Minimum 2-edge connected spanning subgraph of certain graphs

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Abstract

Given an undirected 2-edge connected graph, finding a minimum 2-edge connected spanning subgraph is NP-hard. We solve the problem for Butterfly network, Benes network, Honeycomb network and Sierpiński gasket graph.

1 Introduction

The study of connectivity in graph theory has important applications in the areas of network reliability and network design. In fact, with the introduction of fiber optic technology in telecommunication, designing a minimum cost survivable network has become a major objective in telecommunication industry. Survivable networks have to satisfy some connectivity requirements, this means that they are still functional after the failure of certain links [1]. As pointed out in [1], the topology that seems to be very efficient is the network that survives after the loss of $k-1$ or less edges, for some $k \geq 2$, where k depends on the level of reliability required in the network [2]. In this paper, we concentrate on the minimum 2-edge connected spanning subgraph.

A connected graph $G = (V, E)$ is said to be 2-edge connected if $|V| \geq 3$ and the deletion of any set of less than 2 edges leaves a connected graph. The minimum 2-edge connected spanning subgraph (2-ECSS) problem is defined as follows: Given a 2-edge connected graph G, find efficiently a

*This work is supported by UGC-MANF, No.F1-17.1/2011/MANF-CHR-TAM-2135

spanning subgraph $S(G)$ which is also 2-edge connected and has a minimum number of edges [5]. We denote the number of edges in a graph G by $\varepsilon(G)$ and the edges of minimum 2-edge connected spanning subgraph of G by $\varepsilon(S(G))$.

The minimum 2-edge connected spanning subgraph problem is not yet solved for the interconnection networks. In this paper, we compute the number of edges in the minimum 2-edge connected spanning subgraphs of interconnection networks such as Butterfly network $BF(n)$, Benes network $BB(n)$, Honeycomb $HC(n)$ and Sierpiński gasket graph S_n.

2 Butterfly Network

Definition 2.1. [6] *The n-dimensional butterfly network, denoted by $BF(n)$; has vertex set $V = \{(x;i) : x \in V(Q_n), 0 \leq i \leq n\}$. Two vertices $(x;i)$ and $(y;j)$ are linked by an edge in $BF(n)$ if and only if $j = i + 1$ and either*
(i) $x = y$, or
(ii) x differs from y in precisely the jth bit,
for $x = y$, the edge is said to be a straight edge. Otherwise, the edge is a cross edge. For a fixed i, the vertex $(x;i)$ is a vertex on level i.

From the definition 2.1, the butterfly network $BF(n)$ has $(n+1)2^n$ vertices because $BF(n)$ has $n+1$ levels and there are 2^n vertices in every level. Each vertex on level 0 and n is of degree 2, otherwise, every vertex is of degree 4. It follows that $BF(n)$ has $n2^{n+1}$ edges.

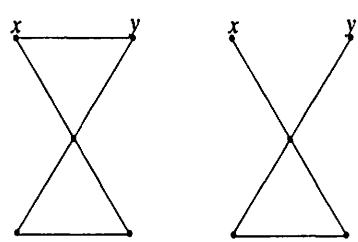

Figure 1: A graph G and its spanning subgraph H

The following Lemma is very interesting and crucial to the proofs of some of our results.

Lemma 2.2. *If one end of every edge of a graph G with $\delta > 1$ is of degree 2 then no proper spanning subgraph of G is 2-edge connected.*

Proof. Let H be a proper spanning subgraph of G. Suppose $(x, y) \notin E(H)$. If $deg_G(x) = 2$, then $deg_H(x) = 1$. Then H is 1-edge connected. See Figure 1. □

The following Lemma 2.3 is an easy consequence of Definition 2.1.

Lemma 2.3. *Let $BF(n), n \geq 3$ be an n-dimensional butterfly network. Then the removal of all crossing edges between levels 1 and $n-1$ disconnects $BF(n)$ into 2^{n-2} components. See Figure 2.*

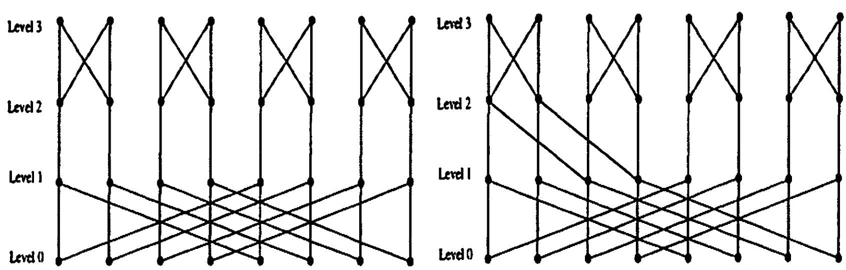

Figure 2: Two disjoint copies in $BF(3)$ after the removal of crossing edges between levels 1 and 2 and Minimum 2-ECSS of $BF(3)$

Theorem 2.4. *Let $BF(r), r \geq 3$ be an r-dimensional butterfly network. Then $\varepsilon(S(BF(r))) = 2^r(r+2) + 2^{r-2}$.*

Proof. Let us prove the theorem by induction on r. When $r = 3$, Lemma 2.2, allows no edges incident to vertices of degree 2 to be removed. Thus the only possible edges that can be removed are the edges between the levels 1 and 2. By Lemma 2.3, removal of all crossing edges between levels 1 and 2 disconnects $BF(3)$ into 2^{3-2} components which is shown in Figure 2. From definition 2.1, adding 2^{3-2} crossing edges between levels 1 and 2 gives a connected minimum 2-edge connected spanning subgraph.

Hence $\varepsilon(S(BF(r = 3))) = \varepsilon(BF(r = 3)) - 6$

$$= 3 \times 2^{3+1} - ((3-2)2^3 - 2^{3-2})$$

$$= 2^3(3+2) + 2^{3-2}$$

Thus the result is true for $r = 3$. We assume that the result is true for $r = k - 1$. When $r = k$, Lemma 2.2, allows no edges incident to vertices of degree 2 to be removed. By Lemma 2.3, removal of all crossing edges between levels 1 and $k-1$ disconnects $BF(k)$ into 2^{k-2} components. Thus, adding 2^{k-2} crossing edges among 2^{k-2} components between levels 1 and $k-1$ gives a minimum 2-edge connected spanning subgraph. Therefore the number of crossing edges removed from $BF(k)$ is $(k-2)2^k - 2^{k-2}$.

Hence $\varepsilon(S(BF(r = k))) = \varepsilon(BF(r = k)) - [(k-2)2^k - 2^{k-2}]$

$$= k2^{k+1} - [(k-2)2^k - 2^{k-2}]$$
$$= 2^k(k+2) + 2^{k-2}$$ □

3 Benes Network

Definition 3.1. [6] *The n-dimensional Benes network consists of back-to-back butterfly, denoted by $BB(n)$. The $BB(n)$ has $2n+1$ levels, each with 2^n vertices. The first and last $n+1$ levels in the $BB(n)$ form two $BF(n)$'s respectively, while the middle level in $BB(n)$ is shared by these butterfly networks.*

From the definition 3.1, the n-dimensional Benes network $BB(n)$ has $(2n+1)2^n$vertices and $n2^{n+2}$ edges. It has only 2-degree vertices and 4-degree vertices, and, thus, is eulerian. The removal of the first and last levels from $BB(n)$ results in two disjoint $BB(n-1)$'s. Benes network is very similar to the butterfly network, in terms of both its computational power and its network structure.

The following Lemma 3.2 is an easy consequence of Definition 3.1.

Lemma 3.2. *Let $BB(n), n \geq 2$ be an n-dimensional benes network. Then the removal of all crossing edges between levels 1 and $2n-1$ disconnects $BB(n)$ into 2^{n-1} components. See Figure 3.*

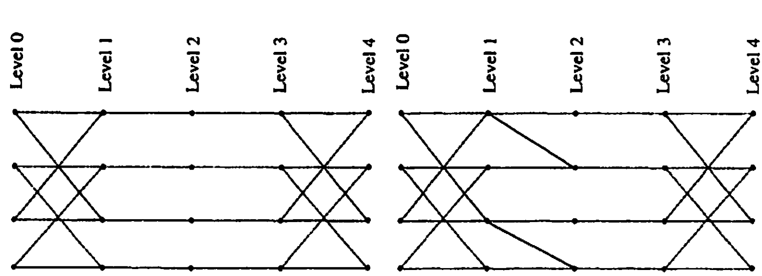

Figure 3: Two disjoint copies in $BB(2)$ after the removal of crossing edges between levels 1 and 3 and Minimum 2-ECSS of $BB(2)$

Theorem 3.3. *Let $BB(r), r \geq 2$ be an r-dimensional benes network. Then $\varepsilon(S(BB(r))) = 2^{k+1}(2^k - 1) + 2^{k-1}$.*

Proof. Let us prove the theorem by induction on r. When $r = 2$, Lemma 2.2, allows no edges incident to vertices of degree 2 to be removed. Thus the only possible edges that can be removed are the edges between the levels 1 and 3. By Lemma 3.2, removal of all crossing edges between levels 1 and 3 disconnects $BB(2)$ into 2^{2-1} components which is shown in Figure 3. The resulting graph is not connected. Therefore using definition of $BB(n)$, adding any two of crossing edges between levels 1 and 3 gives a minimum 2-edge connected spanning subgraph.

Hence $\varepsilon(S(BB(r=2))) = \varepsilon(BB(r=2)) - 6$
$$= 2 \times 2^4 - ((2-1)2^3 - 2)$$
$$= 2 \times 2^4 - 2 \times 2^3 + 2^3 + 2)$$
$$= 2^3(2+1) + 2$$
Thus the result is true for $r = 2$.

We assume that the result is true for $r = k-1$. When $r = k$, Lemma 2.2, allows no edges incident to vertices of degree 2 to be removed. By Lemma 3.2, removal of all crossing edges between levels 1 and $2k-1$ disconnects $BB(k)$ into 2^{k-1} components. Thus adding any two of crossing edges between levels 1 and $2k-1$ gives a minimum 2-edge connected spanning subgraph. Therefore the number of crossing edges removed from $BB(k)$ is $(k-1)2^{k+1} - k$.

Hence $\varepsilon(S(BB(r=k)))= \varepsilon(BB(r=k)) - [(k-1)2^{k+1} - k]$
$$= k2^{k+2} - (k-1)2^{k+1} + k$$
$$= 2^{k+1}(k+1) + k$$

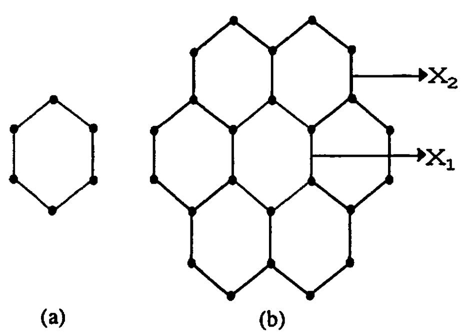

Figure 4: (a) A 1-dimensional honeycomb $HC(1)$ and (b) 2-dimensional honeycomb $HC(2)$

4 Honeycomb Network

Definition 4.1. [4] *A honeycomb network can be built in various ways. The honeycomb network $HC(1)$ is an hexagon; See Figure 4(a). The honeycomb network $HC(2)$ is obtained by adding a layer of six hexagons to the boundary edges of $HC(1)$ as shown in Figure 4(b). Inductively honeycomb network $HC(r)$ is obtained from $HC(r-1)$ by adding a layer of hexagons around* *the boundary of* $HC(n-1)$. *The number of vertices and edges of* $HC(n)$ *are* $6r^2$ *and* $9r^2 - 3r$ *respectively. If* X_1 *denotes the boundary of* $HC(1)$, X_2 *denotes the boundary of* $HC(2)$,..., X_r *denotes the boundary of* $HC(r)$, *then the number of edges in* X_i *is* $12i$-6, $1 \leq i \leq r$.

Lemma 4.2. *Let* $HC(2)$ *be a 2-dimensional honeycomb network. Then* $\varepsilon(S(HC(2))) = \sum_{i=1}^{2} \varepsilon(X_i) + 1$.

Proof. Consider $HC(2)$. Since one end of every edge of X_2 is of degree two, by Lemma 2.2 none of the edges in X_2 can be removed. Removing any two consecutive edges of X_1 gives a 1-edge connected spanning subgraph. See Figure 5(b).

Let $z_1 = (m,n)$, $z_2 = (o,p)$, $z_3 = (q,r)$ and $z_4 = (s,t)$ be the four edges connecting X_1 and X_2 and $z_5 = (u,v)$ be an edge of X_1. See Figure 5 (a). Removing the edges z_1, z_2, z_3 and z_4, no edges incident with z_1, z_2, z_3 and z_4 can be removed, otherwise we get a 1-edge connected spanning subgraph. But, removing an edge $z_5 = (u,v)$ of X_1 which is not incident with z_1, z_2, z_3 and z_4 gives a minimum 2-edge connected spanning subgraph. Therefore the maximum number of edges removed is 5.

Removing all the six edges connecting X_1 and X_2 disconnects $HC(2)$. See Figure 5 (c). Similarly, removing six edges of X_1 gives a 1-edge connected spanning subgraph. See Figure 5 (d). By an easy verification, removing three consecutive edges of X_1 and any three edges connecting X_1 and X_2 (See Figure 5 (e)) or three alternative edges of X_1 and any three edges connecting X_1 and X_2 (See Figure 5 (g)) or removing two consecutive edges of X_1 and any four edges connecting X_1 and X_2 or two alternative edges of X_1 and any four edges connecting X_1 and X_2 (See Figure 5 (j)) or removing five edges connecting X_1 and X_2 and an edge of X_1 vice versa gives either a 1-edge connected spanning subgraph or a disconnected graph. See Figures 5 (f,h,i,k,l). Therefore removing any six edges of $HC(2)$ gives a 1-edge connected spanning subgraph or a disconnected graph. Thus, in the resulting spanning subgraph $S(HC(2))$, an edge of X_1 and every edge except two edges connecting X_1 and X_2 are removed. Hence $\varepsilon(S(HC(2)))$ $= 5 + 18 + 2 = \varepsilon(X_1) - 1 + \varepsilon(X_2) + 2 = \sum_{i=1}^{2} \varepsilon(X_i) + 1$. ∎

Theorem 4.3. *Let* $HC(r)$, $r \geq 3$ *be the honeycomb network of dimension* r. *Then* $\varepsilon(S(HC(r))) = \sum_{i=1}^{r} \varepsilon(X_i) + (r-1)$.

Proof. We prove this theorem by induction on r. When $r = 3$, Consider $HC(3)$. Since one end of every edge of X_3 is of degree two, by Lemma

2.2 none of the edges in X_3 can be removed. Let us construct the minimum 2-edge connected spanning subgraph of $HC(3)$ using minimum 2-edge connected spanning subgraph $S(HC(2))$. It is easy that adding two edges between X_2 of $S(HC(2))$ and X_3 will give a 2-edge connected spanning subgraph. Therefore $\varepsilon(S(HC(3))) = \varepsilon(S(HC(2))) + \varepsilon(X_3) + 2$. But we construct in such a way that adding two edges between X_2 of $S(HC(2))$ and X_3 where the two edges added should have one end at a unique edge of X_2 and deleting that edge gives a minimum 2-edge connected spanning subgraph where no other edges can be removed by Lemma 2.2. See Figure 6.

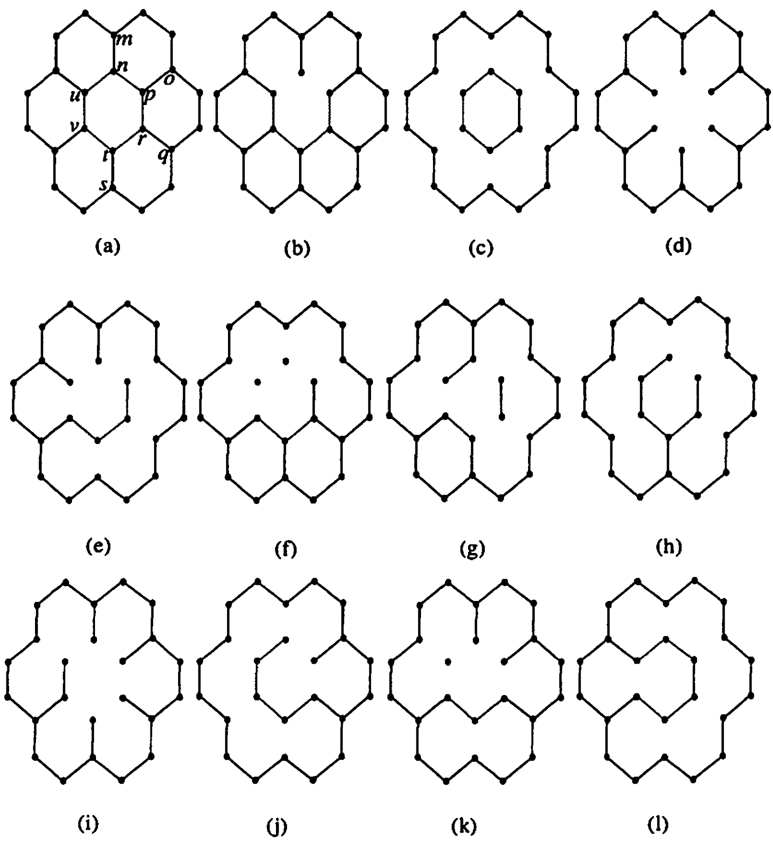

Figure 5: A 2-dimensional honeycomb $HC(2)$

$$\text{Thus } \varepsilon(S(HC(3))) = (\varepsilon(S(HC(2))) + \varepsilon(X_3) + 2) - 1$$
$$= (\sum_{i=1}^{2} \varepsilon(X_i) + 1 + \varepsilon(X_3) + 2) - 1$$

$$= \sum_{i=1}^{2} \varepsilon(X_i) + \varepsilon(X_3) + 2$$
$$= \sum_{i=1}^{3} \varepsilon(X_i) + (3-1).$$

We assume that the result is true for $r = k-1$. Consider $HC(k)$. Since it is not possible to remove any edges in X_r, $S(HC(r))$ can be constructed from $S(HC(r-1))$ and adding two edges between X_{r-1} and X_r where both the edges added have one end at a unique edge of X_{r-1} and delete that edge.

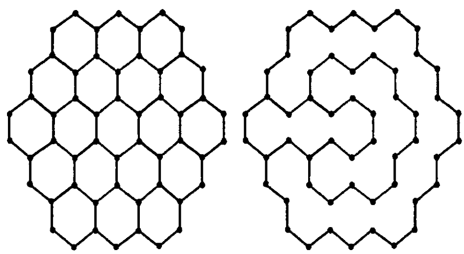

Figure 6: A 3-dimensional honeycomb and its minimum 2-edge connected spanning subgraph

Thus $\varepsilon(S(HC(k))) = (\varepsilon(S(HC(k-1))) + \varepsilon(X_k) + 2) - 1$
$$= (\sum_{i=1}^{k-1} \varepsilon(X_i) + ((k-1)-1) + \varepsilon(X_k) + 2) - 1$$
$$= \sum_{i=1}^{k-1} \varepsilon(X_i) + \varepsilon(X_k) + ((k-1)-1) + 2 - 1$$
$$= \sum_{i=1}^{k} \varepsilon(X_i) + (k-1).$$
□

5 Sierpiński Gasket Graph

Definition 5.1. [3] *The Generalized Sierpiński Graph $S(n,k), n \geq 1, k \geq 1$ is defined in the following way: $V(S(n,k)) = \{1,2,...,k\}^n$, two distinct vertices $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ being adjacent if and only if there exists an $h \in \{1,2,...,n\}$ such that*
(i) $u_t = v_t$, for $t = 1,...,h-1$;
(ii) $u_h \neq v_h$; and
(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h+1,...,n$. For convenience, we write the vertex $(u_1, u_2, ..., u_n)$ as $\langle u_1 u_2 ... u_n \rangle$. The vertices $\langle 1...1 \rangle, \langle 2...2 \rangle, ..., \langle k...k \rangle$ are called the extreme vertices of $S(n,k)$. In the literature, $S(n,3), n \geq 1$ is known as the Sierpiński graph. For i = 1,2,3, let $S(n+1,3)_i$ be the subgraph induced by the vertices of the form $\langle i... \rangle$. Clearly $S(n+1,3)_i$ is

isomorphic to $S(n,3)$ for i = 1,2,3. The Sierpiński gasket graph $S_n, n \geq 1$, can be obtained by contracting all the edges of $S(n,3)$ that do not lie on any triangle. If $\langle u_1u_2...u_rij...j\rangle$ and $\langle u_1u_2...u_rji...i\rangle$ are the end vertices of such an edge, then we will denote the corresponding vertex of S_n by $\langle u_1...u_r\rangle\{i,j\}$, $r \leq n-2$. Thus S_n is the graph with three special vertices $\langle 1...1\rangle, \langle 2...2\rangle$ and $\langle 3...3\rangle$ called the extreme vertices of S_n, together with vertices of the form $\langle u_1...u_r\rangle\{i,j\}$, $0 \leq r \leq n-2$ where all u_k's, i and j are from {1,2,3}. This labeling is called quotient labeling of S_n and $\langle u_1...u_r\rangle$ is called the prefix of $\langle u_1...u_r\rangle\{i,j\}$. S_{n+1} contains three isomorphic copies of S_n that can be described as follows: For i = 1,2,3 let $S_{n,i}$, be the subgraph of S_{n+1} induced by $\langle i...i\rangle, \{i,j\}, \{i,k\}$ where $\{i,j,k\} = \{1,2,3\}$, and all the vertices whose prefix starts with i. S_n has $\frac{3}{2}(3^{n-1}+1)$ vertices and 3^n edges.

Lemma 5.2. *Let S_2 be the 2-dimensional sierpiński gasket graph. Then $\varepsilon(S(S_2)) = 2 \times 3^{2-1}$.*

Proof. From Definition 5.1, S_2 has 3^2 edges. Now label the vertices of S_2 as shown in Figure 7. By Lemma 2.2, edges (1,2), (1,3), (4,2), (4,5), (6,3) and (6,5) cannot be removed. Therefore removing the edges (2,3), (2,5) and (3,5) gives a minimum 2-edge connected spanning subgraph. Thus $\varepsilon(S(S_2)) = 2 \times 3^{2-1}$.

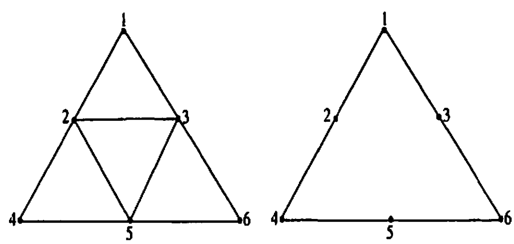

Figure 7: Sierpiński gasket S_2 and its minimum 2-ECSS

Theorem 5.3. *Let $S_r, r \geq 3$ be the r-dimensional sierpiński gasket graph. Then $\varepsilon(S(S_r)) = 2 \times 3^{r-1}$.*

Proof. We prove this theorem by induction on r. When $r = 3$, S_3 contains 3 copies of S_2. Now we construct a minimum 2-edge connected spanning subgraph of S_3 using 3 copies of minimum 2-edge connected spanning subgraph of S_2. Hence $S(S_3) = 3S(S_2) \Rightarrow \varepsilon(S(S_3)) = 3\varepsilon(S(S_2)) = 3 \times 6 = 2 \times 3^{3-1}$. Thus the result is true for $r = 3$.
We assume that the result is true for $r = k-1$. That is, $\varepsilon(S(S_{k-1})) = 3\varepsilon(S(S_{k-2})) = 2 \times 3^{k-2}$. Consider $r = k$. S_k contains three copies of

S_{k-1}. Construct a minimum 2-edge connected spanning subgraph of S_k using 3 copies of minimum 2-edge connected spanning subgraph of S_{k-1}. Thus $S(S_k) = 3S(S_{k-1}) \Rightarrow \varepsilon(S(S_k)) = 3\varepsilon(S(S_{k-1})) = 3 \times 2 \times 3^{k-2} = 2 \times 3^{k-1}$. □

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