Acyclic Coloring of Central and Total Graph of Path P_n and Fan Graph $F_{m,n}$

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Abstract

A proper vertex coloring (no two adjacent vertices have the same color) of a graph G is said to be acyclic if the induced subgraph of any two color classes is acyclic. The minimum number of colors required for acyclic coloring of a graph G is said to be its acyclic chromatic number and is denoted by $a(G)$. In this paper, we find the exact value of the acyclic chromatic number of the central and total graph of path P_n and Fan graph $F_{m,n}$.

1. Introduction

All graphs considered here are simple finite and undirected. Throughout this paper, we use the term coloring for vertex coloring of graphs. A proper coloring of a graph G is a coloring of the vertices of G such that no two neighbors in G are assigned the same color.

Definition 1. *A subgraph H of a graph G is an induced subgraph if it has all the edges that appear in G over the same vertex set. The subgraph induced by the vertex set $v_1, v_2, v_3, \ldots, v_k$ is denoted by $< v_1, v_2, v_3, \ldots, v_k >$.*

Definition 2. *A vertex coloring of a graph is said to be acyclic [9] if the induced subgraph of any two color classes is acyclic. In other words, the subgraph induced by any two color classes is a forest.*

Definition 3. *The minimum number of colors needed to acyclically color the vertices of a graph G is called its acyclic chromatic number and is denoted by $a(G)$.*

The concept of acyclic coloring was introduced by B.Grunbaum in 1973[9]. He has proved that any planar graph is acyclically 9-colorable and he has conjectured that any planar graph can be acyclically vertex colored with 5 colors. This conjecture was later proved by Borodin[5]. Determining the acyclic chromatic number is a hard problem from both theoretical and an algorithmic point of view. More specifically, A.V. Kostochka has proved that it is an NP- complete problem to decide for a given graph G, whether $a(G) \leq 3$[11]. Determining $a(G)$ for the class of bipartite graphs is still an open problem. Alon et al. have given a greedy algorithm to color any graph of maximum degree Δ acyclically, using $\Delta^2 + 1$ colors [2]. Albertson et al. have improved this result and shown that $a(G) \leq \Delta(\Delta - 1) + 2$ [1]. Some of the research works have been carried out by focusing on the family of graphs with small maximum degree. The introductory work of Grunbaum started with a bound of 4 colors on acyclic vertex coloring of a graph with maximum degree 3. Burnstein has shown that any graph with maximum degree 4 can be acyclically vertex colored with 5 colors [6]. Fertin and Raspaud have given $o(n\Delta^2)$ algorithm to color any graph with $\Delta(\Delta - 1)/2$ colors[7]. They have also proved that 9 colors are sufficient to color a graph with maximum degree 4. Skulrattankulchai has given a linear time algorithm to acyclically color the vertices of a subcubic graph with 4 colors [17]. Recently, V.Satish and K.Yadav have proved that graphs with maximum degree 4,5 and 6 can be acyclically vertex colored with 5,8 and 12 colors respectively by providing a linear time algorithm that color these graphs [15, 16, 18]. They have also shown that a graph of maximum degree Δ can be acyclically vertex colored with $(3\Delta^2 + 4\Delta + 8)/8$ colors [14].

Definition 4. *A cycle in a graph G is said to be a bicolored (j, k)-cycle if all its vertices are properly colored with two colors j and k. A graph G is said to be a (j, k)-cycle free graph or C_{jk}-free graph if it does not contain any bicolored (j, k)-cycle.*

Definition 5. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The central graph [13] of G, denoted by $C(G)$, is obtained from G by subdividing each edge exactly once and joining all the non adjacent vertices of G.*

Definition 6. *The total graph [4] of a graph, denoted by $T(G)$, is a graph such that the vertex set of T corresponds to the vertices and edges of G and two vertices are adjacent in T if and only if their corresponding elements are either adjacent or incident in G.*

Definition 7. *A Fan graph $F_{m,n}$ [8] is defined as the graph sum of null graph on m vertices with path on n vertices. Symbolically, $F_{m,n} = \overline{K}_m + P_n$.*

This work is an extended version of [3]. The purpose of this paper is to find the exact value of the acyclic chromatic number of central and total graph of path P_n and Fan graph $F_{m,n}$.

2. Acyclic Coloring of $C(P_n)$

Let P_n be a path on n vertices $v_1, v_2, \ldots, v_n$. Let $x_k (1 \leq k \leq n-1)$ be the newly introduced vertex on the edge joining v_k and v_{k+1}. In this section, we first present a coloring algorithm of $C(P_n)$ and then we prove the coloring is acyclic in the immediate following theorem.

2.1. Structural Properties of $C(P_n)$

1. The vertex v_1 is adjacent with $v_3, v_4, \ldots v_n$ whereas the vertex v_n is adjacent with $v_1, v_2, \ldots, v_{n-2}$.

2. For each $k = 2$ to $n-1$, v_k is adjacent to v_j, for all $j = 1, 2, \ldots, (k-2), (k+2), \ldots, n$.

2.2. Coloring Algorithm

Input : $C(P_n)$
$V \leftarrow \{v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_{n-1}\}$
$v_1, v_2 \leftarrow 1$; $v_3, v_4 \leftarrow 2$;
for $k = 5$ to n
{
$v_k \leftarrow k-2$;
}
end for
for $k = 1$ to $n-1$
{
$x_k \leftarrow n-1$;
}
end for
end procedure
Output: vertex colored $C(P_n)$

Theorem 8. *For any path on n vertices, the acyclic chromatic number is*

$$a[C(P_n)] = n - 1, \qquad \textit{for all} \quad n \geq 4.$$

Proof. Color the vertices of $C(P_n)$ as given in the algorithm 2.2. The color class of 1 is $\{v_1, v_2\}$ and that of 2 is $\{v_3, v_4\}$. The color class of j is

$\{v_{j+2}, 5 \le j \le n-2\}$ whereas the color class of $n-1$ is $\{x_k; 1 \le k \le n-1\}$.
Case(i): Consider the colors 1 and 2. The induced subgraph of these color classes contain a bicolored $(1-2)$-path, $v_2v_4v_1v_3$. Since v_2 and v_3 are non-adjacent, the induced subgraph is a C_{12}-free graph.
Case(ii): Consider the colors 1 and j, $3 \le j \le n-2$. The induced subgraph of these color classes contains a bicolored path $v_2v_{j+2}v_1$. As v_1 and v_2 are non-adjacent, the induced subgraph is C_{1j}-free graph.
Case(iii): Consider the colors 2 and k, $3 \le k \le n-2$. By the same argument as in case (ii), the induced subgraph is a C_{2k}-free graph.
Case(iv): Consider i and $n-1, i = 1, 2$. The induced subgraph of color classes of i and $n-1$ contains only the bicolored $[i-(n-1)]$ path, $v_ix_iv_{i+1}x_{i+1}$, but not bicolored cycle. Hence, the induced subgraph is a $C_{i(n-1)}$-free graph.
Case(v): Consider j and $n-1$, $3 \le j \le n-2$. The induced subgraph of these color classes contains only the bicolored $[j-(n-1)]$-path $x_{j+1}v_{j+2}x_{j+2}$. So, the induced subgraph is a $C_{j(n-1)}$-free graph. Thus, the induced subgraph of any two color classes is acyclic and therefore, the coloring given in the algorithm 2.2 is an acyclic coloring. As minimum $(n-1)$ colors are required for acyclic coloring, we have $a[C(P_n)] = n-1$, for all $n \ge 4$.

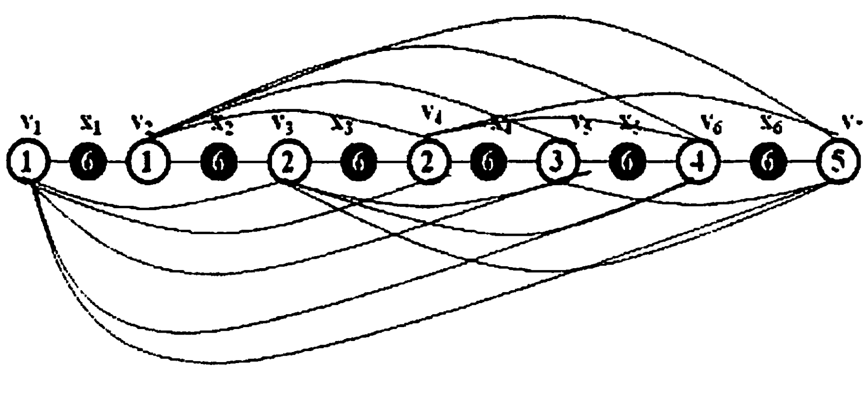

Fig.1. $a[C(P_7)] = 6$

3. Acyclic Coloring of $C[F_{m,n}]$

In Fan graph $F_{m,n}$, let $v_1, v_2, \ldots, v_m$ be the vertices of null graph on m vertices and $w_1, w_2, \ldots, w_n$ be the vertices of path P_n. Let us denote the newly added vertex on the edge joining v_i and w_j by f_{ij} $(1 \le i \le m$, $1 \le j \le n)$. Let us denote the newly added vertex on the edge joining w_k and w_{k+1} by g_k $(1 \le k \le n-1)$.

3.1. Structural properties of $C[F_{m,n}]$

1. $< v_k, k = 1$ to $m >$ form a clique of order $m + 1$
2. For each $1 \leq i \leq m$,$1 \leq j \leq n$, the neighbors of f_{ij} are $\{v_i, w_j\}$.
3. For each $k = 1$ to $n - 1$, the neighbors of g_k are $\{v_k, v_{k+1}\}$.
4. For each $k = 2$ to $n - 1$, the neighbors of v_k are $\{g_{k-1}, g_k, v_1, v_2, \ldots, v_{k-2}, v_{k+2}, \ldots, v_n\}$ and the neighbors of v_1 are $\{g_1, v_3, \ldots, v_n\}$ whereas the neighbors of v_n are $\{g_{n-1}, v_1, v_2, \ldots, v_{n-2}\}$.
5. $< v_i, x_j$; $i = 1$ to n and $j = 1$ to $n - 1>$ is a $C(P_n)$.

3.2 Coloring Algorithm

Consider the acyclic coloring of $C[F_{m,n}]$ as follows. Assign i to v_i, $1 \leq i \leq m$. Assign 1 to w_1, w_2 and 2 to w_3, w_4 and j to w_{j+2}, $3 \leq j \leq n-2$, $(n-1)$ to g_i , $1 \leq i \leq n-1$. Now, assign colors to $f_{ij}, 1 \leq i \leq m$ and $1 \leq j \leq n$ as follows.
Case (i): Suppose v_i and w_j (adjacent vertices of f_{ij}) have different colors[(i-e) i and $j - 2$ are distinct]. Then, assign k to f_{ij} where $k \neq i$, $j - 2$ and $1 \leq k \leq max(m, n - 1)$.
Case (ii): Suppose v_i and w_j have the same color. We have the following cases. (a) Both v_i and w_j have the same color 1.(w_1 and v_1 as well as u_1 and v_2). So, consider f_{11} and f_{12}. Assign 2 to f_{11}, but not to f_{12}. If we assign 2 to both f_{11} and f_{12}, then $v_1 f_{11} w_1 w_4 w_2 f_{12} v_2 v_1$ will be a bicolored $(1-2)$ cycle in $C[F_{m,n}]$. Hence, assign $n-1$ to f_{12}.
(b). Consider f_{23} and f_{24}. Assign 3 to f_{23} and $n-1$ to f_{24}. (c). Both v_i and w_j have the same color, say k, other than 1 and 2. Then, assign r to f_{ij} only if k is not assigned to f_{ij}' whose adjacent vertices have color r.(i-e) suppose f_{ij} is the newly added vertex on edge joining v_i and w_j which have same color k and f_{ij}' is the newly added vertex on the edge joining v_i' and w_j' with same color r . Then, assign r to f_{ij} only if k is not assigned to f_{ij}'. Otherwise, $v_i f_{ij} w_j w'_j f'_{ij} v'_i v_i$ will become a bicolored $(k - r)$ cycle (Refer fig.2).

Theorem 9. For any Fan graph $F_{m,n}$, the acyclic chromatic number is

$$a[C(F_{m,n})] = m, \quad \text{if } m \geq n \text{ and } m \geq 3$$
$$= n - 1, \quad \text{if } m < n \text{ and } n \geq 5.$$

Proof. Color the vertices of $C(F_{m,n})$ as given in the algorithm 3.2.
Case (i): Consider the subgraph induced by the vertices $\{v_1, v_2, \ldots, v_m\}$. As it is a clique, it contains no bicolored cycle.

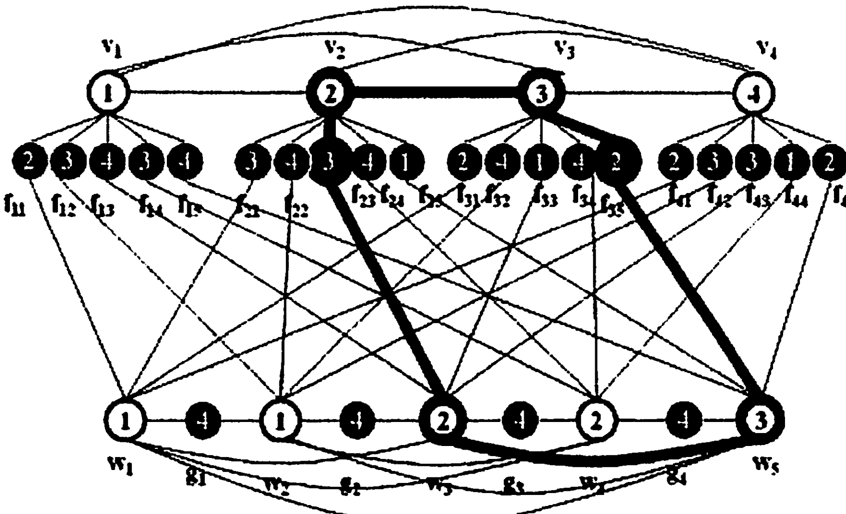

Fig.2. Existance of bicolored cycle

Case (ii):Consider the subgraph induced by $\{w_1, w_2, \ldots, w_n, g_1, g_2, \ldots, g_{n-1}\}$. It is a $C(P_n)$. As the vertices are colored as given in the algorithm 2.2, it does not contain any bicolored cycle by theorem 2.3.
Case (iii): Any bicolored cycle in $C(F_{m,n})$, if such a cycle exists, should contain the path (of length 2) connecting the vertices v_i and w_j with the same color. But, there exists no such a bicolored cycle in the above coloring. Hence, the coloring given in the algorithm 3.2 is acyclic. To determine the acyclic chromatic number of $C(F_{m,n})$, we have the following cases. Case (a): $m \geq n$ and $n > 24$. As $< v_1, v_2, \ldots, v_m >$ form a clique of order m, minimum m colors are required for coloring this induced subgraph and $(n-1)$ colors are required for $C(P_n)$. Since $m > (n-1)$, we have $a[C(F_{m,n})] = m, n > 2$. Case (b): $m < n$ and $n \geq 5$. By the same argument as in case (a), we need minimum $(n-1)$ colors for acyclic coloring of $C(F_{m,n})$ and hence, $a[C(F_{m,n})] = n-1, n \geq 5$.

4. Acyclic Coloring of $T(P_n)$

4.1. Structural Properties of $T(P_n)$

1. For each k, $k = 2$ to $n-1$, the neighbors of v_k are $\{x_{k-1}, x_k, v_{k-1}, v_{k+1}\}$ whereas the neighbors of v_1 and v_n are $\{x_1, v_2\}$ and $\{x_{n-1}, v_{n-1}\}$ respectively.

2. For each k, $k = 2$ to $n-2$, the neighbors of x_k are $\{x_{k-1}, x_{k+1}, v_k, v_{k+1}\}$ whereas the neighbors of x_1 and x_{n-1} are $\{v_1, v_2, x_2\}$ and $\{v_{n-1}, v_n, x_{n-2}\}$ respectively.

4.2. Coloring Algorithm

```
Input : T(P_n)
V ← {v_1, v_2, v_3, ..., v_n, x_1, x_2, ..., x_{n-1}}
for k = 1 to ⌊(n + 2)/3⌋
        {
         v_{3k-2} ← 1 ;x_{3k-2} ← 2 ;
        }
end for
for k = 1to ⌊(n + 1)/3⌋
        {
         v_{3k-1} ← 3; x_{3k-1} ← 1;
        }
end for
for k = 1 to ⌊n/3⌋
        {
         v_{3k} ← 2 ; x_{3k} ← 3;
        }
end for
end procedure.
Outputo: vertex colored T(P_n)
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Theorem 10. *The acyclic chromatic number of* $T(P_n)$ *is*

$$a[T(P_n)] = 3, \qquad n \geq 2.$$

Proof. Color the vertices of $T(Pn)$ as given in the algorithm 4.2. The color class of 1 is $\{v_{3k-2}, x_{3k'-1};\ k = 1 \text{ to } \lfloor (n+2)/3 \rfloor \text{ and } k' = 1 \text{ to } \lfloor (n+1)/3 \rfloor\}$ and the color class of 2 is $\{v_{3k}, x_{3k'-2};\ k = 1 \text{ to } \lfloor n/3 \rfloor \text{ and } k' = 1 \text{ to } \lfloor (n+2)/3 \rfloor\}$. The color class of 3 is $\{v_{3k-1}, x_{3k'};\ k = 1\text{to } \lfloor (n+1)/3 \rfloor \text{ and } k' = 1 \text{ to } \lfloor n/3 \rfloor\}$.

Case (i): The induced subgraph of the color classes of 1 and 2 is a bicolored path,

(a). $v_1 x_1 x_2 v_3 \ldots x_{n-1} v_n$ when $n = 0(mod3)$

(b). $v_1 x_1 x_2 v_3 \ldots v_{n-1} v_n$ when $n = 1(mod3)$

(c). $v_1 x_1 x_2 v_3 \ldots v_{n-1} x_{n-1}$ when $n = 2(mod3)$

In all the cases, the induced subgraph of 1 and 2 is a C_{12}-free graph.

Case (ii): The induced subgraph of the color classes of colors 1 and 3 is a bicolored path

(a). $v_1v_2x_2x_3\dots v_{n-1}x_{n-1}$ when $n = 0(mod3)$
(b). $v_1v_2x_2x_3\dots x_{n-1}v_n$ when $n = 1(mod3)$
(c). $v_1v_2x_2x_3\dots v_{n-1}v_n$ when $n = 2(mod3)$
In all the cases, the induced subgraph of 1 and 3 is a C_{13}-free graph.
Case (iii): The induced subgraph of the color classes of colors 2 and 3 is a bicolored path
(a). $x_1v_2v_3x_3\dots v_{n-1}v_n$ when $n = 0(mod3)$
(b). $x_1v_2v_3x_3\dots v_{n-1}x_{n-1}$ when $n = 1(mod3)$
(c).$x_1v_2v_3x_3\dots x_{n-1}v_n$ when $n = 2(mod3)$
In all the cases, the induced subgraph of 2 and 3 is a C_{23}-free graph. Thus, the induced subgraph of any two color classes is acyclic and therefore the coloring given in the algorithm 4.2 is an acyclic coloring of $T(P_n)$. As $T(P_n)$ has paths $v_1v_2\dots v_n$ and $x_1x_2\dots x_{n-1}$, minimum 3 colors are required for the acyclic coloring. Therefore, we have $a[T(P_n)] = 3, \qquad n \geq 2$.

5. Conclusion

In this paper, we obtain the following results.

1. $a[C(P_n)] = n - 1, n \geq 4$.
2. $a[C(F_{m,n})] = m$, if $m \geq n$ and $m \geq 3$.
   $= n - 1$, if $m < n$ and $n \geq 5$.
3. $a[T(P_n)] = 3, n \geq 2$.

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