Bounds on the 3-rainbow Domination

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Abstract

The k-rainbow domination is a variant of the classical domination problem in graphs and is defined as follows: Given an undirected graph $G = (V, E)$ and a set of k colours numbered $1, 2, \ldots k$, we assign an arbitrary subset of these colours to each vertex of G. If a vertex is assigned the empty set, then the union of colour sets of its neighbours must be k colours. This assignment is called the k-rainbow dominating function of G. The minimum sum of numbers of assigned colours over all vertices of G, is called the k-rainbow domination number of G. In this paper, we present some bounds on the 3-rainbow domination number of circulant networks and grid network.

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1 Introduction

A subset S of the vertex set $V(G)$ of a graph G is called a *dominating set* if every vertex in $V(G)\backslash S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Let G be a graph and $v \in V(G)$. The open neighbourhood of v is the set $N(v) = \{u \in V(G)/uv \in E(G)\}$ and its closed neighbourhood is the set $N[v] = N(v) \cup \{v\}$. Let $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$ be a function that assigns to each vertex of G, a set of colours chosen from the power set of $\{1, 2, ..., k\}$. If for each vertex $v \in V(G)$ with $f(v) = \phi$, we have $\bigcup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$, then the function f is called a *k-rainbow dominating function* (kRDF) of G. The weight of the function, denoted by $w(f)$ is defined as $w(f) = \Sigma_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF is called the *k-rainbow domination number* of G and is denoted by $\gamma_{rk}(G)$. In this paper we consider the 3-rainbow domination, defined as

$f : V(G) \to \mathcal{P}(\{1,2,3\})$ such that for each vertex $v \in V(G)$ with $f(v) = \phi$, we have $\bigcup_{u \in N(v)} f(u) = \{1,2,3\}$. Such a function f is called a *3-rainbow dominating function* (3RDF) and minimum weight of such function is called the *3-rainbow domination number* of G and is denoted by $\gamma_{r3}(G)$.

When a graph is used to model objects or locations which can exchange some resources along its edges, the study of ordinary domination is an optimization problem to determine the minimum number of locations to store the resource in such a way that each vertex either has the resource or is adjacent to one where the resource resides. Imagine a computer network in which some of the computers will be servers and the others are clients. There are k distinct resources and we wish to determine the optimum set of servers, each hosting a nonempty subset of the resources so that any client is directly connected to a subset of servers that together contain all k resources. Assuming all resources have the same cost, we seek to minimize the total number of copies of the k resources. This model leads naturally to the notion of *k-rainbow domination.*

2 Overview of the Paper

The rainbow domination problem has been widely studied recently. Earlier Hartell and Rall investigated the k-rainbow domination for $\gamma(G \times K_k)$ [7]. Bresar, Hennning and Rall [1,2] initiated the study of k-rainbow domination of a graph G and showed that this parameter coincides with the ordinary domination of the cartesian product of G with the complete graph K_k, that is $\gamma_{rk}(G) = \gamma(G \times K_k)$. Rainbow domination problem was studied in generalized Petersen graphs [3,11,12], in trees [2,6], in cartesian product graphs [1]. In [3] Bresar and Kraner Sumenjak showed that the 2-rainbow domination problem is NP complete even when restricted to chordal graphs or bipartite graphs. Later the results were generalized for the case of k-rainbow domination problem by Chang, Wu and Zhu [6] and determined the exact values of paths, cycles and suns. For any generalized Petersen graph $P(n,k)$ where n and k are relatively prime numbers for $k < n$, they showed that $\lceil \frac{4n}{5} \rceil \leq \gamma_{r2}(P(n,k)) \leq n$. They also suspected that there are infinite families of graphs that achieve the bound n and conjectured that $GP(2k+1,k)$ with $k \geq 2$ and $GP(n,3)$ with $n \geq 7$ and $n (mod 3) \neq 0$ are two candidate families. However both of these two candidate families were disproved to achieve the bound n. In [11] Tong Lin Yang and Luo obtained the 2-rainbow domination number of $GP(n,2)$.

The following lemma gives a lower bound for the 3-rainbow domination of any graph G.

Lemma 2.1. [7] *Let G be a graph. Then for any $k \geq 2$,* $\min\{|V(G)|, \gamma(G) + k - 2\} \leq \gamma_{rk}(G) \leq k\gamma(G)$.

In this paper we focus on 3-rainbow domination number for the circulant networks $G(n; \pm\{1,2\})$ and $G(n; \pm\{1,3\})$. Further, we consider the grid network $G_{m,n}$.

3 Circulant Network

Two important parameters in network design are maximum node degree (or number of connections) and incremental extendibility (increase in the number of nodes). Circulant graphs, a family of Cayley graphs, allow for incremental extendibility with the number of connections to each node remaining constant in the networks that they model. Circulant graphs have a vast number of uses and applications to telecommunication network, VLSI design, parallel and distributed computing.

The circulant graphs are an important class of topological structures of interconnection networks. They are symmetric with simple structures and easy extendibility. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [5]. The term circulant comes from the nature of its adjacency matrix; a matrix is circulant if all its rows are periodic rotations of the first one. Circulant matrices have also been employed for designing binary codes. Circulant graphs also constitute the basis for designing certain data alignment networks for complex memory systems. The circulant network is a natural generalization of double loop network, which was first considered by Wong and Coppersmith [13].

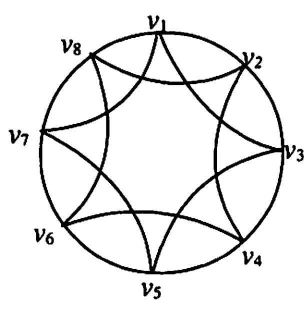

Figure 1: Circulant Network $G(8; \pm\{1,2\})$

Definition: A *circulant undirected graph*, denoted by $G(n; \pm\{1,2,...,j\})$, $1 < j \leq \lfloor \frac{n}{2} \rfloor$, $n \geq 3$ is defined as an undirected graph consisting of the vertex set $V = \{0,1,...,n-1\}$ and the edge set $E = \{(i,j) : |j-i| \equiv s \pmod n)$, $s \in \{1,2,...,j\}$.
It is also clear that $G(n; \pm 1)$ is an undirected cycle C_n and $G(n; \pm\{1,2,..., \lfloor \frac{n}{2} \rfloor\})$ is a complete graph K_n. We observe that $C_n = G(n; \pm 1)$ is a subgraph of

$G(n;\pm\{1,2,...,j\})$ for every j, $1 < j \leq \lfloor\frac{n}{2}\rfloor$. For our convenience, we denote the vertex set V as $\{v_1, v_2, ..., v_n\}$. See Figure 1.

3.1 The 3-rainbow domination number of Circulant network $G(n;\pm\{1,2\})$

To find $\gamma_{r3}(G(n;\pm\{1,2\}))$, we shall begin with the lemmas already known.

Lemma 3.1. [8] *Let G be an r-regular graph on n vertices, then $\gamma(G) \geq \lceil\frac{n}{r+1}\rceil$.*

Lemma 3.2. [8] *Let G be the circulant graph $G(n;\pm\{1,2\})$, then $\gamma(G) \geq \lceil\frac{n}{5}\rceil$.*

The following theorem gives an upper bound for the 3-rainbow domination number of $G(n;\pm\{1,2\})$.

Theorem 3.3. *Let $G(n;\pm\{1,2\})$ be the Circulant graph of dimension n. Then*

$$\lceil\tfrac{n}{5}\rceil + 1 \leq \gamma_{r3}(G(n;\pm\{1,2\})) \leq \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod{12}; \\ \lceil\frac{n}{2}\rceil + 1, & \text{if } n \equiv 1,3,9 \pmod{12}; \\ \lfloor\frac{n}{2}\rfloor + 1, & \text{if } n \not\equiv 0,1,3,9 \pmod{12}. \end{cases}$$

Proof. Label the vertices of $G(n;\pm\{1,2\})$ as $\{v_1, v_2, ..., v_n\}$. We define a function f such that $f : V(G(n;\pm\{1,2\})) \to \mathcal{P}(\{1,2,3\})$ is a 3-rainbow dominating function of G.

Case 1: $n \equiv 0 \pmod{12}$

Let $f(v_i)=\begin{cases} \{1\}, & \text{if n} \equiv 2,7 \pmod{12}; \\ \{2\}, & \text{if n} \equiv 6,11 \pmod{12}; \\ \{3\}, & \text{if n} \equiv 3,10 \pmod{12}; \\ \phi, & \textit{otherwise}. \end{cases}$, for $1 \leq i \leq n-1$ and $f(v_n) = \phi$.

Case 2: $n \equiv 1,2,6,9 \pmod{12}$

Let $f(v_i)=\begin{cases} \{1\}, & \text{if n} \equiv 2,7 \pmod{12}; \\ \{2\}, & \text{if n} \equiv 6,11 \pmod{12}; \\ \{3\}, & \text{if n} \equiv 3,10 \pmod{12}; \\ \phi, & \textit{otherwise}. \end{cases}$, for $1 \leq i \leq n-2$ and $f(v_{n-1}) = \{1\}, f(v_n) = \{2\}$.

Case 3: $n \equiv 3 \pmod{12}$

Let $f(v_i)=\begin{cases} \{1\}, & \text{if n} \equiv 2,7 \pmod{12}; \\ \{2\}, & \text{if n} \equiv 6,11 \pmod{12}; \\ \{3\}, & \text{if n} \equiv 3,10 \pmod{12}; \\ \phi, & \textit{otherwise}. \end{cases}$, for $1 \leq i \leq n-3$ and $f(v_{n-2}) = \{1\}, f(v_{n-1}) = \{1\} f(v_n) = \{2\}$.

Case 4: $n \equiv 4,5,8,11 \pmod{12}$

Let $f(v_i)=\begin{cases} \{1\}, & \text{if n} \equiv 2,7 \pmod{12}; \\ \{2\}, & \text{if n} \equiv 6,11 \pmod{12}; \\ \{3\}, & \text{if n} \equiv 3,10 \pmod{12}; \\ \phi, & \textit{otherwise.} \end{cases}$, for $1 \leq i \leq n-1$ and $f(v_n) = \{2\}$.

Case 5: $n \equiv 7 \pmod{12}$

Let $f(v_i)=\begin{cases} \{1\}, & \text{if n} \equiv 2,7 \pmod{12}; \\ \{2\}, & \text{if n} \equiv 6,11 \pmod{12}; \\ \{3\}, & \text{if n} \equiv 3,10 \pmod{12}; \\ \phi, & \textit{otherwise.} \end{cases}$, for $1 \leq i \leq n-1$ and $f(v_n) = \{1\}$.

Case 6: $n \equiv 10 \pmod{12}$

Let $f(v_i)=\begin{cases} \{1\}, & \text{if n} \equiv 2,7 \pmod{12}; \\ \{2\}, & \text{if n} \equiv 6,11 \pmod{12}; \\ \{3\}, & \text{if n} \equiv 3,10 \pmod{12}; \\ \phi, & \textit{otherwise.} \end{cases}$, for $1 \leq i \leq n-2$ and $f(v_{n-1}) =$ $\{3\}, f(v_n) = \{2\}$. See Figure 2

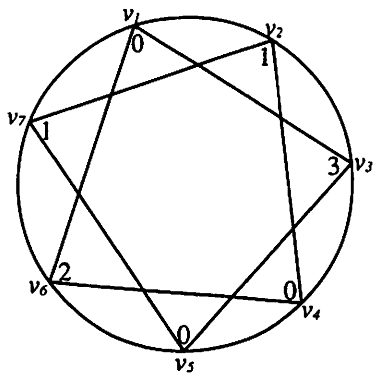

Figure 2: Circulant Network $G(7; \pm\{1,2\})$

Then f is a 3-rainbow dominating function of G. In all the above cases, the weight of f is $w(f) = \begin{cases} \frac{n}{2}, & \text{if n} \equiv 0 \pmod{12}; \\ \lceil \frac{n}{2} \rceil + 1, & \text{if n} \equiv 1,3,9 \pmod{12}; \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{if n} \not\equiv 0,1,3,9 \pmod{12}. \end{cases}$ This implies that, $\gamma_{r3}(G) \leq w(f)$ and from Lemma 2.1, $\gamma_{r3}(G) \geq \gamma(G)+1$. Hence the required result follows from Lemma 3.1 and Lemma 3.2.

□

Proof of correctness: Since the neighbourhood of every v_i with $f(v_i) = \phi$ is such that $\bigcup_{u \in N(v)} f(u) = \{1,2,3\}$, the function yields a 3-rainbow dominating function.

3.2 The 3-rainbow domination number of Circulant network $G(n; \pm\{1,3\})$

To find γ_{r3} of Circulant network $G(n; \pm\{1,3\})$, we begin with the following lemma.

Lemma 3.4. [9] *For any integer* $n \geq 6$,

$$\gamma(G(n; \pm\{1,3\})) \leq \begin{cases} \lceil \frac{n}{5} \rceil + 1, & \text{if } n \equiv 4 \pmod 5; \\ \lceil \frac{n}{5} \rceil, & \text{if } n \not\equiv 4 \pmod 5. \end{cases}$$

The following theorem gives an upper bound for the 3-rainbow domination number of $G(n; \pm\{1,3\})$.

Theorem 3.5. *Let $G(n; \pm\{1,3\})$ be the Circulant graph of dimension n. Then*

$$\gamma(G(n; \pm\{1,3\})) + 1 \leq \gamma_{r3}(G(n; \pm\{1,3\})) \leq \begin{cases} \lceil \frac{n}{2} \rceil + 1, & \text{if } n \equiv 4 \pmod 6; \\ \lceil \frac{n}{2} \rceil, & \text{if } n \not\equiv 4 \pmod 6. \end{cases}$$

Proof. We label the vertices of $G(n; \pm\{1,3\})$ as $\{v_1, v_2, ..., v_n\}$ and define a function f such that $f : V(G(n; \pm\{1,3\})) \to \mathcal{P}(\{1,2,3\})$ is a 3-rainbow dominating function of G.

Case 1: $n \equiv 4 \pmod 6$

Let $f(v_i)= \begin{cases} \{1\}, & \text{if n} \equiv 2 \pmod 6; \\ \{2\}, & \text{if n} \equiv 0 \pmod 6; \\ \{3\}, & \text{if n} \equiv 4 \pmod 6; \\ \phi, & \textit{otherwise}. \end{cases}$, for $1 \leq i \leq n-1$ and $f(v_n) = \{2,3\}$.

Case 2: $n \not\equiv 4 \pmod 6$

Let $f(v_i)= \begin{cases} \{1\}, & \text{if n} \equiv 2 \pmod 6; \\ \{2\}, & \text{if n} \equiv 0 \pmod 6; \\ \{3\}, & \text{if n} \equiv 4 \pmod 6; \\ \phi, & \textit{otherwise}. \end{cases}$, for $1 \leq i \leq n-1$ and $f(v_n) = \{2\}$. See Figure 3.

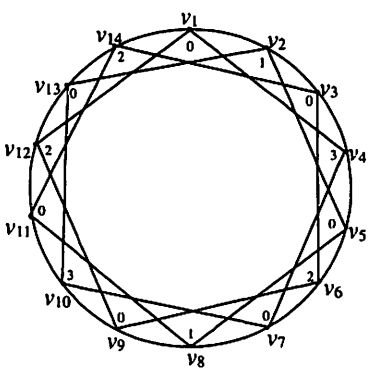

Figure 3: Circulant Network $G(14; \pm\{1,3\})$

From the proof of Theorem 3.3, we can easily observe that the function defined above is 3-rainbow dominating function of G. The weight of f is $w(f) = \begin{cases} \lceil \frac{n}{2} \rceil + 1, & \text{if n} \equiv 4 \pmod 6; \\ \lceil \frac{n}{2} \rceil, & \text{if n} \not\equiv 4 \pmod 6. \end{cases}$ Since $\gamma_{r3}(G) \leq w(f)$, the theorem follows.

□

4 Grid Network

In more recent time, grid graphs have been used to model a variety of routing problems in street networks. Berge [4] relates the problem to keep all the points in a network under surveillance by a set of radar stations. Consequently, the importance of studying the graph-theoretic properties of grid has attracted more interest.

Definition: An $m \times n$ *grid graph* G has the vertex set $V = \{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ with two vertices v_{ij} and $v_{i'j'}$ being adjacent if $i = i'$ and $|j - j'| = 1$ or if $j = j'$ and $|i - i'| = 1$ and is denoted by $G_{m,n}$. The $m \times n$ grid graph can also be presented as a Cartesian product $P_m \times P_n$ of a path of length $m - 1$ and a path of length $n - 1$.

To begin, we start with few known results on domination number of grid network mentioned in [10].

Lemma 4.1. [10] *Let G be the $2 \times n$ grid graph, then $\gamma(G) = \lfloor \frac{n+2}{2} \rfloor$.*

Lemma 4.2. [10] *Let G be the $3 \times n$ grid graph, then $\gamma(G) = \lfloor \frac{3n+4}{4} \rfloor$.*

Theorem 4.3. *Let G be the grid graph of dimension $2 \times n$. Then for any, $n \geq 4$, $\lfloor \frac{n+2}{2} \rfloor + 1 \leq \gamma_{r3}(G_{2,n}) \leq n + 2$.*

Proof. Label the vertices of $G_{2,n}$ as v_{ij}, $1 \leq i \leq 2, 1 \leq j \leq n$. We define a function f such that $f : V(G_{2,n}) \to \mathcal{P}(\{1,2,3\})$ is a 3-rainbow dominating function of G.

For $1 \leq j \leq n-1$, $f(v_{1,j}) = \begin{cases} \{1\}, & \text{if j} \equiv 1 \pmod 6; \\ \{2\}, & \text{if j} \equiv 3 \pmod 6; \\ \{3\}, & \text{if j} \equiv 5 \pmod 6; \\ \phi, & \textit{otherwise}. \end{cases}$, and

$$f(v_{1,n}) = \begin{cases} \{1\}, & \text{if n} \equiv 0,1 \pmod 6; \\ \{2\}, & \text{if n} \equiv 2,3 \pmod 6; \\ \{3\}, & \text{if n} \equiv 4,5 \pmod 6. \end{cases}$$

Also for $1 \leq j \leq n-1$, $f(v_{2,j}) = \begin{cases} \{2,3\}, & \text{if j=2}; \\ \{1\}, & \text{if j} \equiv 4 \pmod 6; \\ \{2\}, & \text{if j} \equiv 0 \pmod 6; \\ \{3\}, & \text{if j} \equiv 2 \pmod 6, j \neq 2; \\ \phi, & \textit{otherwise}. \end{cases}$, and

$$f(v_{2,n})=\begin{cases} \{1\}, & \text{if n} \equiv 3,4 \pmod 6; \\ \{2\}, & \text{if n} \equiv 0,5 \pmod 6; \\ \{3\}, & \text{if n} \equiv 1,2 \pmod 6. \end{cases}$$ See Figure 4.

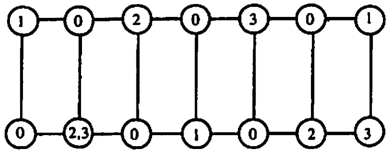

Figure 4: Grid $G_{2,7}$

Thus the above labeling of the function f yields a 3-rainbow dominating function of G. The weight of f is $w(f) = n + 2$, $n \geq 4$. Hence $\gamma_{r3}(G) \leq w(f)$ and also satisfies the relation $\gamma_{r3}(G) \geq \gamma(G) + 1$. This completes the proof. □

Proof of correctness: Since the neighbourhood of every v_i with $f(v_i) = \phi$ is such that $\bigcup_{u \in N(v)} f(u) = \{1, 2, 3\}$, the function yields a 3-rainbow dominating function.

Theorem 4.4. *Let G be the grid graph of dimension $3 \times n$. Then for any, $n \geq 4$,* $\lfloor \frac{3n+4}{4} \rfloor + 1 \leq \gamma_{r3}(G_{3,n}) \leq \begin{cases} \frac{3n}{2} + 2, & \text{if } n \text{ is even;} \\ \lceil \frac{3n}{2} \rceil, & \text{if } n \text{ is odd.} \end{cases}$

Proof. Label the vertices of $G_{3,n}$ as v_{ij}, $1 \leq i \leq 3, 1 \leq j \leq n$. We define a function f such that $f : V(G_{3,n}) \to \mathcal{P}(\{1, 2, 3\})$ is a 3-rainbow dominating function of G.

Case 1: n **is odd**

Let $f(v_{i,j})=\begin{cases} \{1\}, & \text{if j} \equiv 1 \pmod 4,\ i = 1 \text{ and } j \equiv 3 \pmod 4,\ i = 3; \\ \{2\}, & \text{if } j \equiv 1 \pmod 4,\ i = 3 \text{ and } j \equiv 3 \pmod 4,\ i = 1; \\ \{3\}, & \text{if j} \equiv 0 \pmod 2,\ i = 2; \\ \phi, & otherwise. \end{cases}$, for $1 \leq j \leq n$. Then f is a 3-rainbow dominating function of G. Also the weight of f is $w(f) = \lceil \frac{3n}{2} \rceil$ if $n \geq 4$.

Case 2: n **is even**

Let $f(v_{i,j})=\begin{cases} \{1\}, & \text{if j} \equiv 1 \pmod 4,\ i = 1 \text{ and } j \equiv 3 \pmod 4,\ i = 3; \\ \{2\}, & \text{if } j \equiv 1 \pmod 4,\ i = 3 \text{ and } j \equiv 3 \pmod 4,\ i = 1; \\ \{3\}, & \text{if j} \equiv 0 \pmod 2,\ i = 2; \\ \phi, & otherwise. \end{cases}$, for $1 \leq j \leq n - 1$ and $f(v_{2,n}) = \{1, 2, 3\}$. Then f is a 3-rainbow dominating function of G and the weight of f is $w(f) = \frac{3n}{2} + 2$ if $n \geq 4$.

This implies that $\gamma_{r3}(G) \leq w(f)$. By using a similar argument as in Theorem 4.3 we obtain the required result. □

Theorem 4.5. *Let G be a grid graph. Then $\gamma_{r3}(G_{4i+1,4j+1}) \leq 8ij + 2i + 2j + 1$ for i and j are integers and $1 \leq i \leq j$.*

Proof. To achieve this upper bound, we consider 3-rainbow domination of $G_{5,5}$ first. Let Pattern A be a colour assignment of $G_{5,5}$ which consists of $v_{1,1}, v_{2,4}, v_{4,2}, v_{5,5}$ with a colour set $\{1\}$, vertices $v_{1,3}, v_{3,1}, v_{3,3}, v_{3,5}, v_{5,3}$ with a colour set $\{2\}$ and vertices $v_{1,5}, v_{2,2}, v_{4,4}, v_{5,1}$ with a colour set $\{3\}$. Let Pattern B be a colour assignment of $G_{5,5}$ which consist of $v_{1,5}, v_{2,2}, v_{4,4}, v_{5,1}$ with a colour set $\{1\}$, vertices $v_{1,3}, v_{3,1}, v_{3,3}, v_{3,5}, v_{5,3}$ with a colour set $\{2\}$ and vertices $v_{1,1}, v_{2,4}, v_{4,2}, v_{5,5}$ with a colour set $\{3\}$. Figure 5(a) and 5(b) shows Pattern A and B, respectively. It is easy to verify that both Pattern A and B are 3-rainbow dominations of $G_{5,5}$.

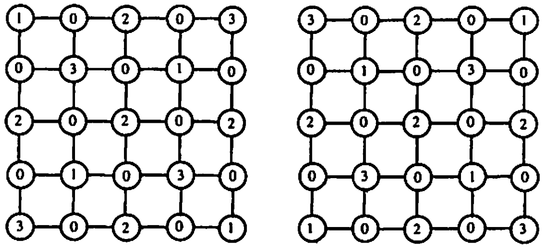

Figure 5(a): Pattern A Figure 5(b): Pattern B

Next, we consider 3-rainbow domination of $G_{5,4j+1}$ for $j \geq 2$. We give a colour assignment which consists of alternate Pattern A and B and the last row of k^{th} Pattern is overlapping with the first row of $(k+1)^{th}$ Pattern for $1 \leq k < j$. The colour assignment of $G_{4i+1,4j+1}$ for $j \geq i \geq 2$ is similar to the above discussion. For $i \geq 2$, we give a colour assignment consisting of alternate Pattern A and B and the last column of k^{th} Pattern is overlapping with the first column of $(k+1)^{th}$ Pattern for $1 \leq k < i$. Sum of the numbers of assigned colours over all vertices of $G_{4i+1,4j+1}$ is $\lceil \frac{(4i+1)(4j+1)}{2} \rceil = 8ij + 2i + 2j + 1$. □

After considering all possible values of m and n we obtain the following result:

Theorem 4.6. *The 3-rainbow domination number of $G_{m,n}$ satisfies,*

$$\gamma_{r3}(G_{m,n}) \leq \begin{cases} \lceil \frac{mn}{2} \rceil, & \text{if } m \text{ and } n \text{ odd;} \\ \frac{mn}{2} + 2, & \text{otherwise.} \end{cases}$$

5 Conclusion

In this paper we find an upper bound for the 3-rainbow domination number of circulant networks $G(n; \pm\{1,2\})$, $G(n; \pm\{1,3\})$ and grid networks. Finding 3-rainbow domination number for other interconnection networks is quite challenging.

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