the edges of type 1 and the rest of type 2 edges, respectively. Otherwise, $h \equiv 2 \pmod 4$ and $n - hv \equiv 2 \pmod 4$. Apply Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$ and $K_h[b_1^1, b_1^h]$ to get a $(C_4, 2K_2)$-packing with a leave of a single edge in each case, say $a_1\ a_2$ and $b_1^1\ b_1^2$ respectively. Those two edges form another copy of $2K_2$.

So, we can assume that h is odd and $v \equiv 1 \pmod 4$. For $1 \leq i \neq j \leq h$, apply Lemma 2.7 to the type 2 (type 3 edges) in $K_{v,v}[b_1^i, b_v^i; b_1^j, b_v^j]$ ($K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$) will produce a $(C_4, 2K_2)$-packing with a leave consisting of a single edge, say $b_1^i\ b_1^j$ (say edge $a_1\ b_1^i$). The collection of those leaves is the complete graph $K_h[b_1^1, \cdots, b_1^h]$ (single edge $a_1\ b_1^i$). Apply Theorem 2.4 to the type 1 edges in $K_{n-hv}[a_1, a_{n-hv}]$ to get either a $(C_4, 2K_2)$-decomposition or a $(C_4, 2K_2)$-packing with a leave consisting of a single edge, say $a_1\ a_2$, depending on the parity of $n - hv$ modulo 4.

If $h \equiv 3 \pmod 4$, then $n - hv \equiv 1 \pmod 4$. Apply Theorem 2.4 to the edges in $K_{h+1}[a_1, b_1^1, \cdots, b_1^h]$ to get a $(C_4, 2K_2)$-decomposition since $h + 1 \equiv 0 \pmod 4$. If $h \equiv 1 \pmod 4$, then $n - hv \equiv 3 \pmod 4$. Apply Theorem 2.4 to the edges in $K_{h+1}[a_1, b_1^1, \cdots, b_1^h]$ to get a $(C_4, 2K_2)$-packing with a leave consisting of a single edge, say $b_1^1\ b_1^2$, since $h + 1 \equiv 2 \pmod 4$. In this case we match the edges $b_1^1\ b_1^2$ and $a_1\ a_2$ to get another copy of $2K_2$.

Case 2: $n \equiv 1 \pmod 4$. If both h and v are even (or h is odd and $v \equiv 0 \pmod 4$), then $n - hv \equiv 1 \pmod 4$. Applying Theorem 2.4 on $K_{n-hv}[a_1, a_{n-hv}]$, we get a $(C_4, 2K_2)$-decomposition that uses all the edges of type 1. For $1 \leq i \neq j \leq h$, we apply Lemma 2.7 to $K_{v,v}[b_1^i, b_v^i; b_1^j, b_v^j]$

and $K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$, to get a $(C_4, 2K_2)$-decomposition that uses all the type 2 and type 3 edges.

If h is even and v is odd, then $n-hv \equiv 1$ or 3 (mod 4). For $1 \leq i \neq j \leq h$, apply Lemma 2.7 to the type 2 edges in $K_{v,v}[b_1^i, b_v^i; b_1^j, b_v^j]$. For each choice of $i \neq j$, the decomposition will have a leave of a single edge. Without loss of generality, we can assume that the leave is the edge $b_1^i\ b_1^j$. The collection of those leave edges is the complete graph $K_h[b_1^1, \cdots b_1^h]$. Apply Lemma 2.7 to the type 3 edges in $K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$ to get a $(C_4, 2K_2)$-packing with a leave of a single edge for each choice of i, say the edge $a_1\ b_1^i$.

If $h \equiv 0$ (mod 4) ($h \equiv 2$ (mod 4)), then $n - hv \equiv 1$ (mod 4) ($n - hv \equiv 3$ (mod 4)). Apply Theorem 2.4 to the complete graphs $K_{h+1}[a_1, b_1^1, \cdots, b_1^h]$ and $K_{n-hv}[a_1, a_{n-hv}]$. If $h \equiv 0$ (mod 4), then $n - hv \equiv 1$ (mod 4). So, we get a $(C_4, 2K_2)$-decomposition of the complete graphs. If $h \equiv 2$ (mod 4), then $n - hv \equiv 3$ (mod 4). In this case, we have a $(C_4, 2K_2)$-packing of the complete graphs $K_{h+1}[a_1, b_1^1, \cdots, b_1^h]$ and $K_{n-hv}[a_1, a_{n-hv}]$ with a leave consisting of a single edge each time, say the edges $b_1^1\ b_1^2$ and $a_1\ a_2$. These 2 edges form another copy of $2K_2$.

So, we can assume that h is odd and $v \equiv 1$ (mod 4) for the rest of this case. Apply Lemma 2.7 to the edges in $K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$ and $K_{v,v}[b_1^i, b_v^i; b_1^j, b_v^j]$ to get a $(C_4, 2K_2)$-decomposition and a $(C_4, 2K_2)$-packing with a leave consisting of a single edge, say the edge $b_1^i\ b_1^j$, for each choice of $i \neq j$. The collection of those leave edges is the complete graph $K_h[b_1^1, \cdots b_1^h]$. Apply Theorem 2.4 to the complete graphs $K_h[b_1^1, \cdots, b_1^h]$ and $K_{n-hv}[a_1, a_{n-hv}]$. If $h \equiv 1$ (mod 4); $n - hv \equiv 0$ (mod 4), then there is a $(C_4, 2K_2)$-decomposition of the complete graphs $K_h[b_1^1, \cdots, b_1^h]$ and $K_{n-hv}[a_1, a_{n-hv}]$. If $h \equiv 3$ (mod 4); $n - hv \equiv 2$ (mod 4), then there is a $(C_4, 2K_2)$-packing of $K_h[b_1^1, \cdots, b_1^h]$ and $K_{n-hv}[a_1, a_{n-hv}]$ with leaves consisting of single edges, say $b_1^1\ b_1^2$ and $a_1\ a_2$ respectively. These two edges form a copy of $2K_2$.

Case 3: $n \equiv 2$ (mod 4) and h is odd. If $v \equiv 2$ (mod 4), then $n - hv \equiv 0$ (mod 4). Apply Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$, and Lemma 2.7 to $K_{v,v}[b_1^i, b_v^i; b_j^1, b_v^j]$ and $K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$ to get $(C_4, 2K_2)$-decompositions, for all $i \neq j$.

So, we can assume that $v \equiv 3$ (mod 4). Apply Lemma 2.7 to $K_{v,v}[b_1^i, b_v^i; b_j^1, b_v^j]$ and $K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$ to get a $(C_4, 2K_2)$-packing with a leave consisting of a single edge, say edges $b_1^i\ b_1^j$ and $a_1\ b_1^i$ respectively, for each choice $i \neq j$. The collection of those leave edges is the complete graph $K_{h+1}[a_1, b_1^1, \cdots b_1^h]$. If $h \equiv 3$ (mod 4), then $n - hv \equiv 1$ (mod 4). Apply Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$ and $K_{h+1}[a_1, b_1^1, \cdots b_1^h]$ to get $(C_4, 2K_2)$-decompositions. Otherwise, if $h \equiv 1$ (mod 4), then $n - hv \equiv 3$ (mod 4).

Apply Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$ and $K_{h+1}[a_1, b_1^1, \cdots b_1^h]$ to get a $(C_4, 2K_2)$-packing with leave consisting of a single edge in each case, say $a_1\ a_{n-hv}$ and $b_1^1\ b_1^2$. These 2 edges form a copy of $2K_2$.

Case 4: $n - hv \equiv 3 \pmod 4$ and h is odd. If $v \equiv 2 \pmod 4$, then $n - hv \equiv 1 \pmod 4$. Apply Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$, and Lemma 2.7 to $K_{v,v}[b_1^i, b_v^i; b_j^1, b_v^j]$ and $K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$ to get $(C_4, 2K_2)$-decompositions, for all $1 \leq i \neq j \leq h$.

So, we can assume that $v \equiv 3 \pmod 4$. Apply Lemma 2.7 to $K_{v,v}[b_1^i, b_v^i; b_j^1, b_v^j]$ to get a $(C_4, 2K_2)$-packing with a leave consisting of a single edge, say edges $b_1^i\ b_1^j$. The collection of those leave edges form the complete graph $K_h[b_1^1, \cdots b_1^h]$. Since $v \equiv 3 \pmod 4$ and h is odd, then $n - hv \equiv 0$ or $2 \pmod 4$. Apply Lemma 2.7 to $K_{n-hv,v}[a_1, a_{n-hv}; b_1^i, b_v^i]$ to get $(C_4, 2K_2)$-decomposition for $1 \leq i \leq h$. If $h \equiv 1 \pmod 4$, then $n - hv \equiv 0 \pmod 4$. Apply Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$ and $K_h[b_1^1, \cdots b_1^h]$ to get $(C_4, 2K_2)$-decompositions. Otherwise, if $h \equiv 3 \pmod 4$, then $n - hv \equiv 2 \pmod 4$. Apply Theorem 2.4 to $K_{n-hv}[a_1, a_{n-hv}]$ and $K_h[b_1^1, \cdots b_1^h]$ to get $(C_4, 2K_2)$-packing with leave consisting of a single edge in each case, say $a_1\ a_2$ and $b_1^1\ b_1^2$. These 2 edges form a copy of $2K_2$.

□

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