Radio Antipodal Number of Gird like Architecture Graphs

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ABSTRACT

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Let $diam(G)$ denote the diameter of G and $d(u, v)$ denote the distance between the vertices u and v in G. An antipodal labeling of G with diameter d is a function f that assigns to each vertex u, a positive integer $f(u)$, such that $d(u,v) + |f(u) - f(v)| \geq d$, for all $u, v \in V$. The span of an antipodal labeling f is $max\{|f(u) - f(v)|: u, v \in V(G)\}$. The antipodal number for G, denoted by $an(G)$, is the minimum span of all antipodal labelings of G. Determining the antipodal number of a graph G is an NP-complete problem. In this paper we determine the antipodal number of certain graphs.

1. Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. A radio k- labeling f of G is an assignment of positive integers to the vertices of G such that $d(u,v) + |f(u) - f(v)| \geq k + 1$ for every two distinct vertices u and v of$G$, where $d(u,v)$is the distance between any two vertices u and v of G. The span of such a function$f$, denoted by $sp(f) = max\{|f(u) - f(v)|: u, v \in V(G)\}$. Radio k −labeling was motivated by the frequency assignment problem [3]. The maximum distance among all pairs of vetices in G is the diameter of G. The radio labeling is a radio k- labeling when $k = diam(G)$. When $k = diam(G) - 1$, a radio k- labeling is called a radio antipodal labeling. In otherwords, an antipodal labeling for a graph G is a function ,$f: V(G) \rightarrow \{0,1,2,\dots\}$ such that $d(u,v) + |f(u) - f(v)| \geq diam(G)$. The radio antipodal number for G, denoted by $an(G)$, is the minimum span of an antipodal labeling admitted by G.

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A radio labeling is a one-to –one function , while in an antipodal labeling, two vertices of distance $diam(G)$ apart may receive the same label.

The antipodal labeling for graphs was first studied by Chartrand et al.[8], in which, among other results, general bounds of $an(G)$were obtained. Khennoufa and Togni [10] determined the exact value of $an(P_n)$ for paths P_n. The antipodal labeling for cycles C_n was studied in [4], in which lower bounds for $an(C_n)$ are obtained. In addition , the bound for the case $n \equiv 2(mod\ 4)$ was proved to be the exact value of $an(C_n)$, and the bound for the case $n \equiv 1(mod\ 4)$ was conjectured to be the exact value as well [7] . Justie Su-tzu Juan and Daphne Der-Fen Liu [9] confirmed the conjecture mentioned above. Moreover they determined the value of $an(C_n)$ for the case $n \equiv 3(mod\ 4)$ and also for the case $n \equiv 0(mod\ 4)$. They improve the known lower bound [4] and give an upper bound. They also conjectured that the upper bound is sharp.

In this paper we obtain the radio antipodal number of the circular ladder, grid and odd torus.

2. The Radio Antipodal Number of Circular Ladder

The circular ladder graph CL_n consists of two concentric n-cycles in which each of the n corresponding vertices is joined by an edge. It is a 3-regular simple graph. See Figure 1.

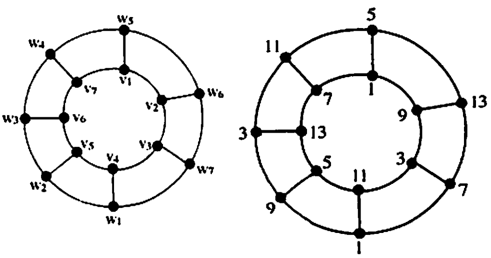

Fig 1: Radio antipodal number of *CLn* with diameter 4

Theorem 1: The radio antipodal number of CLn satisfies $an(CLn) \leq 2\left\lceil\frac{n}{2}\right\rceil\left(\left\lceil\frac{n}{2}\right\rceil - 3\right) + 5$.

Proof: Define a mapping $f: V(CLn) \to N$

$$f(v_{2i-1}) = (d-2)i - (d-3), i = 1,2, \dots \left\lceil\frac{n}{2}\right\rceil$$

$$f(v_{2i}) = (d-2)\left\lceil\frac{n}{2}\right\rceil - (d-3) + (d-2)i, \qquad i = 1,2,\dots\left\lceil\frac{n}{2}\right\rceil - 1.$$

$$f(v_{2j-1}) = (d-2)j - (d-3), \qquad j = 1,2,\dots\left\lceil\frac{n}{2}\right\rceil$$

$$f(v_{2j}) = (d-2)\left\lceil\frac{n}{2}\right\rceil - (d-3) + (d-2)j, \qquad j = 1,2,\dots\left\lceil\frac{n}{2}\right\rceil - 1.$$

We claim that $d(u,v) + |f(u) - f(v)| \geq \left\lceil\frac{n}{2}\right\rceil$ for all $u, v \in V(CLn)$.

Let $v_1, v_2, \dots v_n$ be the vertices label in the inner cycle c_1 and $w_1, w_2, \dots w_n$ be the vertices label in the outer cycle c_2. Let $w_1, w_2, \dots w_n$ be the vertices adjacent to $v_{\left\lceil\frac{n}{2}\right\rceil}, v_{\left\lceil\frac{n}{2}\right\rceil+1}, \dots v_{\left\lceil\frac{n}{2}\right\rceil-3}$.

Case(i): Let u and v lie in the same inner cycle with odd party, say $u = w_{2i}$ and $v = w_{2j}$, then clearly $d(u,v) \geq 2$. By the mapping f defined there exists $f(u)$ and $f(v)$ such that $f(u) = (d-2)i - (d-3)$, $f(v) = (d-2)j - (d-3), i \neq j$.

Now $d(u,v) + |f(u) - f(v)| \geq 2 + |(d-2)(i-j)| \geq \left\lceil\frac{n}{2}\right\rceil$.

Case(ii): Let u and v lie in the same inner cycle with even party, say $u = w_{2i-1}$ and $v = w_{2j-1}$, then clearly $d(u,v) \geq 2$. By the mapping f defined there exists $f(u)$ and $f(v)$ such that $f(u) = (d-2)\left\lceil\frac{n}{2}\right\rceil - (d-3) + (d-2)i$, $f(v) = (d-2)\left\lceil\frac{n}{2}\right\rceil - (d-3) + (d-2)j,\ i \neq j$.

Now $d(u,v) + |f(u) - f(v)| \geq \left\lceil\frac{n}{2}\right\rceil$.

Case(iii): Let u and v lie in the same outer cycle with odd party, say $u = w_{2i}$ and $v = w_{2j}$, then clearly $d(u,v) \geq 2$. By the mapping f defined there exists $f(u)$ and $f(v)$ such that $f(u) = (d-2)i - (d-3)$, $f(v) = (d-2)j - (d-3), i \neq j$.

Now $d(u,v) + |f(u) - f(v)| \geq \left\lceil\frac{n}{2}\right\rceil$.

Case(iv): Let u and v lie in the same inner cycle with even party, say $u = w_{2i-1}$ and $v = w_{2j-1}$, then clearly $d(u,v) \geq 2$. By the mapping f defined there exists $f(u)$ and $f(v)$ such that $f(u) = (d-2)\left\lceil\frac{n}{2}\right\rceil - (d-3) + (d-2)i$, $f(v) = (d-2)\left\lceil\frac{n}{2}\right\rceil - (d-3) + (d-2)j,\ i \neq j$.

Now $d(u,v) + |f(u) - f(v)| \geq \left\lceil\frac{n}{2}\right\rceil$.

Case(v): Let $u = w_{2i-1}$ be a odd vertex and $u = w_{2j}$ be a even vertex in the inner cycle. Then $d(u,v) = 1$, when u and v are adjacent , otherwise $d(u,v) \geq 2$.

By the mapping f. $f(u) = (d-2)i - (d-3)$

$$f(v) = (d-2)\left\lceil \frac{n}{2} \right\rceil - (d-3) + (d-2)j,\ i \neq j.$$

Now $d(u,v) + |f(u) - f(v)| \geq \left\lceil \frac{n}{2} \right\rceil$.

Case(vi): Let $u = w_{2i-1}$ be a odd vertex in the inner cycle and $u = w_{2j}$ be a even vertex in the outer cycle. Then $d(u,v) = 1$, when u and v are adjacent , otherwise $d(u,v) \geq 2$.

By the mapping f. $f(u) = (d-2)i - (d-3)$

$$f(v) = (d-2)\left\lceil \frac{n}{2} \right\rceil - (d-3) + (d-2)j,\ i \neq j.$$

Now $d(u,v) + |f(u) - f(v)| \geq \left\lceil \frac{n}{2} \right\rceil$.

Thus $an(CLn) \leq 2\left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil - 3\right) + 5$.

3. The Radio Antipodal Number of Grid

An n-dimensional mesh $M(d_1, d_2, \dots d_n)$ has$\{(x_1, x_2, \dots x_n): 1 \leq x_i \leq d_i, 1 \leq i \leq n\}$ for its vertex set and vertices $(\dots x_i \dots)$ and $(\dots x_{i+1} \dots)$, $1 \leq i \leq n$ are adjacent in $M(d_1, d_2, \dots d_n)$. A mesh is a bipartite graph. If at least one side has even length, the mesh has a hamiltonian cycle. A Hamiltonian path exists always. Meshes are not regular, but the degree of any vertex is bounded by $2n$. Of course, the degree of a corner vertex is less than the degree of an internal vertex. See Figure 2.

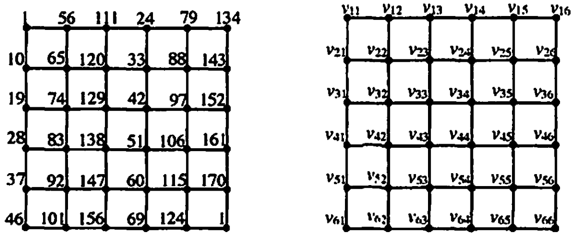

Fig. 2 Radio antipodal number of G_{nxn} with diameter 10

Theorem 2: The radio antipodal number of G_{nxn} satisfies $an(G_{nxn}) \leq (2n-3)\left(\frac{n^2+n-4}{2}\right) - \frac{n}{2} + 2$.

Proof: Define a mapping $f: V(G_{nxn}) \to N$

$$f(a_{ij}) = (d-1)\{n(j-1) + (i-1)\} + j, \qquad i = 1,2,\dots n, \qquad j = 1,2,\dots\frac{n}{2}$$

$$f\left(a_{i,\frac{n}{2}+j}\right) = \frac{n}{2}(d-3) + 2 + (d-1)\{n(j-1) + (i-1)\} + j,$$

$$i = 1,2,\dots n, j = 1,2,\dots\frac{n}{2} - 1.$$

Let $V_1 = \left\{a_{ij}/i = 1,2,\dots n,\ j = 1,2,\dots\frac{n}{2}\right\}$

$V_2 = \left\{a_{i,\frac{n}{2}+j}/i = 1,2,\dots n,\ j = 1,2,\dots\frac{n}{2} - 1\right\}$

Case(i): If $u, w \in V_1$, then $u = a_{kl}$ and $w = a_{st}$,$1 \le k, s \le n$, $1 \le l, t \le \frac{n}{2}$ and $d(u, w) \ge 1$. By the mapping f, $f(u) = (d-1)\{n(l-1) + (k-1)\} + l$

$f(w) = (d-1)\{n(t-1) + (s-1)\} + t$.

Now $d(u, w) + |f(u) - f(w)| \ge 1 + |(d-1)(n+1)(k-l)| \ge 2(n-1)$.

Case(ii): If $u, w \in V_2$, then $u = a_{k,\frac{n}{2}+l}$ and $w = a_{s,\frac{n}{2}+t}$,$1 \le k, s \le n$, $1 \le l, t \le \frac{n}{2} - 1$

and $d(u, w) \ge 1$. By the mapping f, $f(u) = \frac{n}{2}(d-3) + 2 + (d-1)\left\{n\left(\frac{n}{2} + l - 1\right) + (k-1)\right\} + \frac{n}{2} + l$

$f(w) = \frac{n}{2}(d-3) + 2 + (d-1)\left\{n\left(\frac{n}{2} + t - 1\right) + (s-1)\right\} + \frac{n}{2} + t$.

Now $d(u, w) + |f(u) - f(w)| \ge 2(n-1)$.

Case(iii): If $u \in V_1, w \subset V_2$.

Case (iii)a: If $u \in V_l$ and $w \in V_{\frac{n}{2}+l}$,then $d(u, w) \ge \frac{n}{2}$. By the mapping f, $f(u) = (d-1)\{n(l-1) + (k-1)\} + l$ and $f(w) = \frac{n}{2}(d-3) + 2 + (d-1)\left\{n\left(\frac{n}{2} + t - 1\right) + (s-1)\right\} + \frac{n}{2} + t$.

Now $d(u, w) + |f(u) \quad f(w)| \ge 2(n-1)$.

Case (iii)b: If $u \in V_{l+1}$ and $w \in V_{\frac{n}{2}+l}$, then $d(u, w) \ge \frac{n}{2}$. By the mapping f, $f(u) = (d-1)\{n(l-1) + (k-1)\} + l$ and $f(w) = \frac{n}{2}(d-3) + 2 + (d-1)\left\{n\left(\frac{n}{2} + t - 1\right) + (s-1)\right\} + \frac{n}{2} + t$.

Now $d(u,w) + |f(u) - f(w)| \geq 2(n-1)$.

Thus $an(G_{nxn}) \leq (2n-3)\left(\frac{n^2+n-4}{2}\right) - \frac{n}{2} + 2$.

4. The Radio Antipodal Number of Torus

Torus is bipartite if and only if all side length are even. They are Hamiltonian, regular and vertex symmetric. A one-dimensional torus is simply a circle or a ring. A two- dimensional torus network contains *mn* nodes arranged in two dimension with *m,n* nodes per dimension. We denote the *m* x *n* torus by *TR*(*m,n*). It is defined as a graph with vertex set $V = \{(i,j); 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E = \{(i_1,j_1),(i_2,j_2); (i_2 = (i_1+1) mod m \wedge j_1 = j_2) \vee (i_1 = i_2 \wedge j_2 (j_1+1) mod n)\}$. Thus *TR*(*m,n*) as *mn* vertices and 2*mn* edges, it is 4-regular and its diameter is $\left\lfloor\frac{m}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$. See Figure 3.

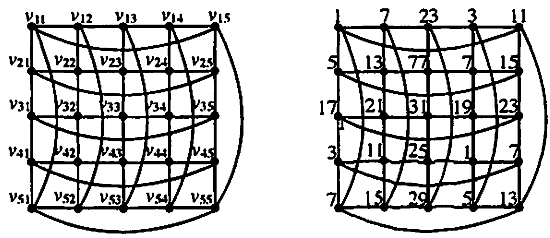

Fig. 3 Radio antipodal number of *TR*(*m*, *n*) with diameter 4

Theorem 3: If n is odd, then the radio antipodal number of the torus TR(m, n) satisfies $an(TR(m,n)) \leq \left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(\frac{n^2+5n-6}{2}\right) - 2\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right) + 3n$.

Proof: A vertical cut together with a horizontal cut of vertices gives 4 meshes. Define a mapping $f: V(TR(m,n)) \to N$.

$$f(a_{ij}) = (j-1)\left(\frac{nd}{2}\right) + (-2j+3) + d(i-1),$$

$$i = 1,2,\ldots\left\lfloor\frac{n}{2}\right\rfloor, j = 1,2,\ldots\left\lfloor\frac{n}{2}\right\rfloor$$

$$f\left(a_{i,\left\lfloor\frac{n}{2}\right\rfloor+j}\right) = (j-1)\left(\frac{nd}{2}\right) + (-2j+3) + d(i-1) + \frac{d}{2},$$

$$i = 1,2,\ldots\left\lfloor\frac{n}{2}\right\rfloor, j = 1,2,\ldots\left\lfloor\frac{n}{2}\right\rfloor$$

$$f\left(a_{\lfloor\frac{n}{2}\rfloor+i,j}\right) = (j-1)\left(\frac{nd}{2}\right) + (-2j+3) + d(i-1) + \frac{d}{2},$$

$$i = 1,2,\dots\left\lfloor\frac{n}{2}\right\rfloor, j = 1,2,\dots\left\lfloor\frac{n}{2}\right\rfloor$$

$$f\left(a_{\lfloor\frac{n}{2}\rfloor+i,\lfloor\frac{n}{2}\rfloor+j}\right) = (j-1)\left(\frac{nd}{2}\right) + (-2j+3) + d(i-1),$$

$$i = 1,2,\dots\left\lfloor\frac{n}{2}\right\rfloor, j = 1,2,\dots\left\lfloor\frac{n}{2}\right\rfloor$$

$$f\left(a_{\lfloor\frac{n}{2}\rfloor+i,\lfloor\frac{n}{2}\rfloor+j}\right) = \left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(\frac{nd}{2}\right) + \left(-2\left\lfloor\frac{n}{2}\right\rfloor+3\right) + d\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right) + \frac{3d}{2} - 2$$
$$+ d(j-1) + (i-1), \quad i = 1, j = 1,2,\dots\left\lfloor\frac{n}{2}\right\rfloor$$

$$f\left(a_{\lfloor\frac{n}{2}\rfloor+i,\lfloor\frac{n}{2}\rfloor+j}\right) = \left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(\frac{nd}{2}\right) + \left(-2\left\lfloor\frac{n}{2}\right\rfloor+3\right) + d\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right) + 2(d-1)$$
$$+ d(j-1) + (i-1), \quad i = 1, j = 1,2,\dots\left\lfloor\frac{n}{2}\right\rfloor$$

$$f\left(a_{i,\lfloor\frac{n}{2}\rfloor+j}\right) = \left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(\frac{nd}{2}\right) + \left(-2\left\lfloor\frac{n}{2}\right\rfloor+3\right) + 2d\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right) + 2(d-1)$$
$$+ d(i-1) + (j-1), \; i = 1,2,\dots\left\lfloor\frac{n}{2}\right\rfloor, j = 1$$

$$f\left(a_{i,\lfloor\frac{n}{2}\rfloor+j}\right) = \left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(\frac{nd}{2}\right) + \left(-2\left\lfloor\frac{n}{2}\right\rfloor+3\right) + 2d\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right) + \left(\frac{5d}{2}-2\right)$$
$$+ d(i-1) + (j-1), \; i = 1,2,\dots\left\lfloor\frac{n}{2}\right\rfloor, j = 1$$

$$f\left(a_{\lfloor\frac{n}{2}\rfloor+i,\lfloor\frac{n}{2}\rfloor+j}\right) = \left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(\frac{nd}{2}\right) + \left(-2\left\lfloor\frac{n}{2}\right\rfloor+3\right) + 2d\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right) + 3d - 2$$
$$+ d\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right) + (i-1) + (j-1), \quad i = 1, \quad j = 1$$

The proof is similar to theorem 2.

Conclusion

The study of radio antipodal number of graphs has gained momentum in recent years. Very few graphs have been proved to have radio antipodal labeling that attains the radio antipodal number. In this paper we have determined the bounds of the radio antipodal number of the circular ladders, grid and odd torus. Further study is taken up for various other classes of graphs.

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