A Note On Distance Graphs

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Abstract

Given a connected (p, q) graph with a number of central vertices, form a new graph G^* as follows: $V(G^*) = V(G)$; Delete all the edges of G. Introduce edge between every central vertex to each and every non central vertex of G; allow every pair of central vertices to be adjacent. In this paper we probed G^* and deducted a number of results.

1 Introduction

The graphs considered in this paper are mostly finite, simple and undirected. Let G be a connected (p, q) graph with p vertices and q edges. To construct a new graph G^* from G first consider its center $C(G)$. Let $C(G) = \{u_1, u_2, \ldots, u_n\}$. Form a graph G^* from G as follows: Delete all the edges of G, and consider p copies of K_1, viz, pK_1. Join u_i with all the non central vertices for $1 \leq i \leq n$ and allow every pair of central vertices to be adjacent.

Let G be a connected (p, q) graph and let v be a vertex of G. By the eccentricity $e(v)$ of v we mean the distance to a vertex farthest from v. So $e(v) = \max\{d(u, v) : u \in V\}$. The radius $r(G)$ of the graph G , is the minimum eccentricity, whereas the maximum eccentricity is the diameter diam(G) of G. A vertex $v \in V(G)$ is called a central vertex if $e(v) = r(G)$. We denote by $C(G)$, the set of central vertices of G.A vertex $v \in V(G)$ is called a peripheral vertex if $e(v) = diam(G)$, and the periphery or peripherian denoted $Peri(G)$ is the set of all peripheral vertices. Note that the central vertices of G in G^* induces a complete subgraph. So if

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$|C(G)| = n$, then K_n is an induced subgraph of G^*. It is easy to see that $deg(v) = n$ if v is a non-central vertex and $deg(v) = p-1$ $(= p-n+n-1)$ if v is a central vertex. Further it is easy to see that $r(G^*) = 1$ and $diam(G^*) = 2$. As any two non central vertices can be reached one from the other through any one of the central vertices, the formation of G^* assumes significance in a variety of real world problems.

2 Some Properties of G^*

The following are some easy properties of G^*. 1) If G is a one centered graph then $G \cong K_{1,p-1}$; 2) If G is a n- centered graph with $n \geq 1$, then $diam(G^*) = 2r(G^*)$; 3) If If G is a 2-centered graph, then G^* is not a tree; 4) If G is a n-centered graph with $n \geq 2$ then G^* is connected but not a tree; 5) $\kappa(G^*) = C(G^*)$, where κ is the vertex connectivity; 6) $\kappa'(G^*) = deg(v)$ where $v \in V(G^*) - C(G^*)$ and κ' stands for the edge connectivity; 7) G^* is not a unicyclic graph; 8) The chromatic number $\chi(G^*)$ of G^* is $n+1$; 9) G^* is a distance regularized graph, where by a distance regularized graph we mean: if for any integer k and any two vertices u and v, the number of neighbors of v which are at distance k from u depends only on $d(u,v)$ and the vertex u; 10) If G is a n- centered graph, then G^* has a circumference of length $n+2$; 11) G^* is a distance heriditary graph and by the same we mean the following; if for all connected induced subgraphs F of G,$d_F(u,v)$=$d_G(u,v)$ for all pairs of vertices $u, v \in F$; 12) G^* is a non hamiltonian graph; 13) G^* is strongly geodetic graph, by which we mean the following; if each pair of vertices u, v of a graph G is joined by at most one path of length not more than the diameter $diam(G)$, then G is called a strongly geodetic graph.

Proposition 2.1 All non-central vertices of G^* has equal status.
Proof Suppose that $C(G^*) = \{u_1, u_2, \ldots, u_n\}$. Let w be any vertex in $G^* - C(G^*)$. Then $s(w) = \sum_{u \in V(G^*)} d(w,u) = \sum_{u \in C(G^*)} d(w,u) + \sum_{u \in V(G^* - C(G^*))} d(w,u)$ $= n + 2(p-n-1) = 2p-n-2$. As w is arbitrary the proof follows. □

Theorem 2.2 Let G be a connected (p,q) graph with n central vertices. Then the G^* graph cannot have more than (p-2) central vertices.
Proof According to Buckley and Harary[2], we have $(p-1) \leq q(G^*) \leq d + \frac{1}{2}[(p-d-1)(p-d+4)$ where d stands for the diameter of G. As diameter of G^* is 2, we have $q(G^*) \leq 2 + \frac{1}{2}(p-3)(p+2)$. Clearly $deg_{v \in C(G^*)}(v) = (n-1) + (p-n) = p-1$ and $deg_{v \in V(G^*) - C(G^*)}(v) = n$. So $2q = n(p-1) + (p-n)n$=$2np - n(n+1)$. Now $2np - n(n+1) \leq 2(2 + \frac{1}{2}(p-3)(p+2)) =$

(p^2-p-2). So $n^2+n(1-2p)+(p^2-p-2) \geq 0$ implies $n = p-2$ or $p+1$. As n cannot be $p+1$, we have $n = p-2$. □

Theorem 2.3 Let G be a connected (p,q) graph with n vertices. Then $M(G^*) = C(G^*)$.

Proof Let $C(G^*) = \{u_1, u_2, \ldots, u_n\}$. Let w be a non central vertex of G^*. Then it is easy to see that $s(w)$=$2(p-n-1)+n$. By Proposition 2.1, it follows that all non central vertices of G^* has status $2(p-1)-n$. Next let u_i be any central vertex of G^*. As u_i is adjacent to all the remaining $(p-1)$ vertices of G^* it follows that $s(u_i) = p-1$. Now by Theorem 2.2 $2(p-1)-n \geq (p-1)$. Hence $M(G^*) = C(G^*)$.□

Theorem 2.4 Let G be any graph with any number of central vertices. Then (i)G^* has the form $K_n \vee G_1$, where $n = |C(G^*)|$. (ii) each of the components of G^* is a complete graph and (iii) the number of components of G_1 is at least two.

Proof Since $r(G^*) = 1$, any central vertex is connected to all other vertices and $< C(G^*) >$ is a complete graph. Let K be a component of the induced subgraph $G_1 =< V(G^*) - C(G^*) >$. If there exists two vertices $x, y \in V(K)$ such that $d_K(x,y) = 2$, then the graph G^* contains a diametral path of length two containing no central vertex, which is a contradiction to the fact that $diam(G^*) = 2$ and to the very construction of G^*, according to which any two non-central vertices are connected by a path of length 2 through any one of the central vertex. Thus K is a complete graph and G_1 has at least two components. □

Consider the G^* graph formed from any given graph with one central vertex. Then we know that $|C(G^*)| = 1$. As the eccentricity of all non-central vertices of G^* is 2 and the $diam(G^*) = 2$ we note that $Peri(G^*)$ consists of all non-central vertices of G^*. Also each non-central vertex of G^* is a center eccentric vertex of the central vertex of G^* we see that $Cep(G^*) = Peri(G^*)$. Hence G^* is a $S-$graph. Observe that if G^* is such that $|C(G^*)| \geq 2$, then $Peri(G^*) = u$: u is a non-central vertex of G^* and $Cep(G^*) = \{u, v :$ u is a non-central vertex and v is any central vertex other than the one to which we find the center eccentric vertices$\}$. (That is any central vertex is a center-eccentric vertex of any other central vertex).So $Peri(G^*) \neq cep(G^*)$. Hence G^* is not a S graph whenever $|C(G^*)| \geq 2$. It is interesting to observe that G^* satisfies another characterization for $S-graphs$. That is , a graph G with $r(G) = 1$ and $diam(G) = 2$ is a $S-graph$ if and only if $|C(G)| = 1$. For, if $|C(G^*)| = \{u\}$, then the eccentricity of any vertex from $V(G^*)-\{u\}$ is equal to 2. So $Peri(G) = cep(G) = V(G^*) - \{u\}$. Conversely, let G^* be a S graph. Let $|C(G^*)| \geq 2$ and $x, y \in C(G^*), x \neq y$. Since $x, y \in Cep(G^*)$ and no vertex from $C(G^*)$ can belong to $Peri(G^*)$, we have a contradiction.

It is easy to see that G^* with one central vertex is an L_1 graph and G^* with $|C(G^*)| \geq 2$ is an L_3 graph. We know that for all graphs of diameter $d \geq 2$ we have $q \leq d+\frac{1}{2}(p-d-1)(p-d+4)$[2]. As G^* with $|C(G^*)| \geq 2$ is connected it has at least $p-1$ edges. So, $(p-1) \leq q(G^*) \leq d+\frac{1}{2}(p-d-1)(p-d+4)$. It is easy to see that G^* with $|C(G^*)| = 1$ is an extremal graph to the graph equation $q(G^*) = p-1$.

Theorem 2.5 Let G be a graph with $|C(G)| = n$. Then $2 \leq \Delta(G^*) \leq p-1$.
Proof If the diameter d of G^* is greater than or equal to two, then G^* contains at least one path of length d and hence $\Delta(G^*) \geq 2$. Bosak, Rosa and Znam proved that for all graphs of diameter $d \geq 2, \Delta(G) \leq p-d+1$[1]. As $diam(G^*) = 2$ we have the upper bound. □

3 Some New Constructions from G^*

Let us now consider the problem of altering (i) the number of centers; (ii) the radius; (iii) the diameter of a given graph G by subjecting it to modifications using certain standard graph edit operations. In all of the following constructions G is a simple graph which is neither a complete graph nor a star. Let $|C(G)| = l$ and consider its G^* graph.

Construction 1 Form a new graph $G^{\circledcirc}$ as follows: Take an even number of copies of G^*. Introduce an edge between any non central vertex of the first copy of G^* with a central vertex of the second copy of G^*, then join a non central vertex of the second copy of G^* with the central vertex of the third copy of G^* and so on and finally join a non central vertex of the last but one copy of G^* with a central vertex of the last copy of G^*.

Construction 2 Form a new graph $G^{\circledcirc}$ as follows: Take any number of copies of G^* and repeat the procedure as in construction-1. Also join the central vertex of the last copy of G^* with the central vertex of the first copy of G^*.

Construction 3 Form a new graph $G^{\circledcirc}$ as follows: Take an even number of copies of G^*. Create a path of length: that many copies of $G^* - 1$ by joining the center vertex of the first copy with the corresponding central vertex of the second copy and so on viz. $u_1^*, u_2^*, \ldots, u_m^*$ where u_i^* is the central vertex of the i−th copy of G^*.

Construction 4 Form a new graph $G^{\circledcirc}$ as follows: Take an even number of copies of G^* and join a non central vertex of the first copy of G^* with a non central vertex of the second copy of G^* and so on. Finally join a non central vertex of the last but one copy of G^* with a non central vertex of the last copy of G^*.

Construction 5 Form a new graph $G^{\circledcirc}$ as follows: Take an odd number of copies of G^* and do the same procedure as in construction-4.

Proposition 3.1 Let G be an S-graph with $|C(G)| = 1$ and let $diam(G) \geq 3$ then $r(G) < diam(G) - 1$.

Note The graphs of Constructions 1 to 5 are both $S-$ graph and L_1 graph.

Proposition 3.2 Suppose that G is an $S - graph$ with $r(G) \neq diam(G)$ then $diam(V(G) - C(G)) \geq diam(G)$, where $C(G)$ is the center of G.

Theorem 3.3 Let G be a graph with diameter four and radius two. Let $Q =< V(G) - C(G) >$, where $C(G)$ is the center of G. Then G is an $S-$ graph if and only if $i)$ $< C(G) >$ is a complete graph. $ii)$ The cardinality of the set $T = \{x \in V(Q) : N_G(x) \bigcap C(G) = \phi\}$ is at least two, and for every $x \in T$ at least one vertex $y \in V(Q)$ such that $d_Q(x, y) \geq 4$ belongs to T.

Note 1 Consider the graph $G^{\circledcirc}$ of Construction 1 with two copies of G^*. Then $r(G^{\circledcirc}) = 2$ and $diam(G^{\circledcirc}) = 4$.

Note 2 Given an $S - graph$ G, we see that an edge $e \in E(G)$ is superfluous in G, if $G - e$ is also an $S - graph$ with the same radius, diameter and peripherian as in G. An $S - graph$ G is said to be critical, if it has no superfluous edge. For example, G is a critical $S - graph$ with one central vertex, radius $r(G)$ and diameter $= 2r(G)$ if and only if it is a tree with one central vertex, radius $r(G)$ and diameter $2r(G)$. Now consider the graphs of Constructions 1 and 5. Note that the graphs $G^{\circledcirc}$ and G^* of these constructions have one central vertex, radius $r(G)$ and diameter $2r(G)$ and they are trees, Hence there graphs are critical $S - graphs$.

Theorem 3.4 Let P be a graph and let a, b be positive integers such that $2 \leq a \leq b \leq 2a$. Then there exists an $L-$ graph H of radius a, diameter b, and containing P as an induced subgraph.

Note 3 The graphs of constructions 2 to 4 satisfy the condition of the above Theorem 3.4.

Theorem 3.5 For any positive integer m with $1 \leq m \leq n, n \in Z^+$, there exists a graph G with rad(G) $= m$, diam(G) $= 2m$.

Proof Let G be any graph with one central vertex. Consider the G^* graph. We know that $r(G^*) = 1, diam(G^*) = 2$, and $|C(G^*)| = 1$. Form a new graph $G^{\circledcirc}$ from m disjoint copies of G^* as follows. Denote by G_i^* the i-th copy and the central vertex of the (i+1)-th copy G_{i+1}^* of G^* of G^* for $1 \leq i \leq m$. Let $u_i, 1 \leq i \leq m$ be the central vertex of the copy of G^*. Let v_j^i be the j-th non central vertex of the i-th copy of G^*. Introduce an edge (v_j^i, u_{i+1}) between the j-th non-central vertex of the i-th copy G_i^* of G^* for $1 \leq i \leq m-1$. Call the resulting graph as $G^{\circledcirc}$. Clearly $V(G^{\circledcirc}) = V(\bigcup_{i=1}^m G_i^*)$ and $E(G^{\circledcirc}) = E(\bigcup_{i=1}^m G_i^*) \bigcup \{(v_j^1, u_2), (v_j^2, u_3), \ldots, (v_j^{m-2}, u_{m-1}), (v_j^{m-1}, u_m)\}$. As $q(G^{\circledcirc}) = \sum_{i=1}^m (p(G_i^*) - 1) + (m-1) = \sum_{i=1}^m p(G_i^*) - 1$, we see that $G^{\circledcirc}$ is connected and a tree. We claim that $r(G^{\circledcirc}) = m$ and diam $(G^{\circledcirc}) = 2m$. As $G^{\circledcirc}$ is a tree, there exists a unique path between any two vertices. Hence one can calculate the eccentricity of the vertices of $G^{\circledcirc}$. It is enough to calculate the eccentricity of the central vertex and any one non-central vertex v_l^i of $G^{\circledcirc}$ as $deg(v_l^i) = 1$ if $l \neq j, 1 \leq i \leq m$. First note that if m is odd then there will be only one middle copy of G^*, namely $G^*_{\lceil \frac{m}{2} \rceil}$. Also $u_{\lceil \frac{m}{2} \rceil}$ is the central vertex of $G^{\circledcirc}$. We know that in any graph G, the central vertex is the one with minimum eccentricity. So we find the eccentricity of $u_{\lceil \frac{m}{2} \rceil}$. Observe that the distance of $u_{\lceil \frac{m}{2} \rceil}$ to $V_l^{\lceil \frac{m}{2} \rceil + 1}$ is 3 along the path $u_{\lceil \frac{m}{2} \rceil} v_j^{\lceil \frac{m}{2} \rceil} u_{\lceil \frac{m}{2} \rceil + 1} v_l^{\lceil \frac{m}{2} \rceil + 1}$. The distance of $u_{\lceil \frac{m}{2} \rceil}$ to $v_l^{\lceil \frac{m}{2} \rceil + 2}$ is 5 along the path $u_{\lceil \frac{m}{2} \rceil} v_j^{\lceil \frac{m}{2} \rceil}$ $u_{\lceil \frac{m}{2} \rceil + 1} v_i^{\lceil \frac{m}{2} \rceil + 1}$ $u_{\lceil \frac{m}{2} \rceil + 2} v_l^{\lceil \frac{m}{2} \rceil + 2}$. Similarly one can find the distance of $u_{\lceil \frac{m}{2} \rceil}$ to $v_l^{\lceil \frac{m}{2} \rceil + 3}$ to be 7 etc., and the distance of $u_{\lceil \frac{m}{2} \rceil}$ v_l^m to be m. As the radius of the graph is defined the minimum eccentricity of the graph we conclude that $r(G^{\circledcirc}) = m$. Further note that the maximum distance occurs between the vertices v_l^1 and v_l^m. By the above argument one can easily see that $d(v_l^1, v_l^m) = d(v_l^1, u_{\lceil m/2 \rceil})$ $+ d(u_{\lceil m/2 \rceil}, v_l^m)$. But as $d(v_l^1, u_{\lceil \frac{m}{2} \rceil} = d(u_{\lceil \frac{m}{2} \rceil}, v_l^m) = m$ we get diam $(v_l^1, v_l^m) = 2m$. That is the maximum eccentricity is 2m. As diameter of a graph is defined as the maximum eccentricity one can conclude that diam($G^{\circledcirc}$) $= 2m$. Next suppose that m is even. Then there are two middle copies of G^* namely $G^*_{\frac{m}{2}}$ and $G^*_{(\frac{m}{2})+1}$. Notice that $v_j^{\frac{m}{2}}$ is the central vertex of G^*. Just as in the odd case above, the distance of $v_j^{\frac{m}{2}}$ to $v_j^{(\frac{m}{2})+1}$, $v_j^{m/2}$ to $v_j^{(\frac{m}{2})+2}$ $v_j^{\frac{m}{2}}$ to $v_j^{(\frac{m}{2})+3}$., etc is $2, 4, 6\ldots$, and hence the distance of $v_j^{(\frac{m}{2})}$ to v_j^m is m. So, $r(G^*) = m$. Also $d(v_j^1, v_j^m) = d(v_j^1, v_j^{\frac{m}{2}}) + d(v_j^{\frac{m}{2}}, v_j^m) = m + m = 2m$. So diam $(G^{\circledcirc}) = 2m$.□

Note To convert mG^* into a connected graph it is not necessary that one should always introduce an edge between (v_j^i, u_{i+1}^*). That is the choice of the non-central vertex in each copy of G^* need not be the same vertex. It may be (v_s^1, u_2) between the first and second copy of G^* and (v_t^2, u_3) between the second and third copy of G^* and so on. In this case also the resulting graph satisfies the above theorem.

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