On Certain Resolving Parameters of Tree Derived Architectures

*Bharati Rajan[1,2], Albert William[1] and S. Prabhu[3]

[1]Department of Mathematics, Loyola College, Chennai, India
[2]School of Electrical Engineering and Computer Science,
The University of Newcastle, NSW 2308, Australia
[3]Department of Mathematics, Velammal Institute of Technology, Chennai, India

pravinovin@gmail.com

Abstract

Let $G(V, E)$ be a graph. A set $W \subset V$ of vertices resolves a graph G if every vertex of G is uniquely determined by its vector of distances to the vertices in W. The metric dimension of G is the minimum cardinality of a resolving set. By imposing conditions on W we get conditional resolving sets.

1 Introduction

A query at a vertex v discovers or verifies all edges and non-edges whose endpoints have different distance from v. In the network verification problem [1], the graph is known in advance and the goal is to compute a minimum number of queries that verify all edges and non-edges. This problem has previously been studied as the problem of placing landmarks in graphs or determining the metric dimension of a graph [2]. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [3, 4, 5].

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For an ordered set $W = \{w_1, w_2 ... w_k\}$ of vertices and a vertex v in a connected graph G, the *code* or *representation* of v with respect to W is the k-vector

$$C_W(v) = (d(v, w_1), d(v, w_2)...d(v, w_k))$$

where $d(x, y)$ is the distance between the vertices x and y. The set W is a *resolving set* for G if distinct vertices of G have distinct codes with respect to W. Equivalently, for each pair of distinct vertices $u, v \in V(G)$ there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. The minimum cardinality of a resolving set for G is called the *resolving number* or *dimension* and is denoted by $dim(G)$.

2 An Overview of the Paper

The concept of resolvability in graphs has previously appeared in literature. Slater [4, 5] introduced this concept, under the name *locating sets*, motivated by its application to the placement of a minimum number of sonar detecting devices in a network so that the position of every vertex in the network can be uniquely determined in terms of its distance from the set of devices. He referred to a minimum resolving set as a reference set and called the cardinality of a minimum resolving set as the *location number.* Independently, Harary and Melter [3] discovered this concept, but used the term metric dimension, rather than location number. Later, Khuller et al. [2] also discovered these concepts independently and used the term metric dimension. These concepts were rediscovered by Chartrand et al. [6] and also by Johnson [7] while attempting to develop a capability of large datasets of chemical graphs.

It was noted in [8] that determining the metric dimension of a graph is *NP*-complete. It has been proved that the metric dimension problem is *NP*-hard [2] for general graphs. Manuel et al. [9] have shown that the problem remains *NP*-complete for bipartite graphs. There are many applications of resolving sets to problems of network discovery and verification [1], pattern recognition, image processing and robot navigation [2], geometrical routing protocols [10], connected joins in graphs [11] and coin weighing problems [12].

Many resolving parameters are formed by combining resolving property with another common graph-theoretic property such as being connected, independent, or acyclic. The generic nature of conditional resolvability in graphs provides various ways of defining new resolving parameters by considering different conditions. In general, a connected graph G can have many resolving sets. It is interesting to study those resolving set whose vertices are located close to one another. In this paper we an independent resolving set in generalized fat trees. We introduce a new resolving parameter called *one star resolving number*. A resolving set W is said to be a one-star resolving set if the subgraph induced by W is a star together with isolated vertices. This paper also introduces a new interconnection network called amalgamation trees and one-star resolving number is investigated for the amalgamation trees.

3 Generalized Fat Trees

Several topologies have been proposed as interconnection networks for multicomputer systems. Among these interconnection networks, the hypercube and mesh topologies are two popular networks from a commercial point of view. However, although the hypercube is an efficient network because of its symmetry, regularity, logarithmic diameter, modularity and fault tolerance, it suffers from wire-ability and packing problems for VLSI implementation due to a non-constant node degree. A good interconnection network must have a relatively small node degree. Therefore a new family of multiprocessor interconnection networks, called generalized fat trees, which includes as special cases the fat trees used for the connection machine architecture CM-5, pruned butterflies, and various other fat trees proposed in the literature have been defined in [14]. This architecture provides a formal unifying concept to design and analyze a fat tree based architecture. Leiserson [15] proposed fat trees as hardware efficient, general-purpose interconnection networks. Several architectures including the Connection Machine CM-5 of Thinking Machines, the memory hierarchy of the KSR-1 parallel machine of Kendall Square Research, and Meiko supercomputers CS-2 are based on the fat trees.

Definition 3.1 *A generalized m-ary fat tree* $GFT(h+1,m,w)$ *is recursively generated from m copies of* $GFT(h,m,w)$, *denoted as* $GFT^j(h,m,w) = G(V_h^j, E_h^j)$, $1 \leq j \leq m$, *and* w^{h+1} *additional nodes such that each top-level node* $(h, k + j.w^h)$ *of each* $GFT^j(h,m,w)$, *for* $0 \leq k \leq w^h - 1$, *is adjacent to w consecutive new top-level nodes, given by* $(h+1, k\cdot w), (h+1, k\cdot w+1), (h+1, k\cdot w+2) \ldots (h+1, (k+1)\cdot w-1)$. *The graph* $GFT^j(h,m,w)$ *is also called a sub-fat tree of* $GFT(h+1,m,w)$. *See Figure 1.*

In the fat tree architecture, the processing elements are located at the leaf nodes and the intermediate nodes serve as routers or switches. Therefore the generalized fat tree $GFT(h+1,m,w)$ of height $h+1$ consists

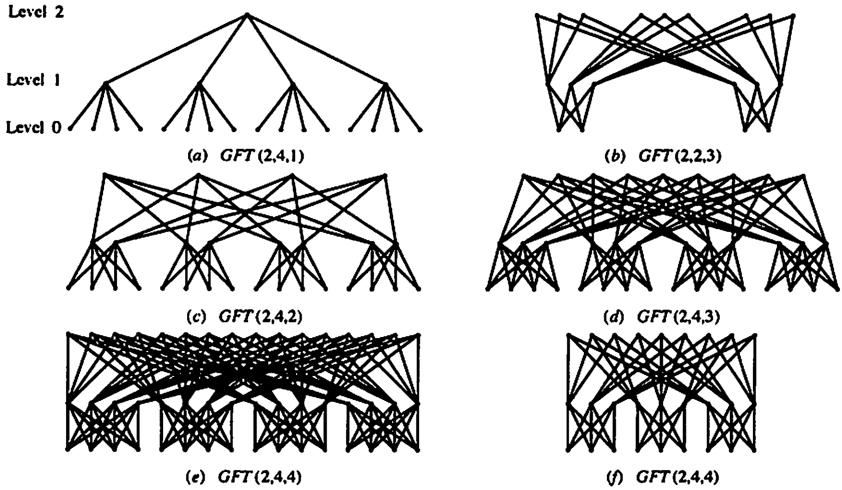

Figure 1: Generalized fat trees of height 2

of m^{h+1} processors in the leaf-level and routers or switching-nodes in the non-leaf levels. Each non-root has w parent nodes and each non-leaf has m children. Leaf nodes are said to be in level 0; a node u is said to be at level l, if there is a vertex v at level 0 such that $d(u,v) = l$. It is clear that $GFT(h+1,m,w)$ has m^{h+1-l} copies of $GFT(l,m,w)$ which are denoted by $GFT^j(l,m,w)$,$1 \le j \le m^{h+1-l}$ and has $m^{h+1-l}.w^l$ vertices in level l.

3.1 Twin vertices and twin sets

In this section we start with the definition of open neighborhood and closed neighborhood and follow the terminology of Caceres et al. [16].

Definition 3.2 *Let u be a vertex of a graph G. The open neighborhood of u is $N(u) = \{v \in V(G) : uv \in E(G)\}$, and the closed neighborhood of u is $N[u] = N(u) \cup \{u\}$.*

Definition 3.3 *Two distinct vertices u, v are adjacent twins if $N[u] = N[v]$, and non-adjacent twins if $N(u) = N(v)$.*

Definition 3.4 *For a graph G, a set $T \in V(G)$ is a twinset of G if u, v are twins in G for every pair of distinct vertices $u, v \in T$.*

Lemma 3.1 [16] *If u and v are twins in a connected graph G, then $d(u,x) = d(v,x)$ for every vertex $x \in V(G) \setminus \{u,v\}$.*

Corollary 1 [16] *Suppose that u and v are twins in a connected graph G and W resolves G. Then u or v is in W. Moreover, if $u \in W$ and v not in W, then $\{W \setminus \{u\}\} \cup \{v\}$ also resolves G.*

Lemma 3.2 [16] *Let u, v, w be distinct vertices in a graph G. If u, v are twins and v, w are twins, then u, w are also twins.*

The next result follows from Lemmas 3.1 and 3.2.

Lemma 3.3 Let G be a connected graph with twin sets T_i, $1 \leq i \leq n$. Then $\beta(G) \geq \sum_{i=1}^{n} |T_i| - n$.

3.2 Independent Resolving Set

In this section we determine the independent resolving number for the generalized fat trees.

Definition 3.5 *A resolving set W of G is independent if the subgraph induced by W is an empty graph.*

Theorem 3.1 *Let G be $GFT(h+1,m,w)$. Then $ir(G) = m^h(m-1) + w^h(w-1)$, $h \geq 2$.*

Proof. $GFT(h+1,m,w)$ has m copies of $GFT(h,m,w)$ which are denoted by $GFT^k(h,m,w)$, $1 \leq k \leq m$. There are m^{h+1} number of level 0 vertices and w^{h+1} number of level $h+1$ vertices. In general, at any level l, G has $m^{h+1-l}.w^l$ vertices and these are the top level vertices of the m^{h+1-l} copies of $GFT(l,m,w)$. Let T_B be the set of all level 0 vertices and T_T be the set of all level $h+1$ vertices. Let $T_{B_i} = \{b_{i_1}, b_{i_2}...b_{i_m}\}$, $1 \leq i \leq m^h$ and $T_{T_j} = \{t_{j_1}, t_{j_2}...t_{j_w}\}$, $1 \leq j \leq w^h$. Consider $b_{i_k}, b_{i_s} \in T_{B_i}$, $k \neq s$. Then it follows from the structure of G that $N(b_{i_k}) = N(b_{i_s})$. This implies that b_{i_k} and b_{i_s} are twins and consequently by Lemmas 3.1 and 3.2 T_{B_i}, $1 \leq i \leq m^h$ and T_{T_j}, $1 \leq j \leq w^h$ are twin sets. Since T_{B_i} contains m vertices, and by Corollary 1 and Lemma 3.3, $m-1$ of these m vertices must belong to any resolving set. For convenience we take the first $m-1$ vertices from any T_{B_i}. This is true for every i. The same argument applies to the set T_{T_j}, $1 \leq j \leq w^h$. Let $W = \{\cup_{i=1}^{m^h} W_{B_i}\} \cup \{\cup_{j=1}^{w^h} W_{T_j}\}$, $W_{B_i} = \{b_{i_1}, b_{i_2}...b_{i_{(m-1)}}\}$,

$1 \leq i \leq m^h$ and $W_{T_j} = \{t_{j_1}, t_{j_2} ... t_{j_{(w-1)}}\}$, $1 \leq j \leq w^h$. Hence $ir(G) \geq |W| = m^h(m-1) + w^h(w-1)$.

We next claim that $ir(G) \leq m^h(m-1) + w^h(w-1)$. To do this we exhibit an independent resolving set of cardinality $m^h(m-1)+w^h(w-1)$. Let $V(G) = \{u^k_{l_p} : 0 \leq l \leq h+1,\ 1 \leq k \leq m^{h+1-l},\ 1 \leq p \leq w^l\}$ where l refers to the level of the vertex, k refers to the position of copies of sub fat trees in level l and p the position of the vertex in level l in the sub-fat tree.

Let $x, y \in V(G) \setminus W$. If x and y are vertices in different levels, then it follows from definition of generalized fat tree that they are resolved by a vertex in some W_{B_i}. So it is sufficient to consider vertices in same level.

Case 1: x and y are level 0 vertices

In this case $x = b_{k_m}$ and $y = b_{s_m}$ for some $k \neq s$. This is because $x, y \in V(G) \setminus W$. Therefore $d(x, b_{k_1}) = 2$ and $d(y, b_{k_1}) > 2$.

Case 2: x and y are level $h+1$ vertices

In this case $x = t_{k_w}$ and $y = t_{s_w}$ for some $k \neq s$ and hence $d(x, t_{k_1}) = 2$ and $d(y, t_{k_1}) > 2$.

Case 3: x and y are level l vertices, $0 < l < h+1$

Case 3.1: x and y belong to different sub fat trees with root nodes in level l. $x = u^{k_1}_{l_{p_1}}$ and $y = u^{k_2}_{l_{p_1}}$, $1 \leq p_1 \leq w^l$, $1 \leq k_1, k_2 \leq m^{h+1-l}$, and $k_1 \neq k_2$

In this case a descendant of x in level 0 lying in W_B resolves x and y.

Case 3.2: x and y belongs to different sub fat trees with root nodes in level l. $x = u^{k_1}_{l_{p_1}}$ and $y = u^{k_2}_{l_{p_2}}$, $1 \leq p_1, p_2 \leq w^l$, $1 \leq k_1, k_2 \leq m^{h+1-l}$, $k_1 \neq k_2$ and $p_1 \neq p_2$.

In this case a descendant of x in level 0 lying in W_B resolves x and y. □

4 Amalgamation Trees

A complete binary tree is a binary tree in which the root is of degree two and every internal vertex is of degree exactly three. A complete binary tree with height h has $2^{h+1} - 1$ vertices and $2^{h+1} - 2$ edges.

In this section we introduce a new architecture called the *amalgamation tree* and list a few of its topological properties. An amalgamation tree $AT(h, m)$ is obtained from m copies of complete binary trees of height h by identifying the corresponding leaf vertices. We shall call these leaf

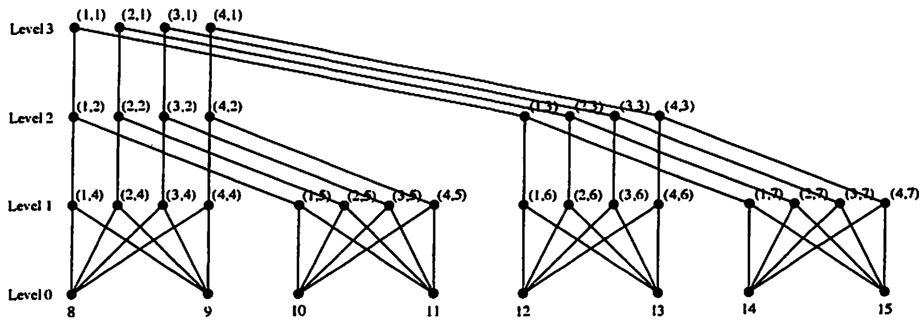

Figure 2: $AT(3,4)$

vertices as *vertices of amalgamation*. Each copy is denoted by T_i^h, $1 \le i \le m$. $AT(h,1)$ is just a complete binary tree of height h. $AT(h,2)$ is called a *diamond tree* in the literature. As in the generalized fat trees, leaf vertices are said to be in level 0. A vertex is said to be in level l, if it is a distance l from a descendent leaf vertex. We propose a labeling of vertices of $AT(h,m)$ using the labeling of the vertices of the complete binary tree. Let $V(AT(h,m)) = \{(i,j) : 1 \le i \le m, 2^{h-l} \le j \le 2^{h-l+1} - 1, 0 \le l \le h\}$. In the ordered pair (i,j), the first component i refers to the i-th complete binary tree T_i^h and the second component j denotes the label of a vertex in T_i^h. See Figure 2.

Theorem 4.1 *Let G be $AT(h,m)$. Then $|V(G)| = 2^h(m+1) - m$ and $|E(G)| = m(2^{h+1} - 2)$.*

Theorem 4.2 *The diameter of $AT(h,m) = 2h$.*

Theorem 4.3 *$AT(h,m)$ is bipartite.*

4.1 One-Star Resolving Number

In this section we introduce a new parameter called *one-star resolving number*.

Definition 4.1 *A resolving set W of G is a one-star resolving set if the subgraph induced by W is a star together with independent vertices.*

The minimum cardinality of a one-star resolving set is called one-star resolving number and is denoted by $os(G)$. A one-star resolving set of cardinality $os(G)$ is called an os-set of G. If G is a connected graph of order n containing an os-set, then it is clear that $2 \leq os(G) \leq n-1$.

Theorem 4.4 *Let G be $AT(h,m)$. Then $os(G) \leq 2^{h-1}+m-1$.*

Proof. We exhibit an os-set of cardinality $2^{h-1}+m-1$. Consider the vertices of amalgamation $\{(i,j) : 1 \leq i \leq m, 2^h \leq j \leq 2^{h+1}-1\}$. Let L be the set of alternate vertices of amalgamation beginning with the first. Since the index i has no role to play, we denote vertices of the set L as $L = \{l_p : l_p = (2^h + 2(p-1)), 1 \leq p \leq 2^{h-1}\}$. Let $S = \{s_q : s_q = (q, 2^{h-1}), 1 \leq q \leq m-1\}$.

We claim that $W = L \cup S$ is a resolving set for G. Let $u, v \in V \setminus W$. We consider different cases depending upon whether u and v belong the same copy of the binary tree or different copies and whether they are in the same level or different level. Let $u = (i_1, j_1)$, $v = (i_2, j_2)$ where $2^{h-l} \leq j_1, j_2 \leq 2^{h-l+1}-1$, $1 \leq i_1, i_2 \leq m$, $0 \leq l \leq h$.

Case 1: u and v belong to the same copy of the binary tree.
In this case $i_1 = i_2$, $1 \leq i_1 = i_2 \leq m$

Case 1.1: u and v are in the same level l , $0 \leq l \leq h$. If $d(u, l_p) = l$, for some $l_p \in L$, then $d(v, l_p) \geq l+2$.

Case 1.2: u and v are in different levels $l_1 < l_2$, $0 \leq l_1, l_2 \leq h$.

If $d(u, l_p) = l_1$, then $d(v, l_p) \geq l_1 + 1$.

Case 2: u and v belong to different copies of the binary tree.
In this case $i_1 \neq i_2$, $1 \leq i_1, i_2 \leq m$

Case 2.1: u and v are in the same level l, $0 \leq l \leq h$.

Case 2.1.1: $j_1 = j_2$

In this case u and v are not resolved by any $l_p \in L$. Now we show some $s_q \in S$ resolves these vertices. If $d(u, s_q) = h - l$, then $d(v, s_q) \geq h - l + 2$.

Case 2.1.2: $j_1 \neq j_2$

If $d(u, l_p) = l$, then $d(v, l_p) \geq l + 2$.

Case 2.2: u and v are in different levels $l_1 < l_2$, $0 \leq l_1, l_2 \leq h$.

If $d(u, l_p) = l_1$, then $d(v, l_p) \geq l_1 + 1$. This proves our claim that W is a resolving set. Further W induces a star S_n and independent vertices. Hence W is an os-set. □

4.2 One-factor and Independent Resolving Number

In this section we determine the one-factor and independent resolving numbers for amalgamation trees. We recall the definitions.

Definition 4.2 *A set W of G is a one factor resolving set for G if $G[W] \cong tK_2$, for some integer t. The minimum t for which $G[W] \cong tK_2$ is called the one factor resolving number of G and it is denoted by $onef(G)$.*

Theorem 4.5 *Let $G = AT(h,m)$. Then $onef(G) \leq 2^{h-1}$ when $m = 2^{h-1}+1$.*

Theorem 4.6 *Let G be $AT(h,m)$. Then $ir(G) \leq 2^{h-1}+m-1$.*

5 Conclusion

In this paper we have exhibited an independent resolving set in generalized fat trees and we have determined three different resolving parameters namely the one-star resolving number, one-factor resolving number and independent resolving number for amalgamation trees. Investigating conditional resolving parameters is an open problem.

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