A Note On Magic Graphs

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Abstract

By a (1,1) edge-magic labeling of a graph $G(V, E)$, we mean a bijection f from $V \cup E$ to $\{1,\ldots,|V| \cup |E|\}$ such that for all edges $uv \in E(G)$, the value of $f(u) + f(v) + f(uv)$ is the same. We give a different proof of a well-known result of Paul Erdos in additive number theory and curiously an interesting application of the same is realized. Also some progress is made with the help of computers towards the conjecture that: "Every graph on $p \geq 9$ vertices can be embedded as a sub graph of some (1,1) edge-magic graph" raised by Yegnanarayanan.

1 Introduction

The graphs considered in this paper are finite, simple and undirected. By a graph labeling we mean an assignment of integers to the elements of a graph such as vertices, or edges or both subject to some conditions. These conditions are usually stated on the basis of the values of some function. That function will produce partial sums of the labeled elements of the graph. The partial sums will be either a et of vertex weights, obtained for each vertex by adding all the labels of a vertex and its adjacent edges, or a set of edge weights, obtained for each edge by adding the labels of an edge and its end vertices. A notable instance is one, when all the edge weights or all the vertex weights are the same. In this paper, we prove certain results concerning a specialized variety of magic labeling. Incidentally we have borrowed some ideas from Number Theory and used computer programming (C++).

By $G(p,q)$ we mean a graph having p vertices and q edges, by $V(G)$ and $E(G)$ the vertex-set and the edge-set of G respectively. A graph $G(p,q)$ is said to be (1,1) edge-magic with the common edge count k_0 if there exists a bijection $f : V(G) \cup E(G) \to \{1,\dots,p+q\}$ such that $f(u)+f(v)+f(e) = k_0$ for all $e = (u,v) \in E(G)$. A graph $G(p,q)$ is said to be (1,1) vertex-magic with the common vertex number count k_1 if there exists a bijection $f : V(G) \cup E(G) \to \{1,\dots,p+q\}$ such that for each $u \in V(G), f(u)+\sum_e f(e) = k_1$ for all $e = (u,v) \in E(G)$ with $v \in V(G)$. A graph $G(p,q)$ is said to be $(1,0)$ edge-magic with the common edge count k_2 if there exists a bijection $f : V(G) \to \{1,\dots,p\}$ such that for all $e = (u,v) \in E(G), f(u)+f(v) = k_2$. A graph $G(p,q)$ is said to be $(0,1)$ vertex-magic with the common vertex count k_3 if there exists a bijection $f : E(G) \to \{1,\dots,q\}$ such that for each $u \in V(G), \sum_e f(e) = k_3$ for all $e = (u,v) \in E(G)$ with $u \in V(G)$. A graph $G(p,q)$ is said to be $(1,0)$ vertex-magic with the common vertex count k_4 there exists a bijection $f : V(G) \to \{1,\dots,p\}$ such that for each $u \in V(G), f(u) + f(v) = k_4$ for all $v \in V(G)$ such that $(u,v) \in E(G)$. A graph $G(p,q)$ is said to be $(0,1)$ edge-magic with the common edge-count k_5 if there exists a bijection $f : E(G) \to \{1,\dots,q\}$ such that for each $e \in E(G), f(e) + f(e_0) = k_5$ for all $e \in E(G)$ such that e and e_0 are adjacent in G. We have made some observations in [4,5, 6] concerning these labeling.

2 Main Results

Theorem 2.1 *If $A = \{a_i : 1 \leq i \leq R$ and $a_1 < \ldots < a_n\}$ is a subset of integers in [1,N] such that $B = \{a_r - a_s : a_r, a_s \in A$ and $s < r\}$ are distinct, then $R \leq N^{\frac{1}{2}} + N^{\frac{1}{4}} + 1$.*

Remark 2.1 *The above result was due to Erdos in [2]. An improvement of this was obtained later by Bose and Chowla in [1]. One can also see [3] for an exhaustive information. We present here a slightly different proof, enabling to compute the 0-constant in the error term better.*

Proof. Let us define $A_j = \{a_{r+j} - a_r : 1 \leq r \leq R-j\}$. Then $|A_j| = R-j$ and $\sum_{b \in A_j} b = \sum_{1 \leq r \leq R-j}(a_{r+j} - a_r) = (a_R - a_1) + (a_{R-1} - a_2) + \ldots + (a_{R-j+1} - a_j) \leq jN$. Let $D = \cup_{1 \leq j \leq k} A_j$. Then $|D| = kR - k(k+1))/2$ and $\sum_{d \in D} d \leq \sum_{1 \leq j \leq k} jN = (k(k+1)/2)N$. Now the elements of D are distinct by definition, and $\sum_{d \in D} d$ is at least the sum of first $|D|$ natural numbers. Hence $\sum_{d \in D} d \geq ((kR - k(k+1)/2(kR - k(k+1)/2 + 1))/2$. Comparing the upper and lower bound of $\sum_{d \in D} d$, we get that $(k(k +$

$1)N)^{\frac{1}{2}} \geq kR - k(k+1)/2$. That is $(N(k+1)/k)^{\frac{1}{2}} \geq R - (k+1)/2$ and $R \leq N^{\frac{1}{2}}(1 + (1/2k)) + (k+1)/2$. Now set $k\lfloor N^{\frac{1}{4}} \rfloor + 1$ to get the result.

Remark 2.2 *Now following Erdos, we shall call any set A as in the Theorem 2.1 as a Siden set.*

Theorem 2.2 *The complete graph K_p is not $(1,1)$ edge-magic if $p \geq 17$.*

Proof. Assume that K_p is (1,1) edge-magic. Then there exists a (1,1) edge-magic labeling f : $V(K_p) \cup E(K_p) \to \{1, \ldots, (p + p(p-1)/2) = N\}$. If $e_i = (u_i, v_i)$ for $1 \leq i \leq 2$ are any two edges, then $c(e_1) = c(e_2)$ implies $f(u_1)$- $f(u_2)$ = $f(v_1)$- $f(v_2)$. Now as $\{f(u) : u \in V(K_p)\}$ is a Siden set in [1,N], we get by the Theorem 2.1, that $p = |V(K_p)| \leq N^{\frac{1}{2}} + N^{\frac{1}{4}} + 1$. This yields a contradiction if $p \geq 17$ is clear. For, put $N = x^4$, so that $p(p+1) = 2x^4$ and $2x^4 \leq (p + (1/2))^2$. Now $p + (1/2) \leq N^{\frac{1}{2}} + N^{\frac{1}{4}} + 3/2$ implies that $\sqrt{2}x^2 \leq x^2 + x + 3/2$ and $x \leq 3.45$.

Problem 2.1 For what values of m and n, is the graph $K_{m,n}$ (0,1) vertex-magic?

Theorem 2.3 *Every graph on p vertices with $9 \leq p \leq 12$ can be embedded as a subgraph of some (1,1) edge-magic graph.*

Comments: Consider K_p for $9 \leq p \leq 12$. Form a new graph G from K_p by adding a number of vertices and joining each of them to appropriate vertices of K_p. We do this using computers to match the following requirement: How many vertices are needed to add to K_p and how their adjacency with the vertices of K_p can be defined so that the resulting graph G is connected and there exists a bijection $f : V(G) \to \{1, \ldots, p(G)\}$ with the induced edge labeling $f_E : E(G) \to Z^+$ forming the set of consecutive integers $\{r, r+1, \ldots, q(G) + r - 1\}$ for some r $(= f_E(e) = f(u) + f(v))$ $\in Z^+$ where $e = (u, v) \in E(G)$. We give a detailed proof for $p = 9$. It is routine to organize the details for $p = 10$ to 12 in a similar manner. But we give the complete vertex labels of the corresponding G's extending K_p for $p = 10$ to 12 in Tables 3 to 5. In these Figures the pendant vertex/vertices with their respective labels adjacent to a vertex of K_p with its label are indicated by an arrow.

Proof. Let $V(K_9) = \{v_i : 1 \leq i \leq 9\}$. Form a new graph $G = K_9(v_1, v_2, \ldots, v_8) \bullet (K_{1,1}, K_{1,2}, K_{1,1}, K_{1,3}, K_{1,2}, K_{1,2}, K_{1,19}, K_{1,21})$, where G is the graph obtained from K_9 by identifying the central vertex of $K_{1,1}, K_{1,2}, K_{1,1}, K_{1,3},$

$K_{1,2}$,$K_{1,2}$, $K_{1,19}, K_{1,21}$, with $v_1, \ldots, v_8$ respectively. Denote the pendant vertices of G by $v_1^1, v_2^1, v_2^2, v_3^1, v_4^1, v_4^2, v_4^3, v_5^1, v_5^2, v_6^1, v_6^2, v_7^i, 1 \leq$ i ≤ 19, v_8^i, 1 $\leq$ i $\leq$ 21. Suppose that f : V(G) $\rightarrow$ (1,...,60} is a bijection defined as follows: $f(v_1) = 1; f(v_2) = 2; f(v_i) = f(v_{i-1}) + f(v_{i-2}); f(v_1^1) = 16; f(v_2^1) = 10; f(v_2^2) = 18; f(v_3^1) = 29; f(v_4^1) = 23; f(v_4^2) = 26; f(v_4^3) = 47; f(v_5^1) = 11; f(v_5^2) = 42; f(v_6^1) = 20;$ $f(v_6^2) = 31; f(v_7^1) = 4; f(v_7^2) = 6; f(v_7^3) = 9; f(v_7^4) = 17; f(v_7^5) = 19; f(v_7^6) = 22; f(v_7^7) = 24; f(v_7^8) = 30; f(v_7^{8+i}) = 31 + i, 1 \leq i \leq 2; f(v_7^{10+i}) = 42 + i$, for $1 \leq i \leq 4; f(v_7^{14+i}) = 55 + i, for 1 \leq i \leq 5; f(v_8^1) = 7; f(v_8^2) = 12; f(v_8^{2+i}) = 13 + i, for 1 \leq i \leq 2; f(v_8^5) = 25; f(v_8^{5+i}) = 26+i, for 1 \leq i \leq 2; f(v_8^{7+i}) = 34+i, for 1 \leq i \leq 7; f(v_8^{14+i}) = 47+i, for 1 \leq i \leq 7$. Then the induced edge-labeling $f_E : E(G) \rightarrow Z^+$ forms the set A of consecutive and distinct integers A $= \cup A_i$ with i = 1 to 9, where A_1 = {3, 4, ..., 11, 13, ..., 16, 18, 21,..., 24, 26, 29, 34,..., 37, 39, 42, 47, 55,..., 58, 60, 63, 68, 76, 89} from the vertex labels of $K_9; A_2$ = {17} from the label of the vertex adjacent to $v_1; A_3$ = {12, 26} from the label of the vertices adjacent to $v_2; A_4$ = {32} from the label of the vertex adjacent to $v_3; A_5$ = {28, 31, 52} from the label of the vertices adjacent to $v_4; A_6$ = {19, 50} from the label of the vertices adjacent to $v_5; A_7$ = {33, 44} from the label of the vertices adjacent to $v_6; A_8$ = {25,27, 30, 38, 40, 43, 45, 51, 53, 54, 64 to 67, 77 to 81} from the label of the vertices adjacent to $v_7; A_9$ = {41, 46, 48, 49, 59, 61, 62, 69 to 75, 82 to 88} from the label of the vertices adjacent to v_8. Now define a bijection $f_1 : V(G) \cup E(G) \rightarrow \{1, \ldots, 147\}$ by $f_1(V(G)) = f(V(G))$ and $f_1(E(G)) = \{147, 146, \ldots, 61\}$. Then one can check that f_1 is a required labeling with the common edge count 150.

Theorem 2.4 *Let G be an $r-$regular $(p, q)(0, 1)$ edge-magic graph. Then a) if $r \equiv 1 (mod\ 2)$, then $p \equiv 2\ (mod\ 4)$; b) if $r \equiv 2\ (mod\ 4)$ and $p \equiv 0\ (mod\ 2)$, then G contains no component of an odd order; c) If $p > 2$, then $r > 2$.*

Proof. First observe that for an r-regular (p, q) graph G that $k_p = 2f(e) = q(q+1), e \in E(G)$ where k is the common edge count and $q = rp/2$ implies $r = r(1 + rp/2)/2$. Now to see that (a) is true, suppose that $p \equiv 0(\text{mod } 2)$. Then $k = r(1 + (rp/2))/2$ implies that k is not an integer. But this is a contradiction to the fact that the common edge count k is a sum of integer labels of the edges of G. Further as the order of an odd regular graph is even it follows that p $\equiv$ 2(mod 4). Now to see that (b) is true, let us assume that G contains a component C_0 of an odd order. Then from k = r(1 + (rp/2))/2 it follows that k is odd. Hence $k|V(C_0)|$ is odd. But then this implies that $k|V(C_0)| = 2\sum f(e), e \in E(C_0)$, a contradiction. Now it is easy to see (c) as a regular graph of degree one is magic if and only if it is connected (ie., p is equal to 2) and a 2-regular graph is never magic.

K_{10}

1	→ 16, 24
2	→ 10, 18, 31
3	→ 29, 50
5	→ 23, 26, 39, 47
8	→ 11, 37, 42, 63, 76
13	→ 60, 68
21	→ 6, 9, 17, 19, 22, 30, 43, 44, 56, 57, 58
34	→ 7, 12, 14, 15, 20, 27, 28, 32, 33, 35, 36, 38, 40, 41, 48, 49, 51 to 54, 59, 61, 62, 64, 65, 69 to 75, 77 to 86, 90 to 107
55	→ 4, 25, 45, 46, 66, 67, 87, 88
89	

Table 1:

K_{11}

1	→ 11, 16, 29, 37
2	→ 18, 23, 26, 31, 52
3	→ 50, 84
5	→ 39, 47, 60, 68, 81
8	→ 24, 42, 58, 63, 71, 73, 76, 97, 110, 131
13	→ 6, 14, 32, 94, 102, 115, 118, 123
21	→ 10, 19, 22, 27, 40, 53, 56, 57, 165
34	→ 7, 12, 15, 17, 28, 33, 35, 36, 38, 48, 49, 54, 59, 61, 69, 70, 74, 75, 77 to 80, 82, 90 to 92, 95, 96, 98 to 101, 103, 116, 117, 119 to 122, 124, 136 to 143, 145, 157 to 164, 166, 178 to 187
55	→ 4, 9, 20, 25, 30, 41, 43 to 46, 51, 62, 64 to 67, 72, 83, 85 to 88, 93, 104 to 109, 111 to 114, 125 to 130, 132 to 135, 146 to 156, 167 to 177
89	
144	

Table 2:

K_{12}	1	→ 16, 29, 42, 50
	2	→ 10, 18, 23, 31, 36, 52
	3	→ 84
	5	→ 26, 39, 44, 47, 60, 68, 81, 136
	8	→ 11, 24, 45, 58, 63, 71, 73, 76, 92, 97, 105, 110, 118, 123, 128, 131
	13	→ 15, 32, 37, 57, 65, 86, 94, 102, 107, 115, 149, 157, 170, 178, 204, 212
	21	→ 6, 19, 53, 139, 152, 165, 183, 186, 191, 199
	34	→ 7, 12, 14, 27, 28, 33, 35, 48, 49, 54, 69, 70, 74, 75, 78, 90, 91, 95, 96, 99, 116, 117, 120, 137, 138, 141, 158 to 160, 162, 179 to 181, 225, 246, 267
	55	→ 4, 9, 17, 20, 22, 25, 30, 38, 40, 41, 43, 46, 51, 56, 59, 61, 62, 64, 66, 67, 72, 77, 79, 80, 82, 83, 85, 87, 88, 93, 98, 100, 101, 103, 104, 106, 108, 109, 111 to 114, 119, 121, 122, 124 to 127, 129, 130, 132 to 135, 140, 142, 143, 145 to 148, 150, 151, 153 to 156, 161, 163, 164, 166 to 169, 171 to 177, 182, 184, 185, 187 to 190, 192 to 198, 200 to 203, 205 to 211, 213 to 224, 226 to 232, 234 to245, 247 to 266, 268 to 321
	89	→
	144	→
	233	→

Table 3:

Acknowledgment
Authors express their sincere thanks to the Stella Maris College, Chennai, India in general and in particular Dr.R. Bharati, Dr.Indra Rajasingh, Dr.Sr.Jasintha Quadras fmm, Dr.P.Christie SJ, the organizers of International Conference On Mathematics in Engineering and Business Manage-

ment ICMEB 2012 (held between 9-10 March 2012) for 1) accepting the paper and 2) having brought the Proceedings of the Conference with ISBN: 978-81-8286-015-5.

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