Acyclic Kernel numberof Biregular Graphs

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ABSTRACT

A *kernel* in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that (u, v) is an arc of D. The problem of existence of a kernel is NP – complete for a general digraph. In this paper we introduce the acyclic kernel problem of an undirected graph G and solve it in polynomial time for uniform theta graph and even quasi – uniform theta graph.

1. Introduction

A *kernel* [7] in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that (u, v) is an arc of D. The concept of kernels in digraphs was introduced in different ways [13,18].
Kernels arise naturally in the analysis of certain two-person positional games. Von Neumann and Morgenstern [18] were the first to introduce kernels when describing winning positions in 2 person games. They proved that any directed acyclic graph has a unique kernel. Not every digraph has a kernel and if a digraph has a kernel, this kernel is not necessarily unique. All odd length directed cycles and most tournaments have no kernels [2, 3].
If D is finite, the decision problem of the existence of a kernel is NP- complete for a general digraph [6, 17], and for a planar digraph with in degrees ≤ 2, out degrees ≤ 2 and degrees ≤ 3 [8]. It is further known that a finite digraph all of whose cycles have even length has a kernel [15], and that the question of the number of kernels is NP- complete even for this restricted class of digraphs [16]. The concept of kernel is widespread and appears in diverse fields such as logic, computational complexity, artificial intelligence, graph theory, combinatorics and coding theory [2, 3]. Efficient routing among a set of mobile hosts is one of the most important functions in ad hoc wireless networks. Dominating set based routing in networks with unidirectional links is proposed in [1, 12]. A new interest for these studies arose due to their applications in finite model theory.

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Indeed, variants of kernel are the best properties to provide counter examples of $0 - 1$ laws in fragments of monadic second order logic [11].
In this paper we view the kernel problem from a different perspective. In the literature, only the existence of kernel of a digraph D and its application are extensively studied [14]. Our aim in this paper is to investigate all acyclic orientations of an undirected graph G and determine the acyclic kernel number of G.

2. Kernel in Oriented Graphs

An orientation of an undirected graph G is an assignment of exactly one direction to each of the edges of G. There are $2^{|E|}$ orientations for G. Let $O_x(G)$ denote the set of all orientations of G. For an orientation $O \in O_x$, let $G(O)$ denote the directed graph with orientation O and whose underlying graph is G.
An orientation O of an undirected graph G is said to be an acyclic orientation if it contains no directed cycles. Let $O_a(G)$ denote the set of all acyclic orientations of G.
Definition 1 [7]: A *kernel* in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that $(\overrightarrow{u,v})$ is an arc of D. u is called the *tail* and v is called the *head* of the arc $(\overrightarrow{u,v})$.
Definition 2: Let $D(V, E)$ be any directed graph. The in-neighborhood of a vertex v, denoted by $N^-(v)$ is the set of tail vertices with head vertex v. The out-neighborhood of a vertex v, denoted by $N^+(v)$ is the set of head vertices with tail vertex v. $|N^+(v)|$ is called the *out-degree* of v and $|N^-(v)|$ is called the *in-degree* of v.
Definition 3 [14]: The *kernel number* κ_x of G is defined as $\kappa_x(G) = min\{\kappa(O): O \in O_x(G)\}$ where $\kappa(O) = min\{|K|: K \text{ is a kernel of } G(O)\}$.
Definition 4 [14]: The *acyclic kernel number* κ_a of G is defined as $\kappa_a(G) = min\{\kappa(O): O \in O_a(G)\}$ where $\kappa(O) = min\{|K|: K \text{ is a kernel of } G(O)\}$.
The acyclic kernel problem: The *acyclic kernel problem* of an undirected graph G is to find a kernel K of $G(O)$ for some acyclic orientation O of G such that $|K| = \kappa_a$.
In this paper we construct a minimum paired dominating set and a minimum total dominating set for infinite diamond lattice. We also demonstrate that the total domatic number of infinite diamond lattice is 4.

3. Topological Ordering

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Graph labeling has a wide range of applications.

For instance, we can find labeling of graphs showing up in x-rays, crystallography, coding theory, radar, astronomy, circuit design and communication network addressing [4,5]. Their theoretical applications are numerous, not only within the theory of graphs but also in other areas of mathematics such as combinatorial number theory, linear algebra and group theory admitting a given type of labeling [9].

Definition 5: Let G be a graph. A labeling f from the vertex set $V(G)$ to $\{1,2,\dots,|V|\}$ is said to induce an *ascent graph* $G(f)$ if $E(G(f))$ satisfies the following condition: $(\overrightarrow{u,v}) \in E(G(f))$ if and only if $f(u) < f(v)$. The labeling is called a *topological ordering* [23].

Theorem 1:[10] A digraph G is an ascent graph if and only if it is acyclic.

In the following section we obtain a lower bound for the acyclic kernel number of biregular graphs. In the next section we prove that the lower bound is tight.

4. Lower Bound on κ_a for Biregular Graphs

Definition 6: A graph G is said to be biregular if there exist integers r_1 and r_2 such that for every vertex V in G, degree of V is either r_1 or r_2.

Lemma 1: Let n, k, r_1 and r_2 be integers such that $(r_1+1) \geq 2(r_2+1)$ and $n \geq (r_1+1)k$. Then for any $t \leq k$, $t + \left\lceil \frac{n-(r_1+1)t}{r_2+1} \right\rceil \geq k + \left\lceil \frac{n-(r_1+1)k}{r_2+1} \right\rceil$.

Proof: If $t = k$, there is nothing to prove.

If $t < k$, we claim that $\left\lceil \frac{n-(r_1+1)t}{r_2+1} \right\rceil - \left\lceil \frac{n-(r_1+1)k}{r_2+1} \right\rceil > k - t$. We are given that $\frac{(r_1+1)}{(r_2+1)} \geq 2$.

Case (i): $(r_2+1)/(n-(r_1+1)t)$ and $(r_2+1)/(n-(r_1+1)k)$

$$\text{L.H.S} = \frac{n-(r_1+1)t}{r_2+1} - \frac{n-(r_1+1)k}{r_2+1}$$
$$= \frac{n}{r_2+1} - \frac{(r_1+1)t}{r_2+1} - \frac{n}{r_2+1} + \frac{(r_1+1)k}{r_2+1}$$
$$= \frac{(r_1+1)(k-t)}{r_2+1} > (k-t).$$

Case (ii): $(r_2+1) \nmid (n-(r_1+1)t)$ and $(r_2+1)/(n-(r_1+1)k)$

$$\text{L.H.S} = \frac{n-(r_1+1)t}{r_2+1} + \alpha - \frac{n-(r_1+1)k}{r_2+1}, 0 < \alpha < 1$$
$$= \frac{n}{r_2+1} - \frac{(r_1+1)t}{r_2+1} - \frac{n}{r_2+1} + \frac{(r_1+1)k}{r_2+1} + \alpha$$
$$= \frac{(r_1+1)(k-t)}{r_2+1} + \alpha > (k-t) + \alpha > (k-t).$$

Case (iii): $(r_2+1)/(n-(r_1+1)t)$ and $(r_2+1) \nmid (n-(r_1+1)k)$

Since $\frac{(r_1+1)}{(r_2+1)} \geq 2$, let $\frac{(r_1+1)}{(r_2+1)} = 1 + x,\ x \geq 1$.

$$\text{L.H.S} = \frac{n-(r_1+1)t}{r_2+1} - \left(\frac{n-(r_1+1)k}{r_2+1} + \beta\right), 0 < \beta < 1$$

$$= \frac{n}{r_2+1} - \frac{(r_1+1)t}{r_2+1} - \frac{n}{r_2+1} + \frac{(r_1+1)k}{r_2+1} - \beta$$

$$= \frac{(r_1+1)(k-t)}{r_2+1} - \beta \;= (1+x)(k-t) - \beta$$

$$= (k-t) + x(k-t) - \beta > (k-t).$$

Since $(k-t) > \beta \Rightarrow x(k-t) > \beta x \Rightarrow x(k-t) - \beta > 0$.

Case (iv): $(r_2+1) \nmid (n-(r_1+1)t)$ and $(r_2+1) \nmid (n-(r_1+1)k)$

L.H.S $= \frac{n-(r_1+1)t}{r_2+1} + \alpha - \left(\frac{n-(r_1+1)k}{r_2+1} + \beta\right), 0 < \beta < 1, 0 < \alpha < 1$

$$= \frac{n}{r_2+1} - \frac{(r_1+1)t}{r_2+1} - \frac{n}{r_2+1} + \frac{(r_1+1)k}{r_2+1} - \beta + \alpha$$

$$= \frac{(r_1+1)(k-t)}{r_2+1} - \beta \; + \alpha$$

Subcase (i): $\alpha = \beta$

L.H.S $= \frac{(r_1+1)(k-t)}{r_2+1} \quad > (k-t)$.

Subcase (ii): $\alpha > \beta$

Here, $\alpha - \beta > 0$.

L.H.S $> \frac{(r_1+1)(k-t)}{r_2+1} \quad > (k-t)$.

Subcase (iii): $\alpha < \beta$

Here, $\alpha - \beta < 0$.

L.H.S $= \frac{(r_1+1)(k-t)}{r_2+1} - \gamma, 0 < \gamma < 1$

Let $\frac{(r_1+1)}{(r_2+1)} = 1 + x,\; x \geq 1$.

Thus L.H.S $= (1+x)(k-t) - \gamma = (k-t) + x(k-t) - \gamma > k - t$.

Since $(k-t) > \gamma \Rightarrow x(k-t) > \gamma x \Rightarrow x(k-t) - \gamma > 0$.

We thus have the following theorem.

Theorem 2: Let G be a graph on n vertices such that every vertex is of degree either r_1 or r_2 and $(r_1+1) \geq 2(r_2+1)$. Let k be the number of vertices of degree r_1, subject to the condition $n \geq (r_1+1)k$. Then $\kappa_a \geq k + \left\lceil \frac{n-(r_1+1)k}{r_2+1} \right\rceil$.

Proof: Every vertex v in G of degree r_1, has r_1 incoming edges in $G(O)$. Suppose all the vertices of degree r_1 are in the kernel and dominate $(r_1+1)k$ vertices in $G(O)$, then the kernel members for the remaining $n-(r_1+1)k$ vertices are to be determined. Thus $\kappa_a \geq k + \left\lceil \frac{n-(r_1+1)k}{r_2+1} \right\rceil$.

5. Kernel in Uniform Theta Graph and Even Quasi – Uniform Theta Graph

Definition 7: A generalized theta graph $\Theta(n, l)$ or simply a theta graph with n vertices has two vertices N and S of degree l such that every other vertex is of degree 2 and lies in one of the l paths joining the vertices N and S. The two vertices N and S are called the North Pole and the South Pole respectively. A path between the North Pole and the South Pole is called a longitude and is denoted by L. $\Theta(n, l)$ has l number of longitudes denoted by $L_1, L_2, \dots, L_l$. See Figure 1.

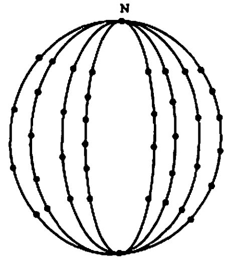

Figure 1: A Generalised Theta graph $\Theta(40,8)$.

Definition 8: A theta graph $\Theta(n, l)$ is uniform if $|L_1| = |L_2| = \cdots = |L_l|$ and quasi-uniform if $|L_1| \geq |L_2| = \cdots = |L_l|$. A quasi - uniform theta graph is said to be even or odd according as $|L_1| + |L_2|$ is even or odd. Clearly a uniform theta graph is an even quasi - uniform theta graph. See Figure 2.

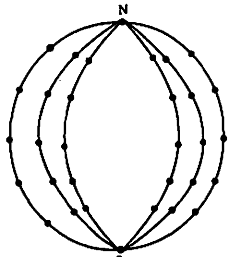

Figure 2: A Uniform Theta graph $\Theta(32,6)$.

We proceed to prove that the acyclic kernel problem is polynomially solvable for uniform theta graph and even quasi - uniform theta graph.

Theorem 3: Let G be an uniform theta graph $\Theta(n, l)$ with $m = 3i - 1, i \geq 2$ number of vertices on each longitude. Then $\kappa_a = 2 + l\left\lceil\frac{m-2}{3}\right\rceil$.

Proof: By Theorem 2, $\kappa_a \geq 2 + \left\lceil\frac{n-2(l+1)}{3}\right\rceil = 2 + \left\lceil\frac{ml+2-2(l+1)}{3}\right\rceil = 2 + \left\lceil\frac{ml-2l}{3}\right\rceil = 2 + l\left\lceil\frac{m-2}{3}\right\rceil$

$\Rightarrow \kappa_a \geq 2 + l\left\lceil\frac{m-2}{3}\right\rceil$.

We proceed to prove the existence of a kernel K of G with $2 + l\left\lceil\frac{m-2}{3}\right\rceil$.

Since each of the l number of longitudes contains m internal vertices, we have $n = ml + 2$. Name the vertices of $\Theta(n, l)$ as follows:

The North Pole and the South Pole are named as N and S. Name the vertices on the longitude L_k as $v_1^k, v_2^k, \dots, v_m^k$ where $1 \le k \le l$. Orient each longitude unidirectionally from North Pole to South. See Figure 3.

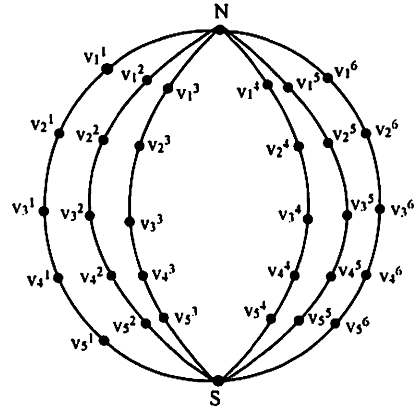

Figure 3: Naming of vertices in an Uniform Theta Graph $\Theta(32,6)$.

Define a labeling $f: V(G) \to \{1,2,\dots,lm+2\}$ by $f(N) = n, f(S) = n-1$ and $f(v_{3j}^k)$ where $1 \le j \le \frac{m-2}{3}$ and $1 \le k \le l$ be labeled arbitrarily as $n-2, n-3, \dots, n-\left(2+l\left\lceil\frac{m-2}{3}\right\rceil-2\right), n-\left(2+l\left\lceil\frac{m-2}{3}\right\rceil-1\right)$. . Let the remaining $n-\left(2+l\left\lceil\frac{m-2}{3}\right\rceil\right)$ vertices be labeled $1,2,\dots,n-\left(2+l\left\lceil\frac{m-2}{3}\right\rceil\right)$ arbitarily by f. Orient the edge (u,v) in G from U to V if $f(u) < f(v)$. This assigns a topological ordering O of $V(G)$ such that the vertices labeled $n-\left(\left(2+l\left\lceil\frac{m-2}{3}\right\rceil\right)-1\right), n-\left(\left(2+l\left\lceil\frac{m-2}{3}\right\rceil\right)-2\right), \dots, n$ have in-degree 2 or l. By Theorem 1, $G(O)$ is acyclic.

We claim that the set K of vertices labeled $n-\left(\left(2+l\left\lceil\frac{m-2}{3}\right\rceil\right)-1\right), n-\left(\left(2+l\left\lceil\frac{m-2}{3}\right\rceil\right)-2\right), \dots, n$ constitute an acyclic kernel set K of G.

For $u \in V \setminus K$ and $v \in K$, we have $f(u) < f(v)$. Therefore, if $(u,v) \in E(G)$ then $\overrightarrow{(u,v)} \in E(G(O))$. The set K constituting vertices named N, S and v_{3j}^k where $1 \le j \le \frac{m-2}{3}$ and $1 \le k \le l$ form an independent set of vertices in G, where G is an uniform theta graph $\Theta(n,l)$ with $m = 3i-1, i \ge 2$ number of vertices on each longitude. Thus K is a kernel of G and hence $\kappa_a = 2 + l\left\lceil\frac{m-2}{3}\right\rceil$. See Figure 4.

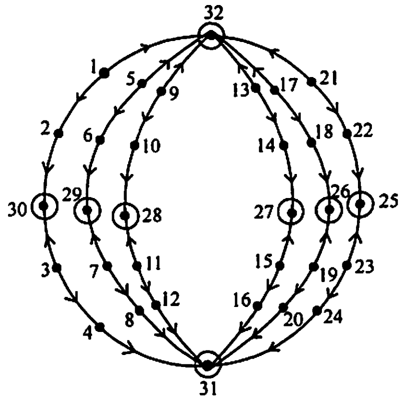

Figure 4: Encircled vertices form a kernel in $\Theta(32,6)$.

Theorem 4: Let G be an even quasi - uniform theta graph $\Theta(n,l)$ with $m = 3i-1, i \geq 2$ number of vertices on each longitude. Then $\kappa_a = 2 + l\left\lceil\frac{m-2}{3}\right\rceil$.

Proof: The proof is similar to that of Theorem 3.

We now have the following theorem.

Theorem 5: The acyclic kernel problem for uniform theta graph and even quasi - uniform theta graph is polynomially solvable.

Conclusion

We have discussed the acyclic kernel number for oriented graphs and also estimated the lower bound for the acyclic kernel number for biregular graphs. In this paper, we have proved that the acyclic kernel problem for uniform theta graph and even quasi – uniform theta graph is polynomially solvable. Further the acyclic kernel number for Butterfly Networks and Benes Networks are under investigation.

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