Embedding Circulant and Grid Based Network

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Abstract

Graph embedding problems have gained importance in the field of interconnection networks for parallel computer architectures. In this paper, we prove that grid and cylinder are the subgraphs of certain circulant networks. Further, we present an algorithm to embed tori into certain circulant networks with dilation 2 and vice-versa.

1 Introduction

In the implementation of any algorithm, it is necessary that the code should be compilable and executable on any machine. However, it is far complicated in the case of parallel algorithms and machines. This is due to the fact that the properties of parallel machines are highly depending on their interconnection structure [1]. Thereby implementation of algorithms is often restricted to a certain class of networks. In order to overcome this dependency, it is necessary to emulate one network by another. Thereby implementation of algorithms is often restricted to a certain class of networks. In order to overcome this dependency, it is necessary to emulate one network by another.

Graph embedding is an important technique that maps a guest graph into a host graph, usually an interconnection network. Many applications

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Guest graph	Host graph	Dilation	Authors
Rectangular Grid	Square Grid	2	Melhem et.al. [5]
Complete Trees	Hypercube	2	Bezrukov [6]
Tori and Grids	Twisted cubes	1	Lai et.al. [7]
Hypercube	Extended Hypercube	2	Manuel et.al. [8]

Table 1: Dilation of Graph Embedding

such as architecture simulations and processor allocations can be modeled as graph embedding [2, 3, 4]. For example, architecture simulation can be modeled as embedding the guest graph into the host graph [2]. Two commonly and extensively studied cost measures of an embedding are the dilation and the congestion [2]. The dilation is defined as the maximum distance in H between two adjacent nodes in G. In general, embedding stretches source edges to paths in the host network. The dilation of an embedding is the maximum length of such paths taken over all source edges [5]. A embedding with a long dilation faces many problems, such as long communication delay, coupling problems and the existence of different types of uncontrolled noise. Therefore, a minimum dilation is a most desirable feature in network embedding. Some of them are listed in Table 1.

Link congestion is defined as the maximum number of paths over an edge in H, where every path represents an edge in G [9]. In general link congestion of an embedding is the maximum number of images of source edges passing through a host edge. It represents the maximum number of inter process communication source channels mapped on one physical host link. Link congestion should be low to minimize link or router buffers contention. Average congestion per node of an embedding is more important than maximum congestion. Graph embedding has been well studied for a number of networks [6, 2, 9, 5]. In general, the embedding problem is NP-complete [10].

The circulant network is a natural generalization of double loop network, which was first considered by Wong and Coppersmith [11]. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [12]. Every circulant graph is a vertex transitive graph [13]. Such graphs are highly desirable, because it allows the use of the same algorithms at each node of the network.

2 Preliminaries

In this section, we give the basic definitions and preliminaries related to embedding problems.

Definition 1. [2] An embedding $< f,p >$of a graph $G(V_G, E_G)$ into a graph $H(V_H, E_H)$ is defined by a mapping f from V_G to V_H, together with a mapping p that maps each edge $(u,v) \in E_G$ onto a path $P_f(u,v)$ in H that connects $f(u)$ and $f(v)$. The load on a node $v \in V_H$ is the number of nodes of G that are mapped onto v, the max-load of an embedding is the maximum load over all nodes of H. The expansion of an embedding f is the ratio of the number of vertices of H to the number of vertices of G.

In this paper, we consider embedding with expansion one and max-load one.

Definition 2. [2] If $e = (u,v) \in E_G$, then the length of $P_f(u,v)$ in H is called the dilation of the edge e. The maximal dilation over all edges of G is called the dilation of the embedding f and denote it by $dil_f(G,H)$. The dilation of G into H is defined as

$$dil(G,H) = min\, dil_f(G,H)$$

The minimum is taken overall embeddings f of G into H. The link congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $C_f(e)$ denote the number of edges (u,v) of G such that e is the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in H. In other words,

$$C_f(e) = |\{(u,v) \in E_G : e \in P_f(u,v)\}|$$

where $P_f(u,v)$ denotes the path between $f(u)$ and $f(v)$in H with respect to f.

Definition 3. [9] The average congestion of an embedding f of G into H is given by

$$AC_f(G,H) = \frac{1}{E(H)} \sum_{(u,v)\in E(G)} d_H(f(u), f(v) = \frac{1}{E(H)} \sum_{e\in E(H)} EC_f(e)$$

where $d_H(f(u), f(v)$ denotes the length of the path $P_f(u,v)$ in H. The average congestion of G into H is defined as

$$AC(G,H) = min\, AC_f(G,H)$$

where the minimum is taken overall embeddings f of G into H.

Definition 4. The $n \times m$ grid graph $M(n \times m)$, has vertex set

$$V(n, m) = \{(i, j) : 0 \leq i < n, 0 \leq j < m\}$$

and edge set

$$E_M(n, m) = \{(i, j), (i', j') : |i - i'| + |j - j'| = 1\}.$$

Let

$$Top_E(n, m) = \{(j, m - 1), (j, 0) : 0 \leq j < n\},$$

$$Side_E(n, m) = \{(n - 1, i), (0, i) : 0 \leq i < m\}.$$

Fat Cylinders $FC(n, m)$, Thin Cylinders $TC(n, m)$ and Tori $T(n, m)$, are graphs with the same vertex set $V(n, m)$ but respective edge sets:

$$E_{FC}(n, m) = E_M(n, m) \cup Side_E(n, m)$$

$$E_{TC}(n, m) = E_M(n, m) \cup Top_E(n, m)$$

$$E_T(n, m) = E_M(n, m) \cup Side_E(n, m) \cup Top_E(n, m)$$

Definition 5. [12] The undirected circulant graph $G(n, S)$, $S \subseteq \pm\{1, 2, \ldots, j\}$, $1 \leq j \leq \lfloor n/2 \rfloor$ is a graph with vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{(i, k) : |k - i| \equiv s(mod n), s \in S$. See Figure 3.1.

3 Embedding Grid Related Networks into Circulant Networks

In this section we prove that grid, fat cylinder and thin cylinder are subgraph of circulant graph. further, we embed tori into circulant graph with dilation 2.

From the definitions, we know that $M(n, m) \subset FC(n, m)/TC(n, m) \subset T(n, m)$. Hence we have the following result.

Lemma 1. *Minimum dilation of an embedding f of G into H*

$$dil(M(n, m), H) \leq dil(FC(n, m)/TC(n, m), H) \leq dil(T(n, m), H).$$

Theorem 1. *Let $m \geq n$ be an integers, then fat cylinder $FC(n, m)$ and thin cylinder $TC(n, m)$ are subgraph of the circulant graph $G(nm, \pm\{1, m\})$ and $G(nm, \pm\{1, n\})$ respectively.*

Algorithm 1 Dilation Algorithm A

Input: The torus $T(n,m), m \geq n$ and the circulant graph $G(nm, \pm\{1,m\})$.
Algorithm: Label the i^{th} row of $T(n,m)$ as $mi+1$, $mi+2$,...,$mi+m$ from left to right, where $0 \leq i < n$. Label the consecutive vertices $G(nm, \pm 1)$ in $G(nm, \pm\{1,m\})$ as $1, 2, \ldots, nm$ in the clockwise sense.
Output: An embedding f of $T(n,m)$ into $G(nm, \pm\{1,m\})$ given by $f(x) = x$ with dilation 2.

Proof. Label the i^{th} row of $FC(n,m)$ as $mi+1$, $mi+2$,...,$mi+m$ from left to right, where $0 \leq i < n$. Label the consecutive vertices $G(nm, \pm 1)$ in $G(nm, \pm\{1,m\})$ as $1, 2, \ldots, nm$ in the clockwise sense. Clearly any edge in the i^{th} row, $0 < i < n$ in $FC(n,m)$ is in $G(nm, 1)$ and any edge in the j^{th} column, $0 < j < m$ in $FC(n,m)$ is in $G(nm, m)$. Hence $FC(n,m)$ is a subgraph of the circulant graph $G(nm, \pm 1, m)$. In a similiar manner $TC(n,m)$ is a subgraph of the circulant graph $G(nm, \pm\{1,n\})$. □

Remark 1. Let $m \geq n$ be an integers, then grid $M(n,m)$ is a subgraph of the circulant graph $G(nm, \pm\{1,m\})$.

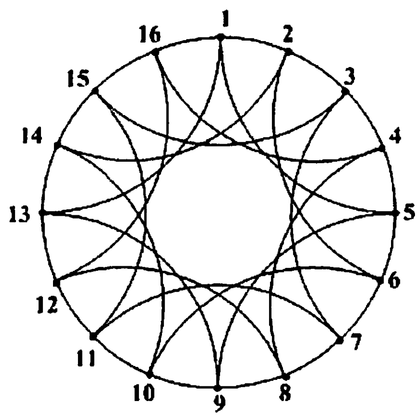

Figure 3.1: $G(16, \pm\{1,4\})$

The following theorem is an easy consequence of Dilation Algorithm A

Theorem 2. *Let $m \geq n$, G be a tori $T(n,m)$ and H be the a circulant graph $G(nm, \pm\{1,m\})$, Then $dil(G,H) = 2$.*

Proof. Let x and y be the label of any two adjacent vertex in torus $T(n,m)$ By Dilation Algorithm A $|x-y| = 1$,$|x-y| = m-1$ or $|x-y| = m$. Again,

Algorithm 2 Dilation Algorithm B

Input: The circulant graph $G(nm, \pm\{1, m\})$ and torus $T(n, m), m \geq n$.
Algorithm: Label the consecutive vertices $G(nm, \pm 1)$ in $G(nm, \pm\{1, m\})$ as $1, 2, \ldots, nm$ in the clockwise sense. Label the i^{th} row of $T(n, m)$ as $mi + 1$, $mi + 2$,...,$mi + m$ from left to right, where $0 \leq i < n$.
Output: An embedding f of $G(nm, \pm\{1, m\})$ into $T(n, m)$ given by $f(x) = x$ with dilation 2.

by the labeling, $|f(x) - f(y)| \subseteq \{1, m\}$ in circulant graph $G(nm, \pm\{1, m\})$. Thus dilation of an embedding f of $T(n, m)$ into $G(nm, \pm\{1, m\})$ given by $f(x) = x$ is 2. □

Remark 2. Let $m \geq n$, tori $T(n, m)$ is the subgraph of circulant graph $G(nm, \pm\{1, m - 1, m\})$.

4 Embedding Circulant Network into Extended Grid

In this section, we embed certain circulant networks into tori, grid with minimum dilation.

4.1 Embedding Circulant Network into Tori

In this section we embed the circulant graph $G(nm, \pm\{1, m\})$ and $G(nm, \pm\{1, m - 1, m\})$ into tori $T(n, m)$ with minimum dilation.

Theorem 3. *Let $m \geq n$, G be a circulant graph $G(nm, \pm\{1, m\})$ and tori $T(n, m)$ and H be the a , Then $dil(G, H) = 2$.*

Proof. Let x and y be the label of any two adjacent vertex in $G(nm, \pm\{1, m\})$. Then $|x-y| = 1$ or m. By Dilation Algorithm B $|f(x)-f(y)| \subseteq \{1, m-1, m\}$ thus dilation of an embedding f of $G(nm, \pm\{1, m\})$ into $T(n, m)$ is 2. □

Lemma 2. *Let G and H be a graph and $|E_G| = x$,$|E_H| = y$/, $x > y$ Then,*

$$dil(G, H) \geq 2,$$

$$AC(G, H) \geq \frac{2x - y}{x}.$$

Algorithm 3 Dilation Algorithm C

Input: The circulant graph $G(nm, \pm\{1, m-1, m\})$ and torus $T(n, m), m \geq n$.

Algorithm: Label the consecutive vertices $G(nm, \pm 1)$ in $G(nm, \pm\{1, m\})$ as $1, 2, \ldots, nm$ in the clockwise sense. Label the i^{th} row of $T(n, m)$ as $mi + 1$, $mi + 2, \ldots, mi + m$ from left to right, where $0 \leq i < n$.

Output: An embedding f of $T(n, m)$ into $G(nm, \pm\{1, m-1, m\})$ given by $f(x) = x$ with dilation and average congestion 2.

Theorem 4. *Let $m \geq n$, G be a circulant graph $G(nm, \pm\{1, m-1, m\})$ and H be a tori $T(n, m)$, Then $dil(G, H) = 2$ and $AC(G, H) = 2$.*

Proof. Let x and y be the label of any two adjacent vertex in $G(nm, \pm\{1, m-1, m\})$ By Dilation Algorithm B $|x - y| = 1$, $x - y| = m - 1$ or $|x - y| = m$. Again, by the labeling of tori $T(n, m)$, $P_f(x, y) \leq 2$. hence dilation of an embedding f of $G(nm, \pm\{1, m\})$ into $T(n, m)$ given by $f(x) = x$ is 2. Also $2nm$ edges mapped as edges and remaining nm edges mapped as path length 2, in other words, dilation and average congestion of embedding $G(nm, \pm\{1, m\})$ into $T(n, m)$ are both equel to 2. □

4.2 Embedding Circulant Network into Grid

The dilation and the wirelength problem are different in the sense that embedding that gives minimum dilation need not give minimum wirelength and vice-versa. In the literature there is no efficient method to compute the exact dilation of graph embedding [2, 5, 6, 7]. In 2012, Manual et. al., obtained a strategy to compute on lower bound for dilation using minimum wirelegth and formulated the result as IPS lemma [8]. In 2013, Rajan et. al., introduced new strategy called dilation lemma to compute the lower bound for dilation without using minimum wirelength [14].

Lemma 3. *[14] Let G be an r-regular graph of order n. Let H be a graph on n vertices that for $u \in V(H)$,$D_\delta(u) \neq \phi$, where $D_\delta(u)$ denotes the set of all diametrically opposite vertices of u in G. If $|D_\delta(u)| + |D_{\delta-1}(u)| + |D_{\delta-2}(u)| + \ldots + |D_{\delta-k}(u)| \geq n - r$, then the dilation of embedding G onto H is at least $\delta - k$, where $k = min_{u \in E(H)} k(u)$ and δ is a diameter of H.*

Theorem 5. *Let G be a circulant graph $G(nm, \pm\{1, 2, \ldots, \lfloor nm/2 \rfloor - 2\})$ and H be the grid $M(n, m), m \geq n$. Then $dil(G, H) = m + n - 4$.*

proof: By Lemma 3, $dil(G, H) \geq m = n - 4$ and by Dilation Algorithm D, $dil(G, H) \leq m + n - 4$. Hence $dil(G, H) = m + n - 4$.

Algorithm 4 Dilation Algorithm D

Input: The circulant graph $G(nm, \pm\{1, 2, \ldots, \lfloor nm/2 \rfloor - 2\})$ and grid $M(n, m), m \geq n$.

Algorithm: Label the consecutive vertices $G(nm, \pm 1)$ in $G(nm, \pm\{1, 2, \ldots, \lfloor nm/2 \rfloor - 2\})$ as $1, 2, \ldots, nm$ in the clock- wise sense. Label the vertices of degree 2 as $0, \lfloor \frac{nm}{4} \rfloor, \lfloor \frac{nm}{2} \rfloor, \lfloor \frac{3nm}{4} \rfloor$ in the clockwise sense in $M(n, m)$ beginning from the top leftmost corner of the grid. Label the open neighbourhood $N(0)$ of 0 as 1 and $nm - 1$, $N(\lfloor nm/4 \rfloor)$ as $\lfloor nm/4 \rfloor + 1$ and $\lfloor nm/4 \rfloor - 1$, $N(\lfloor nm/2 \rfloor)$ as $\lfloor nm/2 \rfloor + 1$ and $\lfloor nm/2 \rfloor - 1$ and $N(\lfloor 3nm/4 \rfloor)$ as $\lfloor 3nm/4 \rfloor + 1$ and $\lfloor 3nm/4 \rfloor - 1$.

Output: An embedding f of $G(nm, \pm\{1, 2, \ldots, \lfloor nm/2 \rfloor\})$ into grid $M(n, m)$ with dilation at most $m + n - 4$.

5 Conclusion

In this paper, we have obtained minimum dilation for embedding grid, cylinder and torus into certain circulant networks. Also, we have obtained minimum dilation of certain circulant networks into torus and grid. Finding minimum average congestion of embedding circulant network $G(nm, \pm\{1, 2, \ldots, j\})$ into grid, cylinder and torus is under investigation.

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