Broadcasting in Sierpinski Gasket Graphs

A. Shanthakumari
Assistant Professor, Department of Mathematics
M.O.P.Vaishnav College for Women (Autonomous), Chennai, India
shanthakumari9775@gmail.com

Abstract

Broadcasting is a fundamental information dissemination problem in a connected graph, in which one vertex called the originator, disseminates one or more messages to all other vertices in the graph. k-broadcasting is a variant of broadcasting in which an informed vertex can disseminate message to at most k uninformed vertices in one unit of time. In general, solving the broadcast problem in an arbitrary graph is NP-complete. In this paper, we obtain the k-broadcast time of the Sierpinski gasket graphs for all $k \geq 1$.

1 Introduction

An essential component of a super-computer based on large-scale parallel processing is the interconnection network. The interconnection network consists of hardware and software entities that are interconnected to facilitate efficient computation and communication. Parallel processing and supercomputing continue to exert great influence in the development of modern science and engineering [8]. Interconnection networks are often modeled by finite graphs or digraphs. The vertices of the graph represent the nodes of the network, the processing elements, memory modules, switches and the edges correspond to communication lines [15].

Fractal antennas have been studied, built, commercialized for a considerable while. Properly synthesized fractal antennas feature multi-band properties. Some of the modern mobile radio communication systems are based on Sierpinski fractals or Sierpinski gasket like structures and have a log-periodic behaviour as far as radiation patterns are concerned. Fractal geometries also have electromagnetic applications [10].

The performance of information dissemination often determines the efficiency of a whole network or a parallel system. There are two approaches to reduce the delay of information dissemination: one is to reduce the amount of data being transferred, while the other is to minimize the delay of information spreading. The first goal can be achieved by data compression or by reducing redundant information and the second can be achieved by designing efficient algorithms and network topologies for gossip or telephonic problem, broadcast problem and their variants [6].

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In this paper, we focus on information dissemination to at most k neighbouring vertices from an orginator u and obtain k-broadcast time of Sierpinski gasket graphs for all $k \geq 1$. In the sequel, we refer to Bondy and Murty [2] for basic concepts in Graph Theory.

2 Preliminaries

Bavelas [1] was the first person to study the effectiveness of different communication patterns in helping small groups of people solve common tasks. In studied tasks, the subjects could communicate with one another according to a given communication pattern by writing messages. Bavelas considered such measures as the number of messages and the time required to complete the task. He showed that for any communication pattern of a certain type among p people, 2(p - 1) messages are required to solve a given task. He also showed that, if any communication pattern is allowed and each message takes unit time, then the time required to complete the task is no more than $\log_2 p$.

Broadcasting, as a major variant of the Gossip Problem, was introduced in 1977 by Slater, Cockayne and Hedetniemi [13], when they studied the minimum time required for one person to transmit one piece of information to everyone in a communication network. A survey of the early results in gossiping and broadcasting was presented by Hedetniemi, Hedetniemi, and Liestman [6] and few results were published by Fraigniaud and Lazard [3]. The k-broadcast time has been obtained for the complete graph, the path graph, the d-grid graph, the d-Torus graph, hypercube, cube connected cycles, butter.y graph, the deBruijn graph, the shuffle-exchange graph, the stargraph, the necklace graph etc. [12].

As processors become faster and more efficient, network communication has become a larger concern for bottlenecks and communication slowdowns. Broadcasting is an information dissemination problem in a connected network, in which one vertex, called the originator, must distribute a message to all other vertices by placing a series of calls along the communication lines of the network. The broadcasting is completed as quickly as possible, subject to the following constrains: (1) A call can involve only one informed vertex and an uninformed vertex, (2) A vertex can call only one of its neighbours who is connected to it, (3) The information is transmitted in one time unit, (4) In one time unit many calls can be performed in parallel.

Given a connected graph G and a message originator u, the broadcast time of vertex u, $b(u)$ is the minimum number of time units required to complete broadcasting from vertex u. It is easy to see that for any vertex u in a connected graph G with n vertices, $b(u) \geq \lceil \log_2 n \rceil$, since the number of informed vertices can at most double during each time unit. The broadcast time of a graph G, $b(G)$ is defined to be the maximum broadcast time of any vertex u in G, i.e. $b(G) = \max \{b(u) : u \in V\}$. For the complete graph K_n with $n \geq 2$ vertices, $b(K_n) = \lceil \log_2 n \rceil$. From application perspective, the minimum broadcast time helps graphs represent the cheapest possible communication networks (having the fewest communication lines) in which broadcasting can be accomplished, from any vertex, as fast as theoretically possible.

A k-broadcasting is an n-vertex communication network that supports a broadcast from any one member to at most k of its neighbours in optimal time [14]. Given a connected graph G and a message orginator u, we defined the k-broadcast time of vertex u, $b_k(u)$ to be the minimum number of time units required to complete k-broadcasting from vertex u. The k-broadcast time of G is defined as $b_k(G) = \max \{b_k(u) : u \in V\}$.

A k-broadcast scheme or k-broadcast schedule is a series of calls that perform k-broadcast. A k-broadcast scheme that finishes the k-broadcasting in $b_k(u)$ is called an optimal k-broadcast scheme. For any vertex u in a connected graph G with n vertices, $b_k(u) \geq \lceil \log_{k+1} n \rceil$, since the informed vertices can at most be multiplied by $k + 1$ during each time unit [9]. In general, solving the broadcast problem in an arbitrary graph is NP-complete [5, 13].

3 Sierpinski Gasket Graphs

The Generalised Sierpinski Graph $S(n,k)$, $n \geq 1$, $k \geq 1$ is defined in the following way:

$V(S(n,k))=\{1, 2, ..., k\}^n$, two distinct vertices $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ being adjacent if and only if there exists an $h \in \{1, 2, ..., n\}$ such that
(i) $u_t = v_t$, for $t = 1, ..., h-1$;
(ii) $u_h \neq v_h$; and
(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, ..., n$.

For convenience we write the vertex $(u_1, u_2, ..., u_n)$ as $(u_1 u_2 ... u_n)$. The vertices (1...1), (2...2), ..., $(k...k)$ are called the extreme vertices of $S(n,k)$. In the literature, $S(n,3)$, $n \geq 1$ is known as the Sierpinski graph. For $i = 1, 2, 3$, let $S(n + 1, 3)_i$ be the subgraph induced by the vertices that have i as the first entry. Clearly $S(n + 1, 3)_i$ is isomorphic to $S(n, 3)$.

The Sierpinski gasket graph S_n, $n \geq 1$, can be obtained by contracting all the edges of $S(n,3)$ that lie in no triangle. For $i = 1, 2, 3$ let $S_{n,i}$ be the subgraph of S_{n+1} induced by $(i...i)$,$\{i,j\}$,$\{i,k\}$ where$\{i, j, k\} = \{1, 2, 3\}$ and all the vertices whose prefix starts with i [9]. The vertices (1...1), (2...2), ..., $(k...k)$ are called the extreme vertices of S_n [10].

Geometrically, S_n is a graph whose vertices are the intersection points of the line segments of the finite sierpinski gasket σ_n and line segment of the gasket as edges. The sierpinski gasket graph S_n is the finite structure obtained by n iterations of the process. The sierpinski graph S_n has $(3/2)(3^{n-1}+1)$ vertices and 3^n edges [14].

The definition of sierpinski graphs $S(n,k)$ originated from the topological studies of the Lipscomb's space and is isomorphic to the graphs of the Tower of Hanoi with n disks. Moreover, sierpinski graphs are the first nontrivial families of graphs of fractal type for which the crossing number is known and several metric invariants such as unique 1-perfect codes, average distance of sierpinski gasket are determined [4, 11]. Teguia and Godbole [14] studied several properties of these graphs such as particular hamiltonicity, pancyclicity, cycle structure, domination number, chromatic number, pebbling number, cover pebbling number. vertex coloring, edge-coloring and total-coloring of sierpinki gaskets have been obtained. S_n is properly three-colourable, that is, $\chi(S_n) = 3$ for each n and its diameter is 2^{n-1} [10].

For convenience, we introduce the following notations : Consider S_n. Structurally, S_n consists of three attached copies of S_{n-1} referred as the top, bottom left and bottom right components of S_n denoted by $S_{n,T}$, $S_{n,L}$ and $S_{n,R}$ respectively. We denote top, left and right vertices of $S_{n,T}$ as $S_{n,TT}$, $S_{n,TL}$ and $S_{n,TR}$. The other vertices in S_n are analogously denoted by $S_{n,LT}$, $S_{n,LL}$, $S_{n,LR}$, $S_{n,RT}$, $S_{n,RL}$ and $S_{n,RR}$ with $S_{n,TL} = S_{n,LT}$, $S_{n,TR} = S_{n,RT}$ and $S_{n,RL} = S_{n,LR}$. See Figure 1.

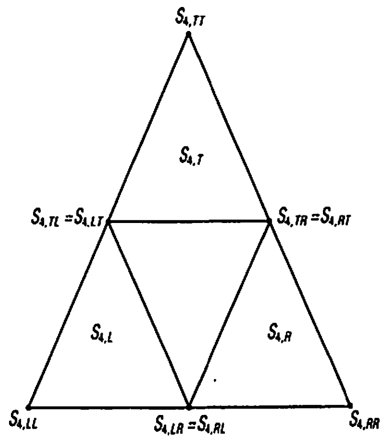

Figure 1: Sierpinski gasket graph S_4

4 Broadcasting in S_n

Lemma 4.1[3]: In any graph G of diameter d, if three different vertices u; v_1; and v_2; with both v_1 and v_2 at a distance d from u exist, then $b(G) \geq d + 1$.

For $n \geq 1$; the broadcast time of S_n, $b(S_n) \geq 2^{n-1} + 1$. We further improve the lower bound of $b(S_n)$ and prove it to be sharp in the following theorem.

Theorem 4.1: For any $n \geq 4$; $b(S_n) \geq 2^{n-1} + (n - 2)$.
Proof: By the structure of S_n, the broadcasting in S_n from the extreme vertex $S_{n,TT}$ to all vertices of S_n must pass through successive lower dimensional Sierpinski gasket graphs, S_i, $1 \leq i \leq n - 1$: Thus, $b(S_n) \geq b(S_{n-1}) + b(S_{n-2}) + \ldots + b(S_1) \geq (2^{n-2} + 1) + (2^{n-3} + 1) + (2^{n-4} + 1) + \ldots + (2^1 + 1) + (2^0 + 1) = (2^{n-2} + 2^{n-3} + \ldots + 2^1 + 1) + (1 + \ldots + 1) = ((2^{n-1} - 1)/(2-1)) + (n - 1) = 2^{n-1} - 1 + n - 1 = 2^{n-1} + (n-2)$.

We prove that the lower bound obtained is sharp by constructing a spanning tree T_n of S_n; $n \geq 4$ inductively as follows:
Step 1: Consider the spanning tree T_4 of S_4 as shown in Figure 2. Let T'_4 denote T_4 with the vertex $S_{4,LL}$ removed. See Figure 2.
Step 2: The spanning tree T_n of S_n, $n \geq 4$ is constructed by extending the spanning tree T_{n-1} of S_{n-1} by attaching a copy of T_{n-1} with its root merged at $S_{n,TL}$ and a copy of T'_{n-1} with its root merged at $S_{n,TR}$.

Label the root of T_n at level 1 as 0. Label its left child as 1 and its right child as 2. We call the shortest path from the top extreme vertex $S_{k,TT}$ to the left extreme vertex $S_{k,LL}$ as left spine and that from $S_{k,TT}$ to the right extreme vertex $S_{k,RR}$ as the right spine of T_n, $k \leq n$. If a vertex is labeled X on the left spine of S_k; $k < n$, then label its children (if they exist) from left to right as $X + 1$ and $X + 2$: On the other hand, if a vertex is labeled X on the right spine of S_k; $k < n$, then label its children (if they exist) from right to left as x + 1 and x+2: By construction, it is clear that $S_{n,LL}$ receives label d_n and that of $S_{n,RR}$ receives label d_{n+1}; where d_n is the diameter of S_n. We now proceed to prove that $n \geq 4$; $b(S_n) \geq 2^{n-1} + (n - 2)$.

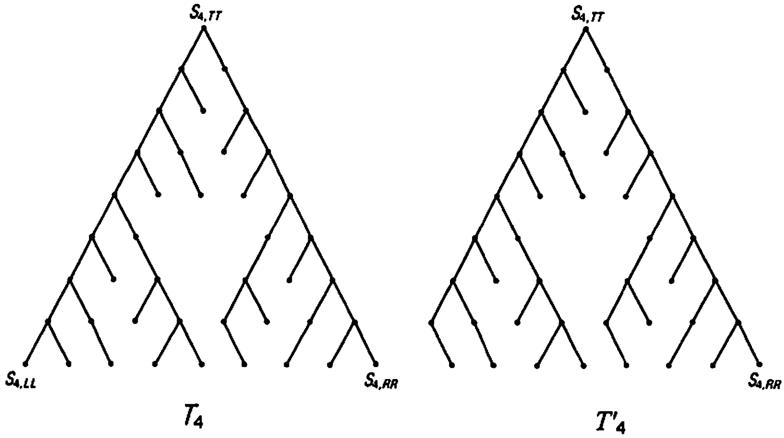

Figure 2: Spanning trees T_4 and T'_4

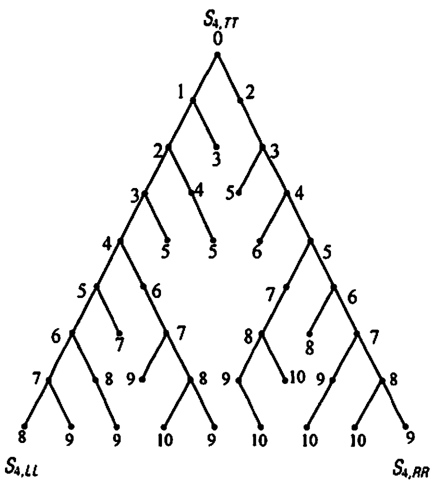

Figure 3: Broadcasting in S_4

We prove this by induction on n. From Figure 3, it is clear that $b(S_4) = 10 = 2^{4-1} + (4 - 2)$. Assume the result to be true for S_{k-1}. Consider S_k. It is clear that, $b(S_k) = t_1 + t_2$ where t_1 is the minimum time unit required for a message to reach $S_{k,TR}$ from

$S_{k,TT}$ and t_2 is the broadcast time of $b(S_{k-1})$. Thus, $b(S_k) = 2^{k-2} + 1 + 2^{k-2} + (k-3)$. $= 2^{k-1} + (k-2)$.

For $n = 1; 2; 3$, $b(S_n)$ are as follows: $b(S_1) = 2$; $b(S_2) = 3$ and $b(S_3) = 6$.

5 k-Broadcasting in S_n, $k \geq 2$, $n \geq 2$

Lemma 5.1[7]: If G is a graph with degree Δ and diameter D, then $b_k(U) = D$ when $k = \Delta$ and $D \leq b_k(U) \leq D + 1$ When $k = \Delta - 1$. Generally speaking, when $k < \Delta$, $D \leq b_k(G) \leq \lceil \frac{\Delta}{k} \rceil$.

Theorem 5.1: For any $k \geq 2$, $n \geq 2$, $b_k(S_n) = 2^{n-1}$.

The following algorithm proves that the k-broadcasting in S_n, $k \geq 2$, $n \geq 2$ is its diameter .

Input: An n-dimensional Sierpinski gasket graph S_n, $n \geq 2$.

Algorithm:

Case 1: If $k = 2$; then construct the BFS tree (Breadth-First Search) of S_n with the two degree vertex $S_{n,TT}$ as the root. The children of the root are in the second level denoted as L_2. Consequently, in general the vertices in level L_{i+1} are the children of vertices in the level L_i. See Figure 4.

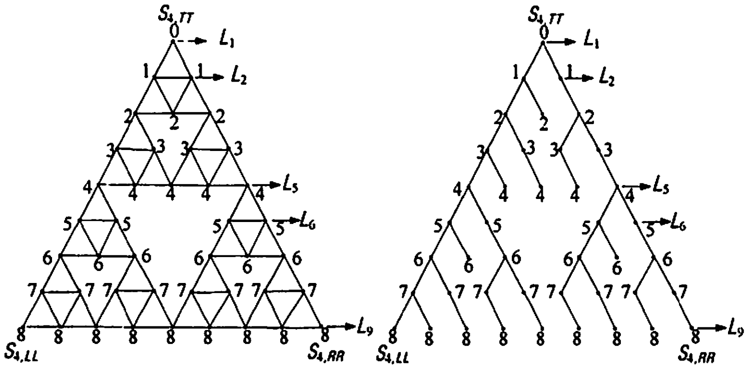

Figure 4: 2-broadcasting in S_4

Case 2: If $k = 3$; then construct the BFS tree (Breadth-First Search) of $S_{n,L}$ with $S_{n,LT}$ as the root. The children of the root are in the second level L_2. Consequently, in general the vertices in level L_{i+1} are the children of vertices in the level L_i.Continue the above procedure in $S_{n,T}$ to obtain a BFS tree with $S_{n,TL}$ as the root. Then construct a BFS tree in $S_{n,R}$ with $S_{n,RT}$ as the root omitting the vertex already traversed in $S_{n,L}$. Thus we obtain a tree rooted at $S_{n,LT}$. See Figure 5.

Output: For $n \geq 2$, the k-broadcast time of S_n; $b_k(S_n) = 2^{n-1}$, $k \geq 2$.

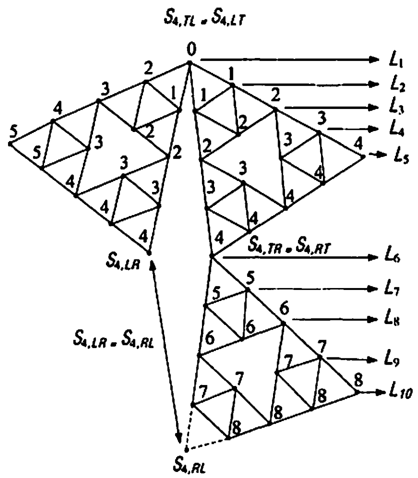

Figure 5: 3-broadcasting in S_4

Proof of Correctness:

Case 1: If $k = 2$; then the BFS tree has at most two children at each vertex. So the vertices in levels L_i, i = 1, 2, 3;..., 2^{n-1} +1 will receive the message from the orginator $S_{n,TT}$ in i - 1 units of time. For n ≥ 1, the BFS tree has 2^{n-1} +1 levels and hence the informed vertices can 2-broadcast message in time $(2^{n-1}+1) - 1 = 2^{n-1} = d_n$, where d_n is the diameter of S_n. See Figure 4.

Case 2: If $k = 3$; then the spanning tree has at most four children at a vertex and only in the first round the originator $S_{n,TL}$ broadcast to three vertices. In all the other rounds the informed vertices broadcast only to at most 2 uninformed vertices. In the first round itself, the message must be sent to the vertex adjacent to the originator $S_{n,TL}$ lying in the path passing through $S_{n,TR}$. So the vertices in levels L_i, i = 1, 2, 3;..., 2^{n-1} +1 will receive the message from the orginator in i - 1 units of time. For n ≥ 1, the spanning tree has 2^{n-1} +1 levels and the informed vertex can 3-broadcast message in time $(2^{n-1}+1) - 1 = 2^{n-1} = d_n$, where d_n is the diameter of S_n. See Figure 6.

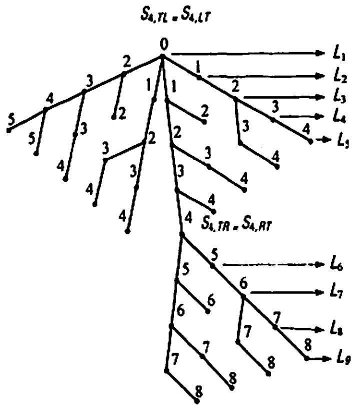

Figure 6: 3-broadcasting in spanning tree of S_4

Corollary 5.1: For $k \geq \Delta(Sn) = 4$, $b_4(S_n) = 2^{n-1} = d_n$, where d_n is the diameter of S_n.

6 Conclusion

In this paper, we obtain the broadcast time of the Sierpinski gasket graphs S_n. It is also proved that the k-broadcast time, $k \geq 2$ of S_n is its diameter. The broadcast time for Sierpinski derived architectures is under investigation.

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