Paired Domination Problems of Infinite Diamond Lattice

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ABSTRACT

A set S of vertices of a graph G(V,E) is a *dominating set* if every vertex of $V\setminus S$ is adjacent to some vertex in S. A dominating set is said to be *efficient* if every vertex of $V\setminus S$ is dominated by exactly one vertex of S. A *paired-dominating set* is a dominating set whose induced subgraph contains at least one perfect matching. A set S of vertices in G is a *total dominating set* of G if every vertex of V is adjacent to some vertex in S. In this paper we construct a minimum paired dominating set and a minimum total dominating set for infinite diamond lattice. The *total domatic number* of G is the size of a maximum cardinality partition of V into total dominating sets. We also demonstrate that the total domatic number of infinite diamond lattice is 4.

1. Introduction

A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in G is a matching such that every vertex of G is incident with an edge of M. A set S of vertices of a graph G(V,E) is a dominating set if every vertex of $V\setminus S$ is adjacent to some vertex in S. A dominating set is said to be *efficient* if every vertex of $V\setminus S$ is dominated by exactly one vertex of S. Domination in graphs has applications to several fields such as sensor networks [16], facility location problems [9] etc. A *paired-dominating set* is a dominating set whose induced subgraph contains at least one perfect matching. See Figures 10 and 11. Paired domination was introduced by Haynes and Slater [8] and it was motivated by the variant of the area monitoring problem in which each guard has another guard as a backup (i.e., pairs of guards protecting each other). Haynes and Slater noted

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that every graph with no isolated vertices has a paired-dominating set. This problem is studied in [2, 4, 5, 10]. The concept of paired dominating sets has applications in mobile ad hoc wireless networks and has been proposed as a virtual backbone for routing in wireless ad hoc networks [6].

The *paired-domination problem* is to find a minimum paired-dominating set of G. The paired-domination problem is known to be NP-complete. It is NP-complete for bipartite graphs [15]. Panda et al. propose a linear time algorithm to compute a minimum paired-dominating set of a chordal bipartite graph, a well-studied subclass of bipartite graphs [15]. Chellali and Haynes [2] provide sharp upper bounds on the total and paired-domination numbers of trees that improve known bounds for some cases. Lappas et al. [13] provide an $O(n)$-time algorithm for the paired-domination problem on permutation graphs. Chen et al [3] study the paired domination problem for block graphs. Cheng et al [4] have solved the paired domination problem for interval and circular arc graphs. Kang et al. [11] have investigated the paired-domination problem in inflated graphs.

A set S of vertices in G is a *total dominating set* of G if every vertex of G is adjacent to some vertex in S. The *total-domination problem* is to find a minimum total-dominating set of G. The *total domatic number* of G is the size of a maximum cardinality partition of the nodes into total dominating sets. A survey article by Henning [7] illustrates the significance and lists the recent development on the total-domination problem.

2. Infinite Diamond Lattice

Graphite sheet is a 2-dimensional hexagonal lattice [14]. See Figure 4. Diamond lattice is a 3 dimensional extension of graphite sheet. A diamond lattice consists of layers of graphite sheets. Infinite diamond lattice consists of infinite layers of infinite graphite sheets. See Figure 1. Each layer consists of sequence of zigzags. Each zigzag consists of a row of red vertices and a row of blue vertices. See Figure 3. In the graph 1, the edges can be classified by the direction of the edges. There are 4 different directions of edges. See Figure 2. In other words, there are 4 types of edges where each type is identified by its direction. Thus the edge set can be partitioned into 4 groups according to its type.

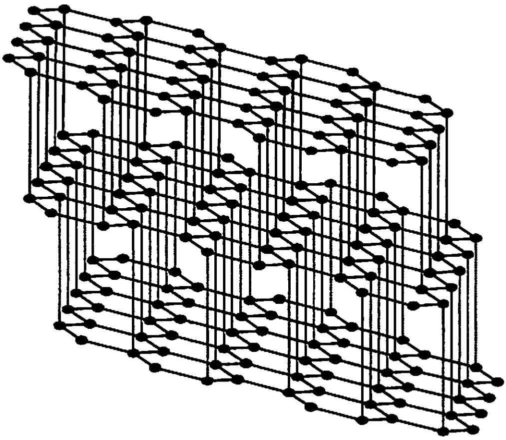


Fig 1: Infinite diamond lattice

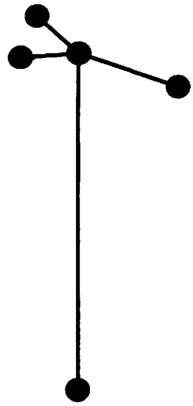


Fig 2: Four types of edges

In this paper we construct a minimum paired dominating set and a minimum total dominating set for infinite diamond lattice. We also demonstrate that the total domatic number of infinite diamond lattice is 4.

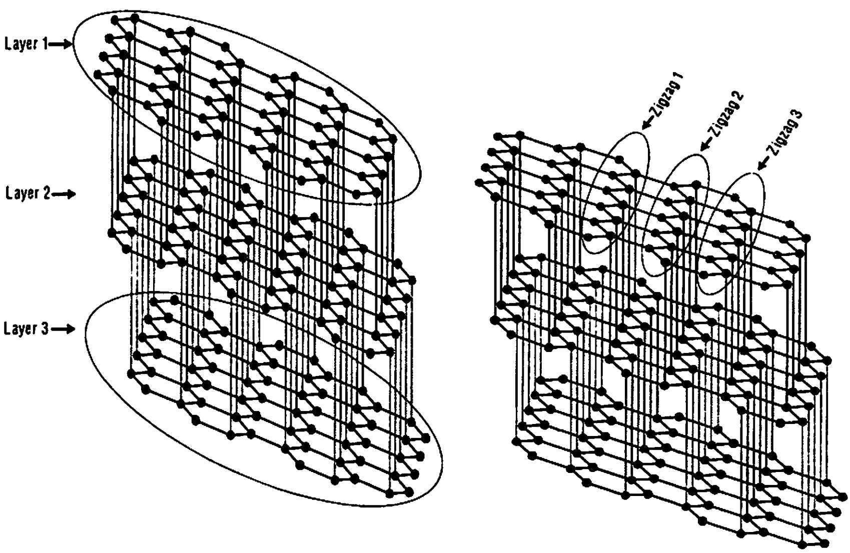
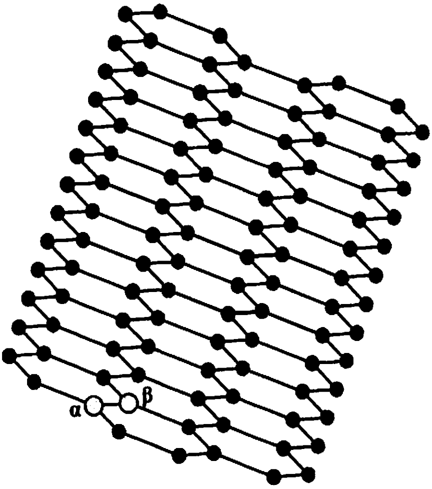
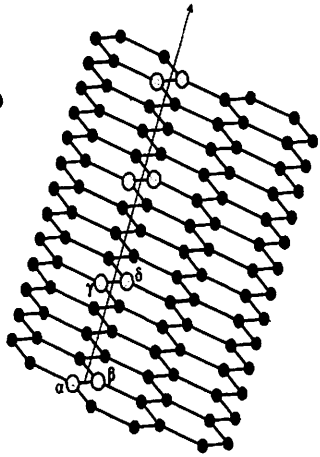


Fig 3: The infinite diamond lattice consists of layers of graphite sheets. Each graphite layer consists of sequence of zigzags. Each zigzag consists of a row of red vertices and a row of blue vertices.



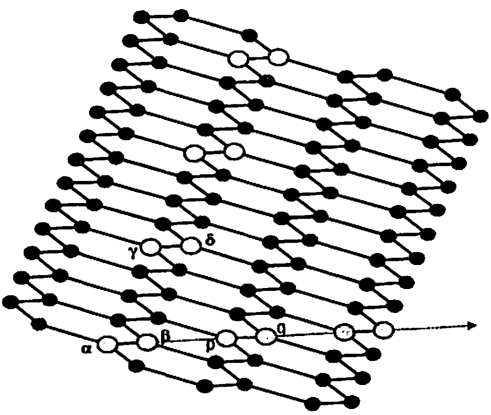


The starting edge (α, β) is called anchoring edge of PD.

Fig 4 Fig 5

3. Paired Dominating set of Diamond Lattice

In this section, we describe the algorithm which constructs minimum paired dominating set of infinite diamond lattice. Let PD denote the set of pairs of vertices constructed by the algorithm. A paired dominating set induces a perfect matching which is a collection of edges. Our algorithm selects only edges and the respective pair of vertices of those edges is added as members of PD.



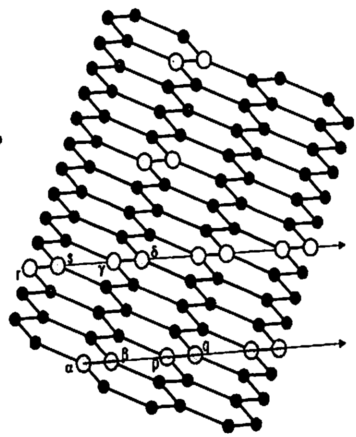


Fig 6 Fig 7

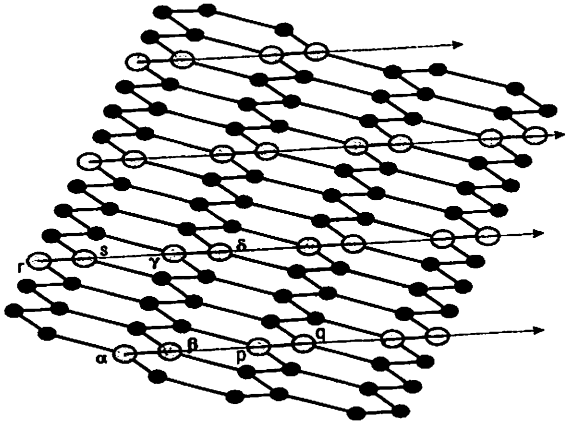


Fig 8

As a first step, the algorithm selects a random edge from the infinite diamond lattice and applies greedy method to select successive edges. The first random edge (α, β) selected by the algorithm is called *Anchoring Edge* of PD. The construction of a paired dominating set is given below:

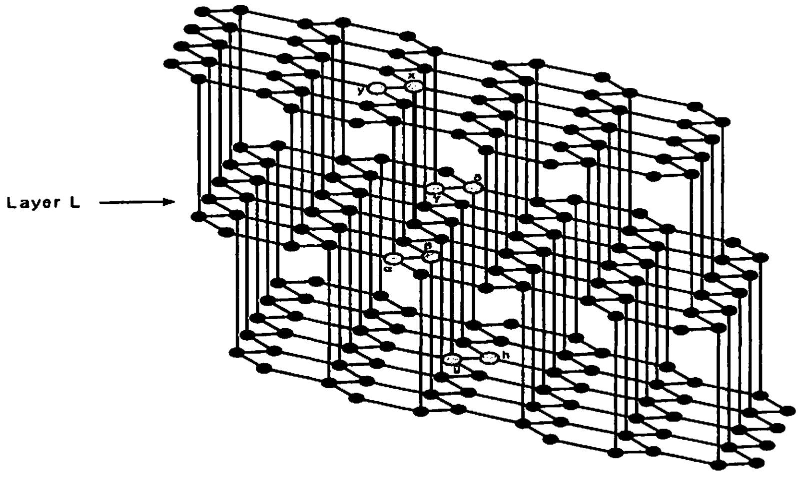


Fig 9

Algorithm A (Construction of paired dominating set PD):

1. Selecting an anchoring edge: As a first step, a random edge (α, β) (See Figure 4) is selected in some layer L (See Figure 9) and the

respective vertices α and β are added to the paired dominating set PD. The edge is called anchoring edge of PD.

2. Processing layer L: Starting at anchoring edge (α, β), edges along the zigzag line are identified and their respective pair of vertices are added to PD. The next edge along the zigzag line is (γ, δ). Between the two edges (α, β) and (γ, δ), there are 6 vertices (3 blue vertices and 3 red vertices). See Figure 5.

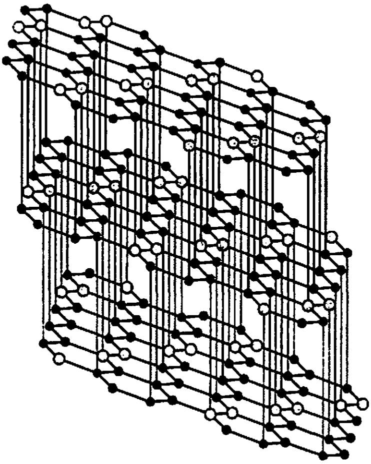
   Starting at the edge (α, β), edges along the direction of edge (α, β) are identified and their respective pair of vertices are added to PD. See Figure 6. In the same way, all the edges along the direction of edge (γ, δ) are identified and their respective pair of vertices are added to PD. See Figure 7. The same process is carried out for other edges in the zigzag line of Figure 5. Now all the pair of vertices of PD of the layer are added to PD. See Figure 8.

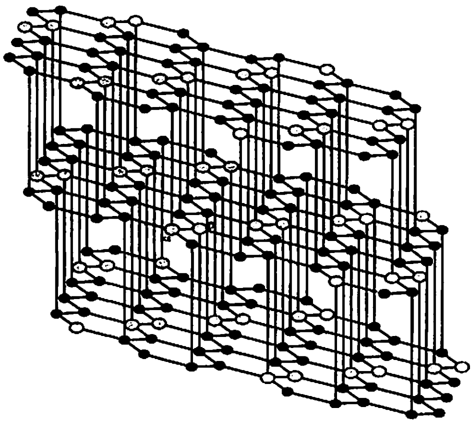
3. Moving downwards from the layer L: The next step is to move downwards to lower layer. It is enough to find one edge in the adjacent layer. An edge (g,h) (along the direction of edge (α, β)) in the lower layer is identified in such a way that the vertex g is adjacent to the middle vertex of β and δ (along the row of red nodes). See Figure 9. In order to identify all the pair of vertices of PD in the lower layer, apply step 2 starting from the edge (g,h).

   There are infinite numbers of layers downwards and the algorithm processes the lower layers one by one sequentially.

4. Moving upwards from the layer L: The next step is to move upwards to upper layer. It is enough to find one edge in the upper layer. An edge (y,x) in the upper layer is selected in such a way that the vertex x is adjacent to the middle vertex of α and γ (along the row of blue nodes). See Figure 9. In order to identify all the pair of vertices of PD in the upper layer, apply step 2 starting from the edge (y,x).

   There are infinite numbers of layers upwards and the algorithm processes the upper layers one by one sequentially.□





The set of green nodes is a paired dominating set.

Fig 10 Fig 11

Theorem 1: The set PD constructed by algorithm A is a minimum paired dominating set.

Proof: Let us prove that the set constructed by the algorithm A is a dominating set. The proof is explained diagrammatically in Figure 12. Let us consider the vertices *a, b, c, d, e* and *f* between the edges (α,β) and (γ, δ). The vertices *a, b, c, d, e* and *f* are dominated by β, p, x, g, s and γ respectively.

Moreover, the subgraph induced by the set PD contains only independent edges. See Figures 10 and 11. Thus PD is a paired dominating set. Since each vertex of V is dominated by only one node of PD, it is a minimum.□

A paired dominating set *S* is said to be *efficient* if for every vertex of *V\S* is dominated by exactly one vertex of *S*.

Corollary 2: The set constructed by algorithm A is an efficient paired dominating set.□

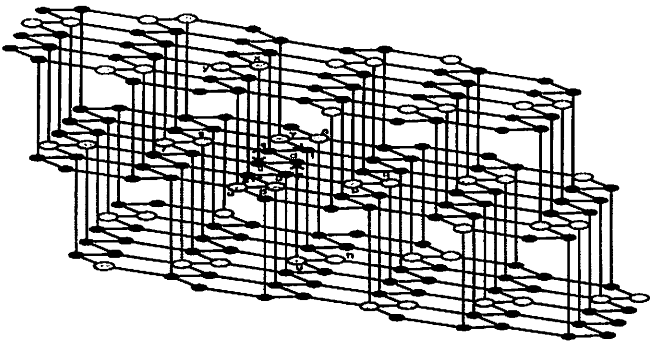


Fig 12: The vertices *a, b, c, d, e and f* are dominated by β, p, x, g, s and γ respectively.

4. Total Dominating set of Diamond Lattice

A total dominating set *S* is said to be *efficient* if every vertex of *V* is dominated by exactly one vertex of *S*. It is straightforward to verify that the set constructed by algorithm A is an efficient total dominating set. Thus, for infinite diamond lattice, efficient paired dominating set and efficient total dominating are the same.

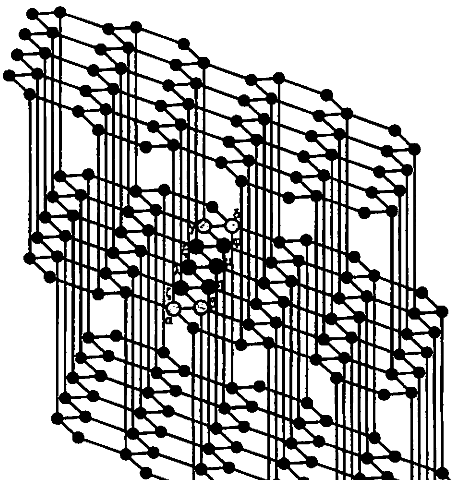


Fig 13

Theorem 3: The set constructed by algorithm A is a minimum total dominating set. .□

Theorem 4: The total domatic partition number of infinite diamond lattice is 4.

Proof: The efficient total dominating set constructed by algorithm A is anchored by(α, β). See Figure 11. Between the edges (α, β) and (γ, δ), there are three edges (ζ, η), (λ, μ) and (ξ, φ) which are of the same type of edge (α, β). See Figure 13. Each edge of (ζ, η), (λ, μ) and (ξ, φ) anchors an efficient total dominating sets.

Let us denote the total dominating set anchored by (α, β) as PD(α, β). In the same way, PD(ζ, η), PD(λ, μ) and PD(ξ, φ) denote the total dominating set anchored by (ζ, η), (λ, μ) and (ξ, φ) respectively. It is straightforward that PD(α, β), PD(ζ, η), PD(λ, μ) and PD(ξ, φ) partition the vertex set of infinite diamond lattice and form total domatic partition. Since each PD is minimum total dominating set, the partition {PD(α, β), PD(ζ, η), PD(λ, μ), PD(ξ, φ)} is maximum. □

5. Conclusion

Two edges having a vertex in common are called *adjacent* edges. We say that an edge *dominates* its adjacent edges. A set of edges M of G(V, E) is called an *edge dominating set* if every edge of E\M is adjacent to an edge of M. Brandstädt et al [1] and Kobler et al [12] list an extensive literature on edge domination problem. Paired domination problem, total domination problem and edge domination problem appear to be similar. Even though paired domination problem and total domination problem have solution for infinite diamond lattice, the edge domination problem seems to be challenging. It will remain as an open problem for diamonds.

Variants of domination such as Roman domination and edge domination for infinite diamond lattice are under investigation. Domination problems on different chemical structure such as Lonsadaleite, Calcium Fluoride crystals etc. are also being studied.

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