Total Edge Irregularity Strength of Hexagonal Networks

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ABSTRACT

Given a graph $G(V, E)$ a labeling $\partial: V \cup E \to \{1, 2 \ldots k\}$ is called an edge irregular total k-labeling if for every pair of distinct edges uv and xy, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. The minimum k for which G has an edge irregular total k-labeling is called the total edge irregularity strength of G. In this paper we examine the hexagonal network which is a well known interconnection network and obtain its total edge irregularity strength.

1. Introduction

A basic feature for a system is that its components are connected together by physical communication links to transmit information according to some pattern. Moreover, it is undoubted that the power of a system is highly dependent upon the connection pattern of components in the system. A connection pattern of the components in a system is called an interconnection network, or network, of the system. Topologically, an interconnection network can essentially depict structural feature of the system. In other words, an interconnection network of a system provides logically a specific way in which all components of the system are connected. *Interconnection networks* (also known as permutation networks) are used for regular interconnections of processors in a parallel computer. In a direct interconnection network, nodes represent processors while edges indicate connections between processors for direct message exchange.

Interconnection networks are becoming increasingly pervasive in many different applications with the operational costs and characteristics of these networks depending considerably on the application. For some applications, interconnection networks have been studied in depth for decades. This is the case for telephone networks, computer networks (telecommunication) and backplane buses. However in the last fifteen years we have seen rapid evolution of the interconnection network technology that is currently being infused into a new generation of multiprocessor systems.

Some interconnection network topologies are designed and some borrow from nature. For example hypercubes, complete binary trees, butterflies and torus networks are some of the *designed architectures*. Grids, hexagonal networks, honeycomb networks and diamond networks, for instance, bear resemblance to atomic or molecular lattice structures. They are called *natural architectures*.

It is known that there exist three regular plane tessellations, composed of the same kind of regular polygons: triangular, square and hexagonal. They are the basis for the designs of direct interconnection networks with highly competitive overall performance.

The advancement of large scale integrated circuit technology has enabled the construction of complex interconnection networks. Graph theory provides a fundamental tool for designing and analyzing such networks. Graph Theory and Interconnection Networks provides a thorough understanding of these interrelated topics. One of the main objectives of researchers is the application of Graph Theory to the study and design of interconnection networks. The problems usually considered include the analysis of characteristic parameters of the network (diameter, connectivity measures, etc.), the study of special substructures (rings, trees, etc), routing algorithms, modularity properties and specific networks (symmetric networks, permutation networks, loop networks, etc).

Motivated by the notion of the irregularity strength of a graph introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba [2] in 1988 and various kinds of other total labelings, Baca, Jendrol, Miller and Ryan [1] introduced the *total edge irregularity strength* of a graph as follows. For a graph $G(V, E)$ a labeling ∂: $V \cup E \rightarrow \{1, 2 \ldots k\}$ is called an *edge irregular total k-labeling* if for every pair of distinct edges uv and xy, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. The minimum k for which G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G. The total edge irregularity strength of G is denoted by $tes(G)$. In this paper we study the total edge irregularity strength of hexagonal networks.

2. Hexagonal Network

Hexagonal networks belong to the family of networks modeled by planar graphs. These networks are based on triangular plane tessellation, or the partition of a plane into equilateral triangles. Hexagonal networks were studied in a variety of contexts [8]. They are applied in chemistry to model benzenoid hydrocarbons [12] in image processing, computer graphics [9] , wireless and interconnection networks. An addressing scheme for processors and corresponding routing and broadcasting algorithms for hexagonal interconnection network were proposed by Chen et al. [7]. The performance of hexagonal networks was further studied in [8,10]. The design, implementation, and evaluation of a distributed real-time architecture called HARTS (hexagonal architecture for real-time systems) are discussed in [8].

Cellular communications have experienced an explosive growth recently. To accommodate more subscribers, the size of cells must be reduced to make more efficient use of the limited frequency spectrum. This, in turn, increases the difficulty level of location management. A number of location management schemes have been reported in literature. Cellular networks are commonly designed as hexagonal networks, where nodes serve as base stations (BSs) to which mobile users must connect to make or receive phone calls.

Definition: Hexagonal network $HX(r)$, where r is the number of vertices on one side of the hexagon[8] has six vertices of degree three which we call as *corner vertices*. There is exactly one vertex v at distance $r - 1$ from each of the corner vertices. This vertex is called the *centre* of $HX(r)$ and is represented by O.

The vertex set V is partitioned into sets inducing concentric cycles around O. Call vertex O as level 1, the first cycle around O as level 2 denoted by C_2 and so on and the last cycle farthest from O as level r denoted by C_r. The level i cycle has $6(i\text{-}1)$ vertices, $i \geq 2$. The subgraph induced by the vertices of C_r and C_{r+1} in $HX(r)$ is called a *circular channel* and is denoted by $CC(r+1)$ (see Figure 1).

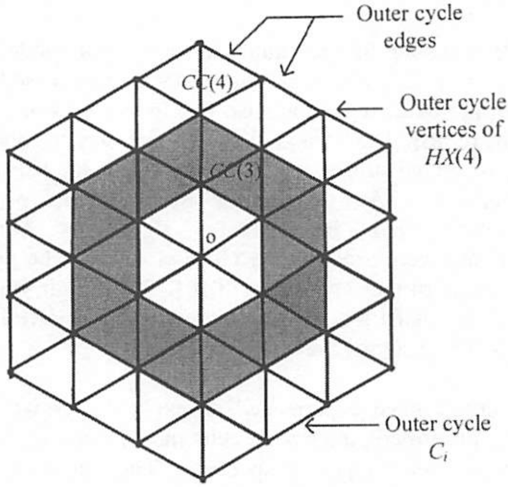

Fig 1: Channels of *HX*(4)

The number of vertices and edges of $HX(r)$ are $3r^2$-$3r$+1 and $9r^2$-$15r$+6 respectively. The diameter is $2r$-2. The edges connecting the cycle C_{r-1} and the cycle C_r, $2 \leq r \leq s$, where s is finite, are called spoke edges. The three lines at the point O which are at mutual angle of 120 degrees between any two of them are considered as three axes as shown in Figure 2. The lines parallel to the three axes are called as α_0, β_0 and γ_0 lines correspondingly.

We begin with few known results on $tes(G)$.

Theorem 1: Every multigraph $G = (V, E)$ without loops of order n, size m, and maximum degree $0 < \Delta < \frac{10^{-3} m}{\sqrt{8n}}$ satisfies $tes(G) = \left\lceil \frac{E+2}{3} \right\rceil$.

Theorem 2: Every graph $G = (V, E)$ of order n, minimum degree $\delta > 0$, and maximum degree Δ such that $\frac{\Delta}{\delta} < \frac{10^{-3}\sqrt{n}}{4\sqrt{2}}$ satisfies $tes(G) = \left\lceil \frac{m+2}{3} \right\rceil$.

Theorem 3: For every integer $\Delta \geq 1$, there is some $n(\Delta)$ such that every graph $G = (V, E)$ without isolated vertices with order $n \geq n(\Delta)$, size m, and maximum degree at most Δ satisfies $tes(G) = \left\lceil \frac{m+2}{3} \right\rceil$.

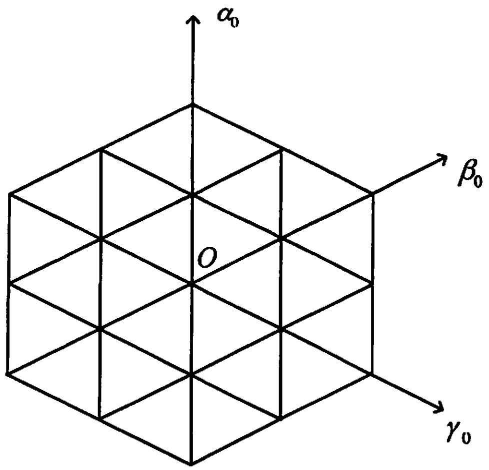

Fig 2: *HX*(3) with α_0, β_0 and γ_0 lines

We now give an algorithm to prove that for the Hexagonal networks *HX*(*n*), the bound on *tes* given in Theorem 1 is sharp. In the proof, by 'edge sum label' of an edge (*u*,*v*) in *HX*(*n*) we mean the sum of the labels of vertices *u*, *v* and the edge (*u*,*v*).

Lemma ***tes*(*HX*(4)) = 31.**

Proof : Let *HX*(4) be labeled as in Figure 3. It is easy to check that *tes*(*HX*(4)) = 31.

Procedure *tes*(*HX*(*r*))

Input:

r-dimensional hexagonal network, *HX*(*r*), $r \geq 5$.

Algorithm:

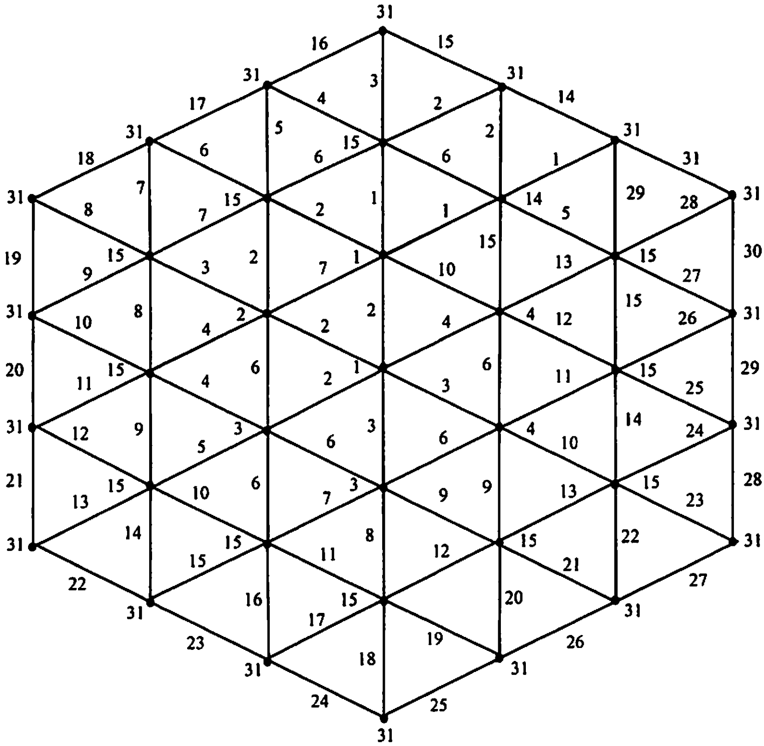

Fig 3: ***tes*****(*****HX*****(4)) = 31**

Let $k(r) = \left\lceil \frac{9r^2 - 15r + 8}{3} \right\rceil$.

(1) Label the vertices and edges of $HX(4)$ as in Lemma.

(2) In this labeling we note that the outer cycle vertices in $HX(4)$ as seen in Figure 3 are labeled as $k(4)$.

Having labeled $HX(r)$, label $HX(r+1)$, $r \geq 4$ as follows:

(a) We begin with labeling all outer cycle vertices of C_{r+1} in $HX(r+1)$ as $k(r+1)$.

(b) Label the outer cycle edges in the anticlockwise direction about the β_0 line (see Figure 2) with consecutive numbers beginning with $k(r) - 1$.

(c) Label the spoke edges e_i starting from the edge along the β_1 line with consecutive numbers in the anticlockwise direction of $CC(r+1)$ beginning with $l(e_i) = 3tes(HX(r)) + i - (l(u_i) + l(v_i))$, where $e_i = (u_i, v_i)$ with vertex labels $l(u_i)$ and $l(v_i)$.

End Procedure *tes*(*HX*(*r*)).

Output: $tes(HX(r)) = \left\lceil \frac{9r^2-15r+8}{3} \right\rceil$.

Proof of Correctness:

We prove the result by induction on *r*. By actual verification, it is easy to check that the labels given in Figure 3 yield *tes*(*HX*(4)) = 31. This proves the result when *r* = 4. Assume the result for *HX*(*r*). Consider *HX*(*r*+1). Since the labeling of *HX*(*r*) is an edge irregular *k*-labeling, it is clear that the labeling of *HX*(*r*+1) by step 2 (b) and (c) is also an edge irregular *k*-labeling. Since the labels are consecutive, the edge sum labels of *HX*(*r*+1) are also consecutive integers which are clearly distinct.□

Theorem Let *HX*(*r*) denote a *r*-dimensional hexagonal network. Then $tes(HX(r)) = \left\lceil \frac{9r^2-15r+8}{3} \right\rceil, r \geq 4$.

Conclusion

In this paper, we have proved that Hexagonal Networks are total edge irregular and we have also obtained the total edge irregularity strength of this network. The problem of determining the total edge irregularity strength is under investigation for certain other architectures like Honeycomb, Honeycomb Torus and Circulant Networks.

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