On γ-Labeling of the Almost-Bipartite Graph $(P_m \Box P_n) + \hat{e}$

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Abstract

In 2004, Blinco et al [1] introduced γ-labeling. A function h defined on the vertex set of a graph G with n edges is called γ-labeling if,

(i) h is a ρ-labeling of G,

(ii) G admits a tripartition (A, B, C) with $C = \{c\}$ and there exist $\bar{b} \in B$ such that $(\bar{b}, c)$ is the unique edge with the property that $h(c) - h(\bar{b}) = n$,

(iii) for every edge $(a, v) \in E(G)$ with $a \in A$, $h(a) < h(v)$.

In [1] they have also proved a significant result on graph decomposition that if a graph G with n edges admits a γ-labeling then the complete graph K_{2cn+1} can be cyclically decomposed into $2cn + 1$ copies of the graph G, where c is any positive integer.

Motivated by the result of Blinco et al [1], in this paper we prove that the well known almost-bipartite graph, the grid with an additional edge, $(P_m \Box P_n) + \hat{e}$, admits γ-labeling. Further, we discuss a related open problem.

Keywords: Gamma labeling, Almost-bipartite graph, Grid plus one edge.

2010 Mathematics Subject Classification Number: 05C78, 05C76.

1 Introduction

In an attempt to settle the Ringel's conjecture that if T is any tree with m edges then the complete graph K_{2m+1} can be decomposed into $2m + 1$

copies of T, Rosa in his classical paper [5] introduced a series of labelings called α, β and ρ-valuation.

Let G be a graph with n edges. A ρ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, 2n\}$ such that $\{min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\}/(u, v) \in E(G)\} = \{1, 2, \ldots, n\}$.

Let G be a graph with n edges. A one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, n\}$ is a β-labeling of G if $\{|f(u) - f(v)|/(u, v) \in E(G)\} = \{1, 2, \ldots, n\}$. A β-labeling f of a graph G with n edges is an α-labeling if there exists an integer k such that $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$ for every edge $uv \in E(G)$. [It is clear that every α-labeling is a β-labeling and every β-labeling is a ρ-labeling.]

Further, Rosa [5] proved the following two significant theorems.

Theorem 1.1. *If G is a graph with n edges then there exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-valuation.*

Theorem 1.2. *Let G be a graph with n edges that has an α-labeling. Then there exists a cyclic G-decomposition of K_{2cn+1}, where c is any positive integer.*

From the definition of α-labeling, it is clear that if a graph G admits α-labeling then it must be bipartite. An almost-bipartite graph is a non-bipartite graph with the property that the removal of a particular single edge renders the graph bipartite. In order to have a cyclic G-decomposition in the complete graph K_{2cn+1}, where G is an almost-bipartite graph, Blinco et al introduced γ-labeling. A function h defined on the vertex set of a graph G with n edges is called γ-labeling if,

(i) h is a ρ-labeling of G,

(ii) G admits a tripartition (A, B, C) with $C = \{c\}$ and there exist $\bar{b} \in B$ such that $(\bar{b}, c)$ is the unique edge with the property that $h(c) - h(\bar{b}) = n$,

(iii) for every edge $(a, v) \in E(G)$ with $a \in A$, $h(a) < h(v)$.

Further, in [1] they have also proved the following significant theorem.

Theorem 1.3. *Let G be a graph with m edges having γ-labeling. Then there exists a cyclic G-decomposition of K_{2cm+1}, where c is any positive integer.*

Motivated by Theorem 1.3 due to Blinco et al [1], the almost-bipartite graphs $K_{m,n}+e$, C_{2k+1}, $C_{2m}+e$, $C_3 \cup C_{4m}$, $C_{2k+1} \cup C_{4n+2}$ are shown to have γ-labeling [Refer [1–3]]. In this paper, we show that the almost-bipartite graph, the grid with an additional edge $(P_m \square P_n)+\hat{e}$ admits γ-labeling and we discuss a related open problem.

2 γ-labeling of $(P_m \square P_n)+\hat{e}$

In this section, we show that the almost-bipartite graph $(P_m \square P_n)+\hat{e}$ admits γ-labeling.

The cartesian product of two graphs G and H, denoted $G \square H$ is a graph having the vertex set $V(G) \times V(H)$ and any two vertices (u_1, v_1) and (u_2, v_2) of $G \square H$ are adjacent if and only if $u_1 = u_2$ and v_1 is adjacent to v_2 in H or $v_1 = v_2$ and u_1 is adjacent to u_2 in G.

The cartesian product of paths P_m and P_n, $P_m \square P_n$, where $P_m : u_1 u_2 \ldots u_m$ and $P_n : v_1 v_2 \ldots v_n$ is a graph with vertex set $\{(u_i, v_j)/1 \leq i \leq m, 1 \leq j \leq n\}$ and two vertices (u_i, v_j) and (u'_i, v'_j) in $P_m \square P_n$ are adjacent if and only if $u_i = u'_i$ and v_j is adjacent to v'_j in P_n or $v_j = v'_j$ and u_i is adjacent to u'_i in P_m. The graph $P_m \square P_n$ is also called grid or mesh. The grid $P_4 \square P_5$ is given in Figure 1.

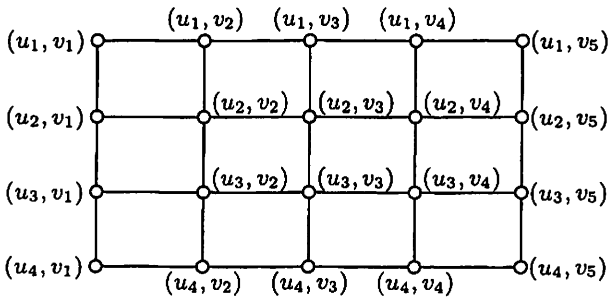

Figure 1: The grid $P_4 \square P_5$

For the convenience, we consider the vertex set of $P_m \square P_n$, $V(P_m \square P_n) = \{v_{ij} = (u_i, v_j)/1 \leq i \leq m, 1 \leq j \leq n\}$. Then, $P_m \square P_n$ can be described as in Figure 2.

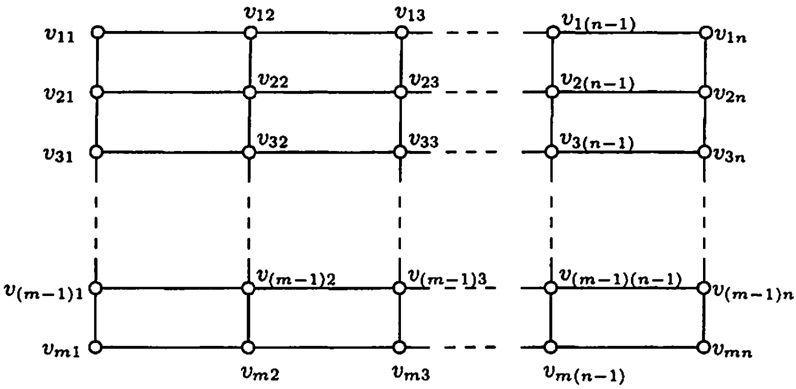

Figure 2: The grid $P_m \Box P_n$

A graph is called *grid with first diagonal edge* if it is obtained from the grid $P_m \Box P_n$ by adding a new edge between the (first) pair of non-adjacent diagonal vertices v_{11} and v_{22} and it is denoted by $(P_m \Box P_n) + \hat{e}$. Then observe that
$V((P_m \Box P_n) + \hat{e}) = V(P_m \Box P_n) = \{v_{ij}/1 \leq i \leq m, 1 \leq j \leq n\}$,
$E((P_m \Box P_n) + \hat{e}) = E(P_m \Box P_n) \cup \{(v_{11}, v_{22})\}$.
It is clear that $(P_m \Box P_n) + \hat{e}$ is an almost-bipartite graph.

In the following Theorem 2.1 we show that grid with first diagonal edge admits γ-labeling.

Theorem 2.1. *Grid with first diagonal edge $(P_m \Box P_n) + \hat{e}$ admits γ-labeling.*

Proof. By definition, the grid with first diagonal edge $(P_m \Box P_n) + \hat{e}$ has
$V((P_m \Box P_n) + \hat{e}) = V(P_m \Box P_n) = \{v_{ij}/1 \leq i \leq m, 1 \leq j \leq n\}$,
$E((P_m \Box P_n) + \hat{e}) = E(P_m \Box P_n) \cup \{(v_{11}, v_{22})\}$.
Thus, $|V((P_m \Box P_n) + \hat{e})| = mn$ and $|E((P_m \Box P_n) + \hat{e})| = 2mn + 1 - (m + n)$.
Let $M = 2mn + 1 - (m + n)$.
Without loss of generality we assume $m \leq n$.
For the convenience of defining γ-labeling on $(P_m \Box P_n) + \hat{e}$ we define a tripartition (A, B, C) on $V((P_m \Box P_n) + \hat{e})$ as given below.
Let $A = \{v_{ij}/1 \leq i \leq m, i \text{ odd and } 2 \leq j \leq n, j \text{ even}\} \cup$
$\{v_{ij}/2 \leq i \leq m, i \text{ even and } 1 \leq j \leq n, j \text{ odd}\}$,

$B = \{v_{1j}/3 \leq j \leq n, j \text{ odd}\} \cup$
$\{v_{ij}/3 \leq i \leq m, 1 \leq j \leq n, i \text{ and } j \text{ are odd}\} \cup$
$\{v_{ij}/2 \leq i \leq m, 2 \leq j \leq n, i \text{ and } j \text{ are even}\}$ and
$C = \{v_{11}\}$.

We define $f : V((P_m \Box P_n) + \hat{e}) \to \{0, 1, 2, \ldots, 2M\}$ in the following way. First we define f on the unique vertex v_{11} of C as $f(v_{11}) = 2M - 1$.
Then we define f on the vertices of A.
For that, we arrange the vertices of A as three different sets of sequences and for each such set of sequences of vertices we give labels in the order of the sequence.

The first set of sequences of vertices of A.
$(v_{1(2k)}, v_{2(2k-1)}, v_{3(2k-2)}, \ldots, v_{\alpha\beta})$, where $1 \le k \le \lfloor \frac{m}{2} \rfloor$,
$\alpha = m$, $\beta = 2k - (m - 1)$ when m is even or
$\alpha = m - 1$, $\beta = 2k - (m - 2)$ when m is odd.
[Note that the sequence is terminated whenever the second index becomes 1.]

Labeling of the first set of sequences of vertices of A.
Define $f(v_{12}) = 1$, $f(v_{21}) = 2$.
For each k, $2 \le k \le \lfloor \frac{m}{2} \rfloor$, define $f(v_{1(2k)}) = f(v_{(2k-2)1}) + 2k - 1$
and for each i, $2 \le i \le 2k$, define $f(v_{i(2k+1-i)}) = f(v_{(i-1)(2(k+1)-i)}) + 1$.

The second set of sequences of vertices of A.
$(v_{1(m+k)}, v_{2(m+k-1)}, v_{3(m+k-2)}, \ldots, v_{m(k+1)})$, where the choice of the index k is defined depending on m is even or odd as below.
(i) If m is even, then $2 \le k \le n - \alpha$ and k even, where $\alpha = m$ when n is even or $\alpha = m + 1$ when n is odd.
(ii) If m is odd, then $1 \le k \le n - \alpha$ and k odd, where $\alpha = m$ when n is even or $\alpha = m + 1$ when n is odd.

Labeling of the second set of sequences of vertices of A.
Define, for odd m, $f(v_{1(m+1)}) = f(v_{(m-1)1}) + m$ and
$f(v_{i(m+2-i)}) = f(v_{(i-1)(m+3-i)}) + 1$ for i, $2 \le i \le m$.
Let k be the index. The choice of k is defined depending on m is even or odd. If m is even then $2 \le k \le n - \alpha$ and k even, where $\alpha = m$ when n is even while when n is odd $\alpha = m + 1$. If m is odd then $3 \le k \le n - \alpha$ and k odd, where $\alpha = m$ when n is even while when n is odd $\alpha = m + 1$.
For every choice of k described above, define
$f(v_{1(m+k)}) = f(v_{m(k-1)}) + m$ and
$f(v_{i(m+k+1-i)}) = f(v_{(i-1)(m+k+2-i)}) + 1$ for i, $2 \le i \le m$.

The third set of sequences of vertices of A.
$(v_{kn}, v_{(k+1)(n-1)}, v_{(k+2)(n-2)}, \ldots, v_{m(n-(m-k))})$, where the choice of the index k is defined depending on n is even or odd as below.
(i) If n is even then $3 \le k \le \alpha$ and k odd, where $\alpha = m - 1$ when m is even or $\alpha = m - 2$ when m is odd.

(ii) If n is odd then $2 \leq k \leq \alpha$ and k even, where $\alpha = m-1$ when m is odd or $\alpha = m-2$ when m is even.

Labeling of the third set of sequences of vertices of A.

Let k be the index. The choice of k is defined depending on n is even or odd. If n is even then $3 \leq k \leq \alpha$ and k odd, where $\alpha = m-1$ when m is even while when m is odd $\alpha = m-2$. If n is odd then $2 \leq k \leq \alpha$ and k even, where $\alpha = m-1$ when m is odd while when m is even $\alpha = m-2$. For every choice of k described above, define

$f(v_{kn}) = f(v_{m(n-(m-k+2))}) + m - k + 2$ and

$f(v_{i(n-j)}) = f(v_{(i-1)(n-(j-1))}) + 1$, $k+1 \leq i \leq m$, $1 \leq j \leq m-k$.

The first set of sequences of vertices of B.

$(v_{1(2k+1)}, v_{2(2k)}, v_{3(2k-1)}, \ldots, v_{(2k+1)1})$, where $1 \leq k \leq \alpha$, $\alpha = \frac{m-1}{2}$ when m is odd or $\alpha = \frac{m-2}{2}$ when m is even.

Labeling of the first set of sequences of vertices of B.

Define $f(v_{13}) = M$, $f(v_{22}) = M-1$, $f(v_{31}) = M-2$.
For each k, $2 \leq k \leq \alpha$, where $\alpha = \frac{m-1}{2}$ when m is odd or $\alpha = \frac{m-2}{2}$ when m is even, define

$f(v_{1(2k+1)}) = f(v_{(2k-1)1}) - 2k$ and

$f(v_{i(2(k+1)-i)}) = f(v_{(i-1)(2k+3-i)}) - 1$ for each i, $2 \leq i \leq 2k+1$.

The second set of sequences of vertices of B.

$(v_{1(m+k)}, v_{2(m+k-1)}, v_{3(m+k-2)}, \ldots, v_{m(k+1)})$, where the choice of the index k is defined depending on m is even or odd as below.
(i) If m is even then $1 \leq k \leq n-\alpha$ and k odd, where $\alpha = m$ when n is odd or $\alpha = m+1$ when n is even.
(ii) If m is odd then $2 \leq k \leq n-\alpha$ and k even, where $\alpha = m$ when n is odd or $\alpha = m+1$ when n is even.

Labeling of the second set of sequences of vertices of B.

Define, for even m, $f(v_{1(m+1)}) = f(v_{(m-1)1}) - m$ and

$f(v_{i(m+2-i)}) = f(v_{(i-1)(m+3-i)}) - 1$ for i, $2 \leq i \leq m$.

Let k be the index. The choice of k is defined depending on m is even or odd. If m is even then $3 \leq k \leq n-\alpha$ and k odd, where $\alpha = m$ when n is odd while when n is even $\alpha = m+1$. If m is odd then $2 \leq k \leq n-\alpha$ and k even, where $\alpha = m$ when n is odd while when n is even $\alpha = m+1$.
For every choice of k described above, define

$f(v_{1(m+k)}) = f(v_{m(k-1)}) - m$ and

$f(v_{i(m+k+1-i)}) = f(v_{(i-1)(m+k+2-i)}) - 1$ for i, $2 \leq i \leq m$.

The third set of sequences of vertices of B.

$(v_{kn}, v_{(k+1)(n-1)}, v_{(k+2)(n-2)}, \ldots, v_{m(n-(m-k))})$, where the choice of the index k is defined depending on n is even or odd as below.

(i) If n is even then $2 \leq k \leq \alpha$ and k even, where $\alpha = m-1$ when m is odd or $\alpha = m-2$ when m is even.

(ii) If n is odd then $3 \leq k \leq \alpha$ and k odd, where $\alpha = m-1$ when m is even or $\alpha = m-2$ when m is odd.

Labeling of the third set of sequences of vertices of B.

Let k be the index. The choice of k is defined depending on n is even or odd. If n is even then $2 \leq k \leq \alpha$ and k even, where $\alpha = m-1$ when m is odd while when m is even $\alpha = m-2$. If n is odd then $3 \leq k \leq \alpha$ and k odd, where $\alpha = m-1$ when m is even while when m is odd $\alpha = m-2$. For every choice of k described above, define

$f(v_{kn}) = f(v_{m(n-(m-k+2))}) - (m-k+2)$ and

$f(v_{i(n-j)}) = f(v_{(i-1)(n-(j-1))}) - 1,\ k+1 \leq i \leq m,\ 1 \leq j \leq m-k.$

Now, $f(v_{mn}) = \begin{cases} f(v_{m(n-1)}) - 1 & \text{if } v_{mn} \in A \\ f(v_{m(n-1)}) + 1 & \text{if } v_{mn} \in B \end{cases}$

By definition, we observe that $f(v_{ij})$, $v_{ij} \in A$ form a monotonic increasing sequence and $f(v_{ij})$, $v_{ij} \in B$ form a monotonic decreasing sequence. Further, when m and n are of same parity, we have,

$max\{f(v_{ij})/v_{ij} \in A\} = \frac{M-1}{2}$ and $min\{f(v_{ij})/v_{ij} \in B\} = \frac{M+1}{2}$.

When m and n are of opposite parity, we have

$max\{f(v_{ij})/v_{ij} \in A\} = \frac{M}{2} + 2$ and $min\{f(v_{ij})/v_{ij} \in B\} = \frac{M}{2} + 3$.

Thus, $min\{f(v_{ij})/v_{ij} \in B\} > max\{f(v_{ij})/v_{ij} \in A\}$.

Also, $f(v_{11})$ is greater than the labels of all the vertices in the sets A and B. Thus the vertex labels are distinct.

To understand the edge values of the edges of $(P_m \Box P_n) + \hat{e}$ more clearly, we decompose the edge set of $(P_m \Box P_n) + \hat{e}$ into sequence of paths in $(P_m \Box P_n) + \hat{e}$ as given below.

$P_0 : v_{m(n-1)} v_{mn} v_{(m-1)n}$,

$P_1 : v_{12} v_{11} v_{21}$,

$P_2 : v_{11} v_{22}$,

$P_3 : v_{13} v_{12} v_{22} v_{21} v_{31}$,

$P_4 : v_{14} v_{13} v_{23} v_{22} v_{32} v_{31} v_{41}$,

$P_5 : v_{15} v_{14} v_{24} v_{23} v_{33} v_{32} v_{42} v_{41} v_{51}$,

$\vdots$

$P_m : v_{1m} v_{1(m-1)} v_{2(m-1)} v_{2(m-2)} \cdots v_{(m-1)1} v_{m1}$,

$P_{m+1} : v_{1(m+1)} v_{1m} v_{2m} v_{2(m-1)} \cdots v_{m2} v_{m1}$,

$\vdots$

$P_n : v_{1n}v_{1(n-1)}v_{2(n-1)}v_{2(n-2)}\cdots$
$\cdots v_{(m-1)(n-(m-2))}v_{(m-1)(n-(m-1))}v_{m(n-(m-1))}v_{m(n-m)},$

$P_{n+1} : v_{1n}v_{2n}v_{2(n-1)}v_{3(n-1)}\cdots$
$\cdots v_{(m-1)(n-(m-3))}v_{(m-1)(n-(m-2))}v_{m(n-(m-2))}v_{m(n-(m-1))},$

$P_{n+2} : v_{2n}v_{3n}v_{3(n-1)}v_{4(n-1)}\cdots$
$\cdots v_{(m-1)(n-(m-4))}v_{(m-1)(n-(m-3))}v_{m(n-(m-3))}v_{m(n-(m-2))},$

$\vdots$

$P_{n+m-3} : v_{(m-3)n}v_{(m-2)n}v_{(m-2)(n-1)}v_{(m-1)(n-1)}v_{(m-1)(n-2)}v_{m(n-2)}v_{m(n-3)},$

$P_{n+m-2} : v_{(m-2)n}v_{(m-1)n}v_{(m-1)(n-1)}v_{m(n-1)}v_{m(n-2)}.$

The following Table 1 gives the edge labels of the edges of the corresponding P_i, $0 \leq i \leq n+m-2$ given in the above path sequence of $(P_m \Box P_n) + \hat{e}$.

Table 1: Edge labels of edges in the sequence of paths in $(P_m \Box P_n) + \hat{e}$

Path	Edge labels of the edges in the above path of decomposition of $(P_m \Box P_n) + \hat{e}$ (in the order of edges appear in the path)
P_0	1, 2
P_1	3, 4
P_2	M
P_3	$M-1, M-2, M-3, M-4$
P_4	$M-5, M-6, M-7, M-8, M-9, M-10$
P_5	$M-11, M-12, M-13, M-14, M-15, M-16,$ $M-17, M-18$
$\vdots$	$\vdots$
P_{n+m-4}	22, 21, 20, 19, 18, 17, 16, 15
P_{n+m-3}	14, 13, 12, 11, 10, 9
P_{n+m-2}	8, 7, 6, 5

It is clear from the Table 1 that the edge labels are distinct and the edge labels ranges from 1 to M. Hence f is a γ-labeling.

Corollary 2.2. *For every grid with first diagonal edge $(P_m \Box P_n) + \hat{e}$, there exists a cyclic decomposition of the complete graph K_{2cq+1} into subgraphs isomorphic to $(P_m \Box P_n) + \hat{e}$ where $q = |E((P_m \Box P_n) + \hat{e})|$ and c is an arbitrary positive integer.*

Illustration:

γ-labeling of $(P_6 \Box P_9) + \hat{e}$ is illustrated in Figure 3.

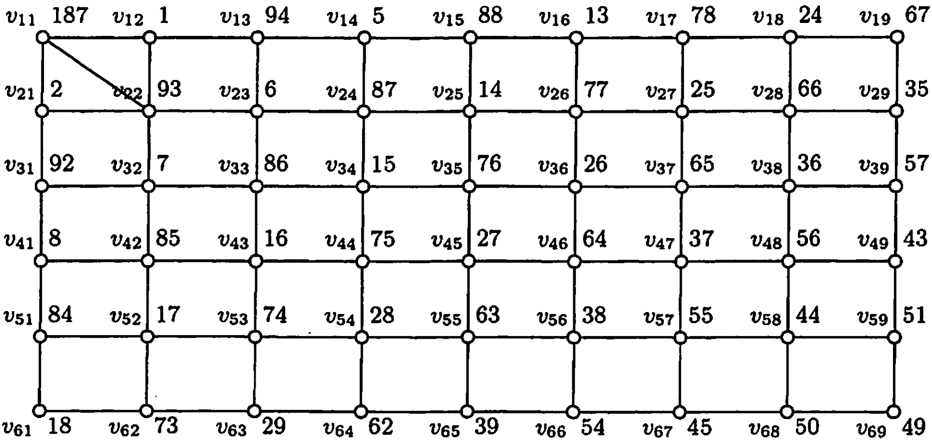

Figure 3: γ-labeling of $(P_6 \Box P_9) + \hat{e}$

3 Discussion

We observe from the Figure 4(a) through Figure 4(d) that the almost-bipartite graph $(P_m \Box P_n) + \hat{e}$ admits γ-labeling even if the edge $\hat{e}$ is added between a pair of non-adjacent vertices of $P_m \Box P_n$ randomly. Thus we feel that the edge $\hat{e}$ can be added between any two non-adjacent vertices of $P_m \Box P_n$ to get the almost-bipartite graph $(P_m \Box P_n) + \hat{e}$ that can have γ-labeling. Thus we pose the following conjecture and an open question.

Conjecture: The almost-bipartite graph $(P_n \Box P_n) + uv$ always admit γ-labeling for every edge uv which is added between any two non-adjacent vertices u and v of $P_n \Box P_n$.

Question: Is it true that the almost-bipartite graph $(P_m \Box P_n) + uv$ admits γ-labeling, for every edge uv which is added between any two non-adjacent vertices u and v of $P_m \Box P_n$ for all values of m and n?

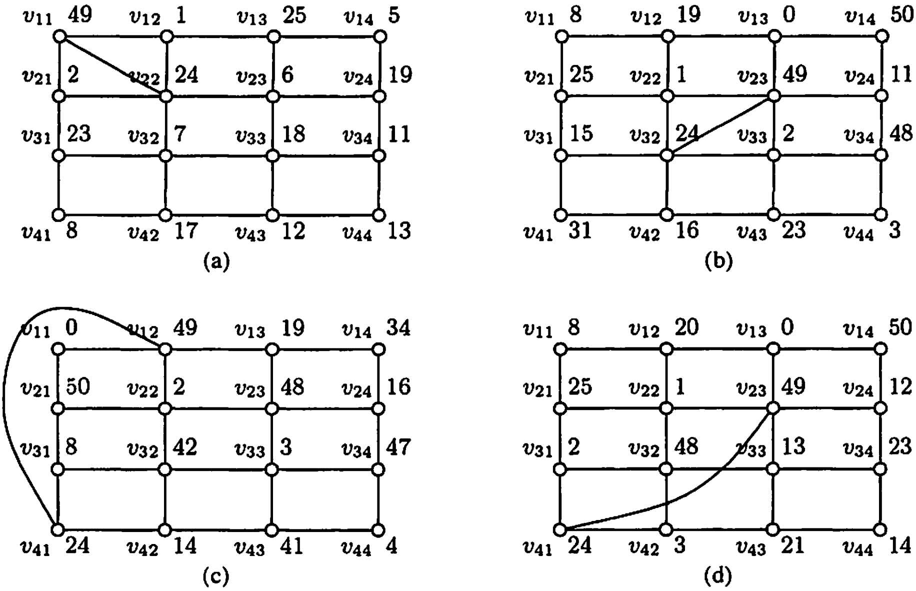

Figure 4: (a) γ-labeling of $(P_4 \Box P_4) + (v_{11}v_{22})$,
(b) γ-labeling of $(P_4 \Box P_4) + (v_{23}v_{32})$,
(c) γ-labeling of $(P_4 \Box P_4) + (v_{12}v_{41})$,
(d) γ-labeling of $(P_4 \Box P_4) + (v_{23}v_{41})$.

References

[1] A. Blinco, S.I. El-Zanati and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, *Discrete Math.*, **284** (2004), 71–81.

[2] S.I. El-Zanati, C. Vanden Eynden, On Rosa-type labelings and cyclic decomposition, *Mathematica Slovaca*, **59**(1) (2009), 1–18.

[3] S.I. El-Zanati, C. Vanden Eynden and N. Punin, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas J. Combin.*, **24** (2001), 209–219.

[4] J.A. Gallian, A dynamic survey of graph labeling, The *Electronic Journal of Combinatorics*, **19** (2012), # DS6.

[5] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs*, (International symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris, (1967), 349–355.