Chromatic Layout Number of Paths and Cycles

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Abstract

A linear layout, or simply a layout, of an undirected graph $G = (V, E)$ with $n = |V|$ vertices is a bijective function $\phi : V \rightarrow \{1, 2, \ldots, n\}$. A k-coloring of a graph $G = (V, E)$ is a mapping $\kappa : V \rightarrow \{c_1, c_2, \ldots, c_k\}$ such that no two adjacent vertices have the same color. A graph with a k-coloring is called a k-colored graph. A colored layout of a k-colored graph (G, κ) is a layout ϕ of G such that for any $u, x, v \in V$, if $(u, v) \in E$ and $\phi(u) < \phi(x) < \phi(v)$ then $\kappa(u) \neq \kappa(x)$. Given a k-colored graph (G, κ), the problem of deciding whether there is a colored layout ϕ of (G, κ) is NP-complete. In this paper we introduce the concept of chromatic layout of G and determine the chromatic layout number for paths and cycles.

1 Introduction

Sequence reconstruction problem occurs in Molecular Biology at different levels of DNA mapping. It is currently not possible to sequence large parts of DNA or proteins at a time [6, 7, 12, 14, 15, 17]. Therefore the sequence is cut into smaller parts, which are called fragments or colors, which can then be sequenced. However, the order of the different fragments in the large sequence is lost during the fragmentation processes. The reconstruction of this order is called sequence reconstruction and a colored layout of a graph plays an important role in the process.

2 Preliminaries

A proper vertex coloring of a graph G is an assignment of colors to the vertices of G, so that adjacent vertices are colored differently. A k-coloring of a graph $G = (V, E)$ is a mapping $\kappa : V \to \{c_1, c_2, \ldots, c_k\}$ such that no two adjacent vertices have the same color. Let $\kappa(v)$ denote the color of vertex v in G. A k-colored graph is a graph together with a k-coloring. If V_i is the set of all vertices in G colored c_i, then each nonempty set $V_i, i = 1, 2, \ldots, k$ is called a color class and the sets $V_1, V_2, \ldots, V_k$ produce a partition of $V(G)$. Since no two adjacent vertices are assigned the same color, each nonempty color class V_i is an independent set of vertices of G [8].

A graph G is k-colorable if there exists a k-coloring of G. The minimum positive integer k for which G is k-colorable is the chromatic number of G and is denoted by $\chi(G)$. The chromatic number of G is therefore the minimum number of independent sets into which $V(G)$ can be partitioned. A graph G with chromatic number k is a k-chromatic graph. A graph is k-colorable if and only if $\chi(G) \leq k$ [8].

A linear layout, or simply a layout, of an undirected graph $G = (V, E)$ with $n = |V|$ vertices is a bijective function $\phi : V \to \{1, 2, \ldots, n\}$. A linear layout is also called a linear ordering [1], a linear arrangement [18], a numbering [9] or a labeling [16] of the vertices of a graph. The set of all layouts is denoted by $\Phi(G)$.

Definition 2.1. *A colored layout of a k-colored graph (G, κ) is a layout ϕ of G such that for any $u, x, v \in V$, if $(u, v) \in E$ and $\phi(u) < \phi(x) < \phi(v)$, then $\kappa(u) \neq \kappa(x)$. ϕ is called a k-colored layout of graph G* [10].

Given a k-colored graph (G, κ), deciding whether there is a colored layout ϕ of (G, κ) is called a Colored Layout Problem [3, 10]. In 1998, Alvarez [2] showed that the colored layout problem is NP-complete for caterpillars with hairs of length at most 2 and the problem can be solved in NC for caterpillars with hairs of length at most 1. In 2001 [4], it was shown that the problem is still NP-complete for four colored caterpillars with unbounded hair length.

Given a k-colored graph (G, κ), deciding whether there is an edge superset E' such that the graph $G' = (V, E')$ is an interval graph and κ is still a proper coloring of G' is called the Interval Colored Graph Problem [4, 5, 10]. The Interval Colored Graph Problem is a special case of the Interval Sandwich Problem [13], and has received attention as a simplified model for reconstructing the ordering in physical DNA mapping problems [12]. Moreover, the problem was shown to be NP-complete [12, 14] and it is polynomial time equivalent to the colored layout problem [2, 11].

In this paper we introduce chromatic layout of G as follows:

Definition 2.2. *A k-chromatic layout of G is a k-colored layout of G such that for any t-colored layout of G, $t \geq k$. k is called the chromatic layout number of G and is denoted by $\chi_L(G)$.*

3 Main Results

In this section, we obtain a lower bound for $\chi_L(G)$ and prove that the bound is sharp for paths and cycles.

Theorem 3.1. *Let G be a graph. Then $\chi_L(G) \geq \chi(G)$.*

Proof. Since a colored layout of G requires G to be a k-colored graph, $\chi_L(G) \geq k$. But $\chi(G) \leq k$. Hence $\chi_L(G) \geq \chi(G)$. □

Theorem 3.2. *Let G be a graph and H be a subgraph of G. Then $\chi_L(H) \leq \chi_L(G)$.*

Proof. Consider any chromatic layout ϕ of G and remove the label $\phi(v)$ if v is not in $V(H)$. Continue the process until no such vertex exists. The resultant layout is a colored layout of the subgraph H with at most $\chi_L(G)$ colors. Hence $\chi_L(H) \leq \chi_L(G)$. □

It is clear that for a connected graph G, $\chi_L(G) = 1$ if and only if G is isomorphic to P_1.

3.1 The chromatic layout of Paths

Theorem 3.3. *Let P_n denote the path on n vertices. Then $\chi_L(P_n) = 2, n \geq 2$.*

Proof. Let $P_n : v_1, v_2, v_3, \ldots, v_n, n \geq 2$. Since $\chi(P_n) = 2$, $\chi_L(P_n) \geq 2$. We define a layout ϕ of P_n as $\phi(v_i) = i$. In any chromatic coloring of P_n, $\kappa(v_i) = \kappa(v_{i+2})$, $i = 1, 2, \ldots, n$. For every edge $(u, v) \in E(P_n)$, either $\phi(u) = \phi(v) + 1$ or $\phi(v) = \phi(u) + 1$. Hence ϕ is a chromatic layout of P_n and $\chi_L(P_n) = 2$. □

Lemma 3.4. *In any chromatic layout of $P_n : v_1, v_2, v_3, \ldots, v_n, n \geq 5$, $\phi(v_i) \neq 1$, for $i \neq 1, 2, n-1, n$.*

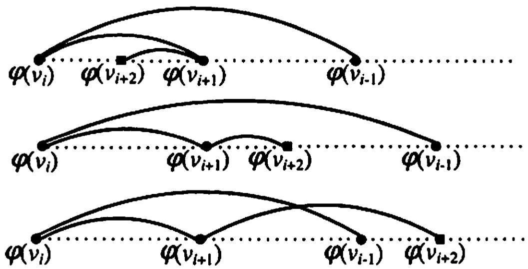

Figure 1: Illustration of Lemma 3.4 : (a) Case 1, (b) Case 2 and (c) Case 3.

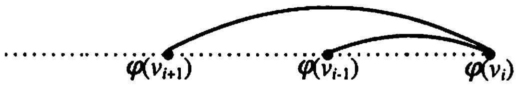

Figure 2: Illustration of Lemma 3.5.

Proof. Suppose there exists a v_i such that $\phi(v_i) = 1$, for $i \neq 1, 2, n-1, n$. Without loss of generality, let $\phi(v_{i+1}) < \phi(v_{i-1})$ (See Figure 1).

Case 1: $\phi(v_{i+2}) < \phi(v_{i+1})$. Then $\phi(v_i) < \phi(v_{i+2}) < \phi(v_{i+1})$ and $\kappa(v_i) = \kappa(v_{i+2})$.

Case 2: $\phi(v_{i+1}) < \phi(v_{i+2}) < \phi(v_{i-1})$. Then $\phi(v_i) < \phi(v_{i+2}) < \phi(v_{i-1})$ and $\kappa(v_i) = \kappa(v_{i+2})$.

Case 3: $\phi(v_{i+2}) > \phi(v_{i-1})$. Then $\phi(v_{i+1}) < \phi(v_{i-1}) < \phi(v_{i+2})$ and $\kappa(v_{i+1}) = \kappa(v_{i-1})$.

All three cases lead to a contradiction to the definition of colored layout. Hence $\phi(v_i) \neq 1$, for $i \neq 1, 2, n-1, n$. ☐

Lemma 3.5. *In any chromatic layout of $P_n : v_1, v_2, v_3, \ldots, v_n, n \geq 5$, $\phi(v_i) \neq n$, for $i \neq 1, 2, n-1, n$.*

Proof. Suppose there exists a v_i such that $\phi(v_i) = n$, for $i \neq 1, 2, n-1, n$. Without loss of generality, let $\phi(v_{i+1}) < \phi(v_{i-1})$. We have $\phi(v_{i+1}) < \phi(v_{i-1}) < \phi(v_i)$ and $\kappa(v_{i+1}) = \kappa(v_{i-1})$ (See Figure 2), a contradiction to the definition of colored layout. Hence $\phi(v_i) \neq n$, for $i \neq 1, 2, n-1, n$. ☐

Lemma 3.6. *In any chromatic layout of $P_n : v_1, v_2, v_3, \ldots, v_n, n \geq 5$, the adjacency is maintained for the vertices v_i, $i \neq 1, 2, n-1, n$.*

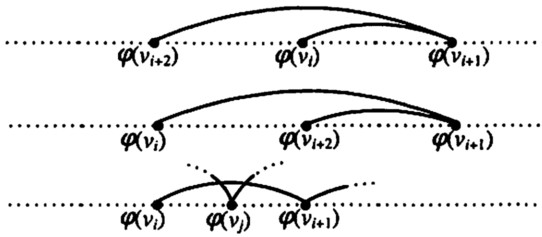

Figure 3: Illustration of Lemma 3.6: (a) Case 1, (b) Case 2 and (c) Case 3.

Proof. Suppose the adjacency is not maintained. Then we have the following three cases.

Case 1: $\phi(v_{i+2}) < \phi(v_i) < \phi(v_{i+1})$ and $\kappa(v_{i+2}) = \kappa(v_i)$.

Case 2: $\phi(v_i) < \phi(v_{i+2}) < \phi(v_{i+1})$ and $\kappa(v_i) = \kappa(v_{i+2})$.

Case 3: $\phi(v_i) < \phi(v_j) < \phi(v_{i+1})$, $j \neq i, i+1$ then either $\kappa(v_i) = \kappa(v_j)$ or $\kappa(v_j) = \kappa(v_{i+1})$.

All the cases contradict the definition of colored layout. See Figure 3. □

Lemma 3.7. *In any chromatic layout of $P_n : v_1, v_2, v_3, \ldots, v_n, n \geq 5$, if $\phi(v_i) \leq 2$, $i = 1, 2$ then $\phi(v_j) = j$, $j = n-1, n$ and if $\phi(v_i) \leq 2$, $i = n-1, n$ then $\phi(v_j) = n - j + 1$, $j = 1, 2$.*

Proof. **Case 1**: Given $\phi(v_i) \leq 2$, $i = 1, 2$, by Lemma 3.5, $\phi(v_j) \neq n$, for $j \neq 1, 2, n-1, n$. If $\phi(v_n) < \phi(v_{n-1})$, then $\phi(v_{n-2}) < \phi(v_n) < \phi(v_{n-1})$ and $\kappa(v_{n-2}) = \kappa(v_n)$ giving a contradiction to the definition of colored layout. Hence $\phi(v_j) = j$, $j = n-1, n$.

Case 2: Given $\phi(v_i) \leq 2$, $i = n-1, n$, by Lemma 3.5, $\phi(v_j) \neq n$, for $j \neq 1, 2, n-1, n$. If $\phi(v_1) < \phi(v_2)$, we have $\phi(v_3) < \phi(v_1) < \phi(v_2)$ and $\kappa(v_3) = \kappa(v_1)$ giving a contradiction to the definition of colored layout. Hence $\phi(v_j) = n - j + 1$, $j = 1, 2$. □

Theorem 3.8. *There are four chromatic layouts to any path $P_n : v_1, v_2, v_3, \ldots, v_n$, $n \geq 3$.*

Proof. By Lemma 3.4 and Lemma 3.5, we have $\phi(v_i) < \phi(v_{i+1})$ or $\phi(v_{i+1}) < \phi(v_i)$, $3 \leq i \leq n-2$ and by Lemma 3.6 and Lemma 3.7, we have the following 4 layouts.

Case 1: $\phi(v_1) < \phi(v_2) < \phi(v_i) < \phi(v_{i+1}) < \phi(v_{n-1}) < \phi(v_n)$

Case 2: $\phi(v_2) < \phi(v_1) < \phi(v_i) < \phi(v_{i+1}) < \phi(v_{n-1}) < \phi(v_n)$

Case 3: $\phi(v_n) < \phi(v_{n-1}) < \phi(v_{i+1}) < \phi(v_i) < \phi(v_2) < \phi(v_1)$

Case 4: $\phi(v_{n-1}) < \phi(v_n) < \phi(v_{i+1}) < \phi(v_i) < \phi(v_2) < \phi(v_1)$ □

3.2 The chromatic layout of Cycles

Theorem 3.9. *Let C_n be a cycle on n vertices. Then $\chi_L(C_n) = 3, n \geq 3$.*

Proof. When n is odd, $\chi(C_n) = 3$ and hence by Theorem 1, $\chi_L(C_n) \geq 3$. When n is even, $\chi(C_n) = 2$ and hence by Theorem 1, $\chi_L(C_n) \geq 2$.

Suppose that $\chi_L(C_n) = 2$, when n is even, consider C_n with vertices $v_1, v_2, v_3, \ldots, v_n$. Consider a linear layout ϕ of C_n, where the vertices $v_1, v_3, \ldots, v_{n-1}$ are assigned the color c_1 and the vertices $v_2, v_4, \ldots, v_n$ are assigned the color c_2. Suppose ϕ is a colored layout of C_n with $\chi_L(C_n) = 2$. Without loss of generality, let $\phi(v_i) = 1$, $\phi(v_{i+1}) < \phi(v_{i-1})$.

Case 1: $\phi(v_{i+1}) < \phi(v_{i-1}) < \phi(v_{i+2})$ with $\kappa(v_{i+1}) = \kappa(v_{i-1})$, which is a contradiction to the definition of colored layout.

Case 2: $\phi(v_i) < \phi(v_{i+2}) < \phi(v_{i-1})$ with $\kappa(v_i) = \kappa(v_{i+2})$ which is a contradiction to the definition of colored layout.

Both the cases lead to a contradiction to the assumption that $\chi_L(C_n) = 2$ and hence $\chi_L(C_n) \geq 3$.

Therefore we redefine ϕ on C_n as $\phi(v_i) = i$ with $\kappa(v_1) = c_3$, $\kappa(v_i) = c_1$, $i = 3, 5, \ldots$ and $\kappa(v_i) = c_2$, $i = 2, 4, \ldots$. Since the adjacency is maintained in the layout for all the edges except for the edge $v_1 v_n$ and the color c_3 is assigned to the vertex v_1 alone, ϕ is a colored layout. Therefore ϕ is a chromatic layout and $\chi_L(C_n) = 3$. □

4 Conclusion

In this paper we have obtained lower bound for $\chi_L(G)$ and proved that the bound is sharp for paths and cycles.

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