Oriented Diameter of Grids

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Abstract

A grid is a large-scale geographically distributed hardware and software infra-structure composed of heterogeneous networked resources owned and shared by multiple administrative organizations which are coordinated to provide transparent, dependable, pervasive and consistent computing support to a wide range of applications. One of the major problems in graph theory is to find the oriented diameter of a graph G, which is defined as the smallest diameter among the diameter of all strongly connected orientations. The problem is proved to be NP-complete. In this paper we obtain the oriented diameter of grids.

Keywords: Directed graph, strongly connected orientation, oriented diameter, grid.

1 Introduction

A network is a graph with a collection of nodes interconnected by edges. The distance $d(u, v)$ between two vertices u and v in a graph G is the length of a shortest path between them. The diameter $d(G)$ of a graph G is the maximum of the distances between pairs of vertices. An orientation O of an underlying undirected graph G is a directed graph $G(O)$ whose arcs correspond to assignment of directions to the edges of G. A vertex v of a simple digraph is said to be reachable from u if there is a directed path from u to v. In general, the condition that u is reachable from v does not imply that v is also reachable from u. A directed graph G is said to be strongly connected, if for every pair of vertices of the graph, both the vertices of the pair are reachable from one another. An orientation O is strongly connected if the induced digraph is strongly

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connected. The oriented diameter $\bar{d}(G)$ of a directed graph G is the smallest diameter among the diameters of all strongly connected orientations of G.

In 1939, Robbins proved that every undirected graph G admits a strongly connected orientation if and only if G is connected and bridgeless [3]. Strongly connected orientations of graphs have been studied by Chvatal and Thomassen [3, 4, 12]. If a graph G is thought of as the plan of the system of two-way streets, then the orientations of G can be viewed as arrangements of one-way streets. Applications also appear in network routing, broadcasting and gossip problems [3, 7, 9]. A variety of interrelated diameter problems are discussed in the literature [7]. Among interconnection network topologies, the grid has been extensively studied because it has many advantages over other topologies, such as short diameter, short average distance, simple connection method, ease of routing, node symmetry and edge symmetry [6].

The term "Grid" was coined in the mid 1990s to denote a proposed distributed computing infrastructure for advanced science and engineering. Considerable progress has since been made on the construction of such an infrastructure (e.g., [1, 10]), but the term grid has also been conflated, at least in popular perception, to embrace everything from advanced networking to artificial intelligence [6]. Finding oriented diameter of a graph is NP-complete [8]. Fedor et al. have shown that finding minimum oriented diameter of chordal and split graphs remain NP-complete and obtained approximated oriented diameter of chordal graphs [5]. They have derived linear bounds for the oriented diameter of AT-free graphs. Earlier, Chvatal and Thomassen studied the problem of finding the largest oriented diameter among graphs of diameter d [4]. In this paper we obtain the oriented diameter for grid graph.

2 Oriented Diameter of Grids

An orientation of an undirected graph G is an assignment of exactly one direction to each of edges of G. There are $2^{|E|}$ orientation for G. Let $O_x(G)$ denote the set of all orientations of G. For an orientation $O \in Ox$, let $G(O)$ denote the directed graph with orientation O and whose underlying graph is G.

An orientation O of an undirected graph G is said to be strongly connected if for any two vertices x, y of the directed graph $G(O)$, there are both (x, y)-path and (y, x)-path in $G(O)$. Let $O_s(G)$ denote the set of all strongly connected orientations of G [2].

Let $\bar{d}(G)$ denote the diameter of the directed graph G and $d(G)$ denote the diameter of the undirected graph G.

Definition 1. The oriented diameter $\bar{d}(G) = min\{\bar{d}(G(O)) : O \in O_s(G)\}$

Theorem 2. *For any graph G, $\bar{d}(G) \geq d(G)$.*

Theorem 2 gives a lower bound for the oriented diameter of a graph G. In this paper, we prove that the bound is sharp for grid networks.

Grid graph is a well known architechture with an infrastructure that bonds and unifies globally remote and diverse resources in order to provide computing

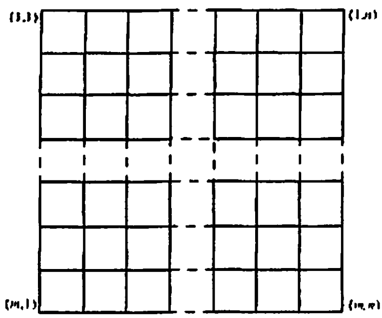

Figure 2.1: Grid $M(m \times n)$

support for a wide range of applications. It has been used for parellel computing, distributed supercomputing, high-throughput computing support, on-demand computing support, collaborative computing support, data-intensive computing support, multimedia computing support and VLSI layout [2].

Definition 3. [11] Let m and n be positive integers such that $m \leq n$. An $m \times n$ grid graph $M(m,n)$ is a graph where $V = \{(i,j)|1 \leq i \leq m, 1 \leq j \leq n\}$ and $E = \{((i,j),(i,j+1)) : 1 \leq i \leq m, 1 \leq j \leq n-1\} \bigcup \{((i,j),(i+1,j)) : 1 \leq i \leq m-1, 1 \leq j \leq n\}$. See Figure 2.1.

Notations:

Rows of $M(m \times n)$ are oriented either from left to right ($\rightarrow$) or from right to left ($\leftarrow$) and the columns of $M(m \times n)$ are oriented either from top to bottom ($\downarrow$) or from bottom to top ($\uparrow$)). If $v = (i,j)$, then $v(\rightarrow,\downarrow)$ indicates that row i is oriented from left to right and column j is oriented from top to bottom. Similarly $v(\rightarrow,\uparrow), v(\leftarrow,\downarrow)$ and $v(\leftarrow,\uparrow)$ are defined. Let $\bar{d}(u,v)$ be the distance from u to v in a directed graph.

Grid Orientation Algorithm :

Input: The grid graph $M(m \times n)$, m,$n \geq 6$.

Algorithm: Orient the edges in $M(m \times n)$ as follows:

Stepl: Orient the rows 1 & 2 and m & $(m-1)$ from left to right and right to left respectively. Orient the columns 1 & 2 and n & $(n-1)$ from bottom to top and top to bottom respectively.

Step2: Orient the rows 3 to $(m-2)$ alternately from left to right and right to left beginning with left to right.

Step3: Orient columns 3 to $(n-2)$ alternately from top to bottom and bottom to top beginning from bottom to top, see Figure 2.2.

Output: Oriented diameter of $m \times n$ grid graph is $\vec{d}(M_{(m\times n)}) = d((M_{(m\times n)})=$ $n+m-2$.

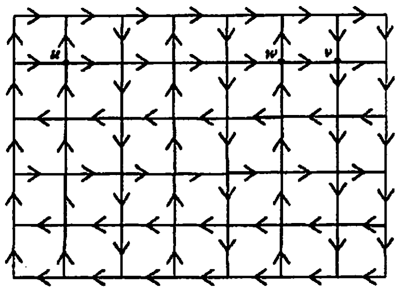

Figure 2.2: Orientation of $M(6,8)$, $\vec{d}(v,u) = \vec{d}(u,v)+2$ and $\vec{d}(w,u) = \vec{d}(u,w)+4$.

Proof of Correctness:

Let u and v be the two distinct vertices in $M(m \times n)$.

Case (i): u and v are in the same row (column)

In this case, $d(u,v) \leq n-1 \leq m+n-2$. Next to prove $\vec{d}(v,u) \leq n+m-2$. Let j be the column and l be the distance between any two vertices u and v in $M(m \times n)$.

If $j=1$ or 2 and $j+l=n$ or $n-1$. Then $l+2 \leq \vec{d}(v,u) \leq l+4$. Therefore, $\vec{d}(v,u) \leq n+3 \leq n+m-2$. if $j=1$ or 2 and $j+l < n-1$, then $\vec{d}(u,v)=l$ and $\vec{d}(v,u) \leq l+6 \leq n+m-2$.

On the otherhand, if $j \neq 1$ or 2 and $j+l \neq n$ or $n-1$, then $l \leq n-2$ and $\vec{d}(v,u) \leq l+8 \leq n+m-2$. Thus, $d(u,v)$ and $d(v,u)$ are at most $m+n-2$.

Case (ii): u and v are in distinct rows (columns)

Let $u=(i,j)$ and $v=(i+k,j+l)$ be the two vertices in G with orientations $u(\rightarrow,\uparrow)$ and $v(\leftarrow,\downarrow)$, respectively. Let P_1 be the path along the i^{th} row from $u\equiv (i,j)$ to $(i,j+l)$; P_2 be the path along the $(j+l)^{th}$ column from $(i,j+l)$ to $(i+k,j+l)$. Again let Q_1 be the path along the $(i+k)^{th}$ row from $v \equiv (i+k,j+l)$ to $(i+k,j)$ and Q_2 be the path along j^{th} column from $(i+k,j)$ to $u\equiv (i,j)$. Now, $P=P_1 \circ P_2$ and $Q=Q_1 \circ Q_2$ are the paths from (i,j) to $(i+k,j+l)$ and from $(i+k,j+l)$ to (i,j) respectively. Then $\vec{d}_{M(m\times n)} = d((i,j),(i,j+l))+d((i,j+l),(i+k,j+l)) = l+k \leq n+m-2$. Similarly, $d(v,u) \leq n+m-2$.

Let $u=(i,j)$ and $v=(i+k,j+l)$ be two vertices in G with orientations $u(\rightarrow,\downarrow)$ and $v(\leftarrow,\uparrow)$, respectively. Let R_1 be the path along the i^{th} row from $u\equiv (i,j)$ to $(i,j+l+1)$; R_2 be the path along the $(j+l+1)^{th}$ column from $(i,j+l+1)$ to $(i+k,j+l+1)$. Again let S_1 be the path along the $(i+k)^{th}$row from $(i+k,j+l+1)$ to $(i+k,j-1)$ and S_2 be the path along $(j-1)^{th}$ column from$(i+k,j-1)$ to $(i,j-1)$. Now, $R=R_1 \circ R_2 \circ ((i+k,j+l+1),(i+k,j+l))$ and $S=S_1 \circ S_2 \circ ((i,j-1),(i,j))$ are the paths from (i,j) to $(i+k,j+k)$

and from $(i+k, j+k)$ to (i, j) respectively. Then $\bar{d}_{M(m\times n)} = d((i,j),(i,j+l+1))+d((i,j+l),(i+k,j+l+1)) \leq n+m-2$. Similarly, $\bar{d}(v,u) \leq n+m-2$.

The worst case arises when $d(u,v)$ is calculated when $u(\rightarrow,\uparrow)$ and $v(\rightarrow,\uparrow)$ are in first row. In this case, u cannot be in the first 3 columns and v cannot be in the $n-1$ or n^{th} column. Then $\bar{d}(u,v) = \bar{d}(v,u) \leq n+m-2$.

Proceeding in the same way, we see that $\bar{d}(u,v)$ and $\bar{d}(v,u)$ are at most $n+m-2$ in all other cases.

The following theorem is an easy consequence of Grid Orientation Algorithm

Theorem 4. *Let G be a $M(m \times n)$ grid, $m,n \geq 6$. Then $\bar{d}(G) = d(G)$.*

Proof. By Theorem 2, $\bar{d}(G) \geq d(G) = m+n-2$. By Grid Orientation Algorithm, we have $\bar{d}(G) \leq m+n-2$. Hence $\bar{d}(G) = d(G)$. □

3 Conclusion

In this paper we have been discussed oriented diameter of grids and proved that oriented diameter of grid is exactly equal to the diameter of grid. The oriented diameter problem is still open for multi-dimension of grid, cylinder and torus.

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