Eternal m-security in Certain Classes of Graphs

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Abstract

Eternal 1-secure set of a graph $G = (V, E)$ is defined as a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of single guard shifts along edges of G. That is, for any k and any sequence $v_1, v_2, \ldots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2, \ldots, u_k$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $u_iv_i \in E$, such that each set $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$ is dominating. It follows that each S_i can be chosen to be an eternal 1-secure set. The *eternal 1-security number*, denoted by $\sigma_1(G)$, is defined as the minimum cardinality of an eternal 1-secure set. This parameter was introduced by Burger et al. [3] using the notation γ_∞. The *eternal m-security* number $\sigma_m(G)$ is defined as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. A suitable placement of the guards is called an *eternal m-secure* set. It was observed that $\gamma(G) \leq \sigma_m(G) \leq \beta(G)$. In this paper we obtain specific values of $\sigma_m(G)$ for certain classes of graphs namely circulant graphs,generalized Petersen graphs, binary trees and caterpillars.

Keywords: Eternal security, domination number.

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1 Introduction

Burger et al. [2, 3], introduced a dynamic form of domination which has, according to Goddard et al. [7], been designated *eternal security*.The concept calls for a fixed number of guards which are positioned on the vertices of a graph $G = (V, E)$, at most one to a vertex. A guard on a vertex w can respond to an attack at a vertex v by moving along an edge from w to v (assuming v does not already have a guard). Informally, if such a response can be made no matter what vertex is attacked and if the changing position of the guards can continue to respond forever, they say that the guards form an *eternally secure set.*

Two versions of the eternal security problem were considered. In the first version, which they call *1-security*, only one guard moves in response to an attack; in the second, which they call *m-security* all guards can move in response to an attack. The first version was introduced by Burger et al. [2, 3], though being able to withstand two attacks with a single-guard movement was explored in [4, 5, 11, 13, 14]. On the other hand, the idea that all guards may move in response to an attack appears to have been considered only in [14].

They defined an *eternal 1-secure* set of a graph $G = (V, E)$ as a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of single-guard shifts along the edges of G. That is, for any k and any sequence $v_1, v_2, \ldots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2, \ldots, u_k$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $u_iv_i \in E$, such that each set $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$ is dominating. It follows that each S_i can be chosen to be an eternal 1-secure set. They defined the *eternal 1-security number*, denoted by $\sigma_1(G)$, as the minimum cardinality of an eternal 1-secure set. This parameter was introduced by Burger et al. [3] using the notation γ_∞.

In order to reduce the number of guards needed for eternal security, they consider allowing more guards to move. Suppose that in responding to each attack, every guard may shift along an incident edge. The *eternal m-security* number $\sigma_m(G)$ is defined as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. A suitable placement of the guards is called an *eternal m-secure set*, they call such a set a σ_m-set of G. They observed that $\sigma_m(G) \leq \sigma_1(G)$, for all graphs G.

A set S is a *dominating set* if $N[S] = V(G)$ or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of minimum cardinality is called a γ-set of G. A set S is a *2-dominating set* if every vertex in $V - S$ is dominated by at least two vertices in S. The minimum cardinality of a 2-dominating set is called the *2-domination number* $\gamma_2(G)$. A set S of vertices is called *independent* if no two vertices in S are adjacent. The independence number $\beta(G)$ is the maximum cardinality of a independent set in G.

Wayne Goddard et al. [7] have proved that $\gamma(G)$ and $\beta(G)$ are lower and upper bounds of $\sigma_m(G)$ respectively for any graph G. They have also proved that the *2-domination* number $\gamma_2(G)$ of a graph is also an upper bound for $\sigma_m(G)$. Further they have found the value of $\sigma_m(G)$ when G is a path, cycle, complete graph, and complete bipartite graph. More results related to these parameters $\sigma_1(G)$ and $\sigma_m(G)$ are found in [1, 9, 10]. In [15] we have characterized trees and split graphs for which $\sigma_m(G) = \gamma(G)$. Further we have also characterized trees, unicylic graphs and split graphs for which $\sigma_m(G) = \beta(G)$. Wayne Goddard et al. [7] also have proved that $\sigma_m(G) = \gamma(G)$ when G is a Cayley graph and they have mentioned that $\sigma_m(G) = \gamma(G)$ is probably true for any vertex transitive graph. In this paper we disprove this statement by way of proving that certain generalized Petersen graphs are such that $\sigma_m(G) > \gamma(G)$.

In this paper we obtain specific values of $\sigma_m(G)$ for certain classes of graphs namely,circulant graphs, generalized Petersen graphs, binary trees and caterpillars.

2 Notations

Let $G = (V, E)$ be a simple and connected graph of order $|V| = n$. For graph theoretic terminology we refer to Harary [8]. For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood is $N[S] = N(S) \cup S$. The *external private neighbourhood* $epn(v, S)$ of a vertex $v \in S$ is defined by $epn(v, S) = \{u \in V - S : N(u) \cap S = \{v\}\}$.

A vertex of degree one in a graph is a *pendant vertex* (a leaf). A vertex of G adjacent to pendant vertices is called a *support*. We call a support vertex adjacent to exactly one pendant vertex a *weak support* and a support vertex adjacent to at least two pendant vertices a *strong support*.

A connected graph having no cycle is called a *tree*. A *rooted tree* is a tree in which one of the vertices is distinguished from others. The distinguished vertex is called the *root* of the tree.

A graph G is *k-partite*, $k \geq 1$ if it is possible to partition $V(G)$ into k subsets. $V_1, V_2, \ldots, V_k$ (called partite set) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. If G is a 1-*partite* graph of order n, then $G = \overline{K_n}$. For $k = 2$, such graphs are called *bipartite* graphs.

3 Circulant Graphs

The *circulant graph* $C_n(S_c)$ is the graph with the vertex set
$V(C_n(S_c)) = \{v_i : 0 \leq i \leq n-1\}$ and the edge set

$E(C_n(S_c)) = \{v_iv_j : 0 \le i, j \le n-1, (i-j)(mod\ n) \in S_n\}$.
$S_c \subseteq \{1, 2, 3, \dots, \lceil \frac{n}{2} \rceil\}$ where subscripts are taken modulo n.In this section we find the value of σ_m for the circulant graphs $C_n(1,2)$ and $C_n(1,3)$.

Theorem 3.1. *For any integer $n \ge 5$, $\sigma_m(C_n(1,2)) = \lceil \frac{n}{5} \rceil$.*

Proof. Let $G = C_n(1,2)$ and $V(G) = \{v_1, v_2, \dots, v_n\}$.
We know that $\gamma(G) = \lceil \frac{n}{5} \rceil$. Therefore, $\sigma_m(G) \ge \lceil \frac{n}{5} \rceil$. Define $S = \{v_{5k-4} : k = 1, 2, 3, \dots, \lceil \frac{n}{5} \rceil\}$.
Let $v_i \in S$. If there is an attack at v_{i+1} or v_{i-1}, then the guard at v_i will respond to it and the guards at the other vertices in S say v_j will move to v_{j+1} or v_{j-1} accordingly so that one can eternally respond to any attack. If there is an attack at v_{i+2} or v_{i-2}, then the guard at v_i will respond to it and the guards at the other vertices in S say v_j will move to v_{j+2} or v_{j-2} accordingly so that one can eternally respond to any attack. Hence we see that S is a σ_m-set of G. Therefore $\sigma_m(G) = \lceil \frac{n}{5} \rceil$.

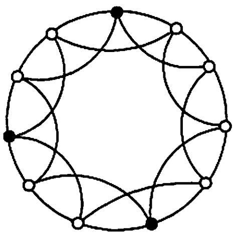

Figure 1: $\sigma_m[C_{11}(1,2)] = 3$

Now we state a theorem proved by Nader Jafari Rad [12].

Theorem 3.2. *[12] For any integer $n \ge 6$,*

$$\gamma(C_n(1,3)) = \begin{cases} \lceil \frac{n}{5} \rceil, & n \not\equiv 4(mod\ 5) \\ \lceil \frac{n}{5} \rceil + 1, & n \equiv 4(mod\ 5). \end{cases}$$

Theorem 3.3. *For any integer $n \ge 5$,*

$$\sigma_m(C_n(1,3)) = \begin{cases} \lceil \frac{n}{5} \rceil + 1, & n \equiv 4(mod\ 5) \\ \lceil \frac{n}{5} \rceil, & otherwise. \end{cases}$$

Proof. Let $G = C_n(1,3)$ and $V(G) = \{v_1, v_2, \dots, v_n\}$. By Theorem 3.2

$$\sigma_m(G) \ge \begin{cases} \lceil \frac{n}{5} \rceil + 1, & n \equiv 4(mod\ 5) \\ \lceil \frac{n}{5} \rceil, & otherwise. \end{cases}$$

Case (i) $n \equiv 4(mod\ 5)$.
Define $S = \{v_{5k-4} : k = 1, 2, 3, \ldots, \lceil \frac{n}{5} \rceil\} \cup \{v_{n-1}\}$. Let $v_i \in S$. If there is an attack at v_{i+1} or v_{i-1},then the guard at v_i will respond to it and the guards at the other vertices in S say v_j will move to v_{j+1} or v_{j-1} accordingly so that one can eternally respond to any attack.

If there is an attack at v_{i+3} or v_{i-3}, then the guard at v_i will respond to it and the guards at the other vertices in S say v_j will move to v_{j+3} or v_{j-3} accordingly so that one can eternally respond to any attack. Hence we see that S is a σ_m-set of G. Therefore $\sigma_m(G)) = \lceil \frac{n}{5} \rceil + 1$.

Case (ii) $n \not\equiv 4(mod\ 5)$.
Define $S = \{v_{5k-4} : k = 1, 2, 3, \ldots, \lceil \frac{n}{5} \rceil\}$. As in Case (i) we see that S is a σ_m-set of G. Hence $\sigma_m(G)) = \lceil \frac{n}{5} \rceil$.

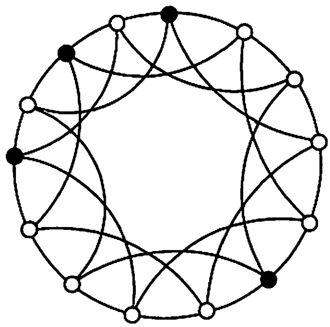

Figure 2: $\sigma_m(C_{14}(1,3)) = 4$

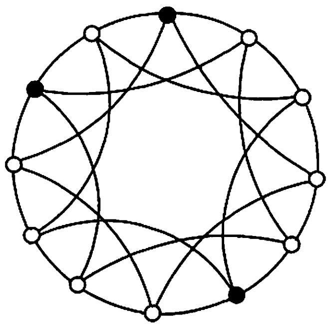

Figure 3: $\sigma_m(C_{12}(1,3)) = 3$

Corollary 3.4. *If G is $C_n(1,2)$ or $C_n(1,3)$, then $\sigma_m(G) = \gamma(G)$.*

Proof. The proof follows from Theorems 3.1 to 3.3.

4 Generalized Petersen Graphs

The generalized Petersen graph $P(n,k)$ is defined to be a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(P(n,k)) = \{v_iv_{i+1}, u_iv_i, u_iu_{i+}$ $1 \leq i \leq n\}$ subscripts taken modulo $n+1$. In this section we find the value of σ_m for the generalised Petersen graphs $P(n,1)$ and $P(n,2)$.

We state the following theorems proved by B.J. Ebrahimi et al. [6].

Theorem 4.1. [6] *For any integer $n \geq 3$, $\gamma(P(n,1)) = \lceil \frac{n}{2} \rceil + 1$ when $n \equiv 2(mod\ 4)$ and $\gamma(P(n,1)) = \lceil \frac{n}{2} \rceil$ when $n \equiv 0,1,3(mod\ 4)$.*

Theorem 4.2. [6] *For any integer $n \geq 4$, $\gamma(P(n,2)) = \lceil \frac{3n}{5} \rceil$.*

Theorem 4.3. *For the generalized Petersen graph $G = P(n,1)$,*

$$\sigma_m(G) = \begin{cases} \lceil \frac{n}{2} \rceil + 1, & n \equiv 2(mod\ 4) \\ \lceil \frac{n}{2} \rceil, & otherwise. \end{cases}$$

Proof. Let C_1 and C_2 be the two disjoint cycles of G, and let $V(C_1) = v_1, v_2, \ldots, v_n$ and $V(C_2) = u_1, u_2, \ldots, u_n$. Since $\sigma_m(G) \geq \gamma(G)$, by Theorem 4.1 we have,

$$\sigma_m[G] \geq \begin{cases} \lceil \frac{n}{2} \rceil + 1, & n \equiv 2(mod\ 4) \\ \lceil \frac{n}{2} \rceil, & otherwise. \end{cases}$$

Let $S_1 = \{v_{4t+1} : 0 \leq t < \frac{n}{4}\}$, $S_2 = \{v_{4t+3} : 0 \leq t < \frac{n}{4}\}$.
Define S as follows:

$$S = \begin{cases} S_1 \cup S_2, & n \equiv 0(mod\ 4) \\ S_1 \cup S_2 \cup \{v_n\}, & n \equiv 1(mod\ 4) \\ S_1 \cup S_2 \cup \{v_{n-1}, u_n\}, & n \equiv 2(mod\ 4) \\ S_1 \cup S_2 \cup \{u_n\}, & n \equiv 3(mod\ 4). \end{cases}$$

Now, we claim that S is a σ_m-set of G. Let $v \in S \cap C_1$. If v has to respond to an attack at a vertex on C_1, then the guard at v moves either in the clockwise or anticlockwise direction in C_1 whereas all the guards at the vertices of $S \cap C_1$ and $S \cap C_2$ move in the clockwise or anticlockwise direction respectively. If v has to respond to an attack at a vertex in C_2, then the guards at $S \cap C_1$ move to the cycle C_2 whereas the guards at $S \cap C_2$ move to the cycle C_1. Hence we see that the guards at the vertices

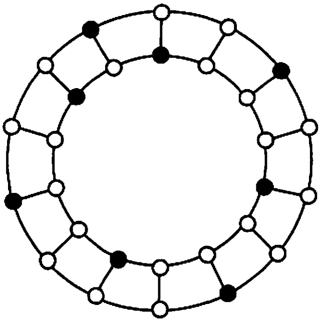

Figure 4: $\sigma_m(P(14,1)) = 8$

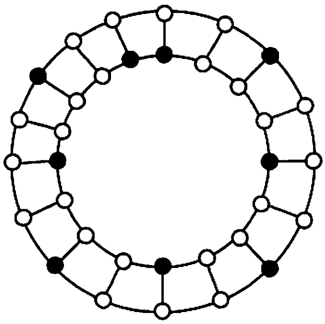

Figure 5: $\sigma_m(P(17,1)) = 9$

of S can eternally respond to any attack. Hence when $n \equiv 2(mod\ 4)$, $\sigma_m(G) = |S_1| + |S_2| + 2 = \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{4} \rfloor + 2 = \lceil \frac{n}{2} \rceil + 1$ and when $n \not\equiv 2(mod\ 4)$, $\sigma_m(G) = \lceil \frac{n}{2} \rceil$.

Corollary 4.4. *For the generalized Petersen graph $G = P(n,1)$, $\sigma_m(G) = \gamma(G)$.*

Proof. The proof follows from Theorems 4.1 to 4.3.

Theorem 4.5. *For the generalized Petersen graph $G = P(n,2)$,*

$$\sigma_m[G] = \begin{cases} \lceil \frac{3n}{5} \rceil + 1, & n \equiv 0, 3(mod\ 5) \\ \lceil \frac{3n}{5} \rceil, & otherwise. \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$, where $u_iv_i \in E(G)$, $1 \leq i \leq n$. Now, by Theorem 4.2, $\sigma_m[G] \geq \lceil \frac{3n}{5} \rceil$.

When $n \equiv 0, 3(mod\ 5)$, let $S = \{v_{5t+1}, u_{5t}, u_{5t+2}, : 0 \leq t < \lfloor \frac{n}{5} \rfloor\}$ where $u_0 = u_n$ be a γ-set of G. Here u_1 is a non private neighbour of S. If there is an attack at u_1, then the guard at v_1 or u_n or u_2 responds to it.

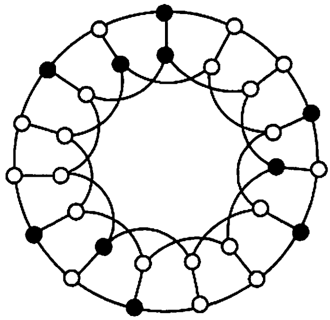

Figure 6: $\sigma_m(P(15, 2)) = 10$

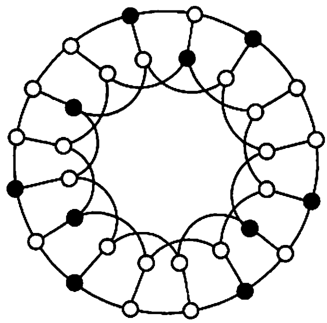

Figure 7: $\sigma_m(P(16, 2)) = 10$

If the guard at u_2 responds to it, then $u_2 \to u_1$, $u_n = u_0 \to u_{n-1}$; $v_1 \to v_3$.

If the guard at u_n responds to it, then $u_n \to u_1$, $v_1 \to v_{n-1}$, $u_2 \to u_3$.

If the guard at v_1 responds to it, then $v_1 \to u_1$, $u_n \to u_{n-1}$, $u_2 \to u_3$.

In all the above movements of guards, we see that when $n \equiv 0(mod\ 5)$, v_2 and v_n are undefended and when $n \equiv 3(mod\ 5)$, v_2 is undefended.

Hence $\sigma_m(G) > \lceil \frac{3n}{5} \rceil$.

Now define $S' = \begin{cases} S \cup \{v_2\}, & n \equiv 0, 3(mod\ 5) \\ S, & \text{otherwise.} \end{cases}$

We claim that S' is a σ_m-set of G.

Let $v_i \in S'$. Let $V_1 = S' \cap \{v_1, v_2, \dots, v_n\}$ and $V_2 = S' \cap \{u_1, u_2, \dots, u_n = u_0\}$. If there is an attack at v_{i+2} or v_{i-2} and the guard at v_i has to respond

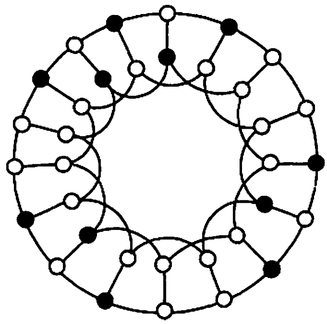

Figure 8: $\sigma_m(P(17,2)) = 11$

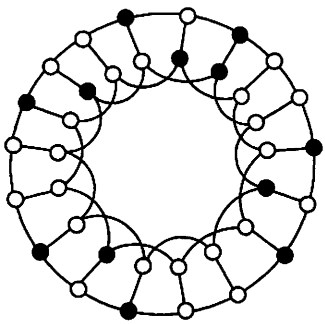

Figure 9: $\sigma_m(P(18,2)) = 12$

it and the guards at V_1 will move either clockwise or anticlockwise along one edge each, whereas the guards at V_2 (if necessary) will move in such a way that v_s, u_{s-1}, u_{s+1}, $1 \leq S \leq \lceil \frac{3n}{5} \rceil$ are equipped with guards. In this way one can eternally respond to any attack.

Similarly, if there is an attack at $u_j \in S'$ then the guards in S' move in such a way that v_s, u_{s-1}, u_{s+1}, $1 \leq S \leq \lceil \frac{3n}{5} \rceil$ are equipped with guards so that one can eternally respond to any attack.

Corollary 4.6. *For the generalized Petersen graph $G = P(n, 2)$,*

$$\sigma_m(G) = \begin{cases} \gamma(G) + 1, & n \equiv 0, 3 (mod\ 5) \\ \gamma(G), & otherwise. \end{cases}$$

Proof. The proof follows from Theorem 4.2 and Theorem 4.5.

5 Binary Trees

A *complete binary tree* is a rooted tree in which all leaves have the same depth and all internal vertices have degree three except the root vertex which is of degree two. If T is a complete binary tree with root vertex v, the set of all vertices with depth k are called *vertices at level k*. In this section, we find the value of $\sigma_m(T)$ for a complete binary tree.

Theorem 5.1. *For any binary tree T of level k,*

$$\sigma_m(T) = \begin{cases} \frac{4}{7}(2^k - 1) & \text{for } k \equiv 0(mod\ 3) \\ \frac{4}{7}(2^k - \frac{1}{4}) & \text{for } k \equiv 1(mod\ 3) \\ \frac{4}{7}(2^k - \frac{1}{2}) & \text{for } k \equiv 2(mod\ 3). \end{cases}$$

Proof. Let S_i be the set of vertices in level i. Then $|S_i| = 2^{i-1}$. Now at least 2^{k-1} guards are needed to safeguard the set of vertices $S_k \cup S_{k-1} \cup S_{k-2}$ and at least 2^{k-4} guards are needed to safeguard the set of vertices $S_{k-3} \cup S_{k-4} \cup S_{k-5}$ and so on.
Now we define

$$S = \begin{cases} S_k \cup S_{k-3} \cup \cdots \cup S_3 & \text{if } k \equiv 0(mod\ 3) \\ S_k \cup S_{k-3} \cup \cdots \cup S_4 \cup S_1 & \text{if } k \equiv 1(mod\ 3) \\ S_k \cup S_{k-3} \cup \cdots \cup S_5 \cup S_2 & \text{if } k \equiv 2(mod\ 3) \end{cases}$$

Then S is a σ_m-set of T.
Hence

$$\sigma_m(T) = \begin{cases} 2^{k-1} + 2^{k-4} + \cdots + 2^2 & \text{if } k \equiv 0(mod\ 3) \\ 2^{k-1} + 2^{k-4} + \cdots + 2^0 & \text{if } k \equiv 1(mod\ 3) \\ 2^{k-1} + 2^{k-4} + \cdots + 2^1 & \text{if } k \equiv 2(mod\ 3). \end{cases}$$

Case (i): $k \equiv 0(mod\ 3)$.

$$\begin{aligned} \sigma_m(T) &= 2^{k-1} + 2^{k-4} + \cdots + 2^2 \\ &= \frac{2^2}{7}(2^k - 1) \\ &= \frac{4}{7}(2^k - 1). \end{aligned}$$

Case (ii) $k \equiv 1 (mod\ 3)$.

$$\begin{aligned}\sigma_m(T) &= 2^0 + 2^3 + \cdots + 2^{k-4} + 2^{k-1} \\ &= \frac{(2^3)^{\frac{k+2}{3}} - 1}{2^3 - 1} \\ &= \frac{2^{k+2} - 1}{7} \\ &= \frac{4}{7}(2^k - \frac{1}{4}).\end{aligned}$$

Case (iii) $k \equiv 2 (mod\ 3)$.

$$\begin{aligned}\sigma_m(T) &= 2^{k-1} + 2^{k-4} + \cdots + 2^4 + 2^1 \\ &= \frac{2((2^3)^{\frac{k+1}{3}} - 1}{2^3 - 1} \\ &= \frac{4}{7}(2^k - \frac{1}{2}).\end{aligned}$$

6 Caterpillar

A *caterpillar* is a tree T with the property that the removal of the end-vertices of a tree results in a path. This path is referred to as the *spine* of the caterpillar. We now proceed to find the value of $\sigma_m(T)$ when T is a caterpillar.

First we state a Theorem proved by Wayne Goddard et al. [7]

Theorem 6.1. *[7]* $\sigma_m(P_n) = \lceil \frac{n}{2} \rceil$.

Theorem 6.2. *Let T be a caterpillar such that every vertex on the spine is a support. Then*

$$\sigma_m(T) = \begin{cases} t, & \text{if every vertex on the spine is a weak support} \\ t+1, & \text{otherwise,} \end{cases}$$

where t is the number of vertices on the spine of T.

Proof. Let T be a caterpillar such that every vertex on the spine is a support and t be the number of vertices on the spine of T. Suppose every vertex on the spine is a weak support, then clearly $\sigma_m(T) = t$.

Suppose at least one vertex on the spine is a strong support.Let v be a strong support of T. Let S be the set of all supports of T. Now S is a γ-set of T. Since $|epn(v, S)| \geq 2$. The guard at v cannot respond to two successive attacks at the eternal private neighbours of v. Hence

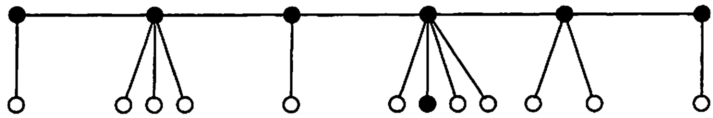

Figure 10: A caterpillar T with $\sigma_m(T) = 7$

$\sigma_m(T) > \gamma(T)$. Now we claim that $S \cup \{z\}$ where z is a leaf adjacent to a strong support in T is a σ_m-set of T.

Now, if there is an attack at any leaf, then the guard at the corresponding support vertex responds to the attack and the remaining t guards move in such a way that all the spine vertices are equipped with guards (Refer Figure 10). Hence one can eternally respond to any attack. Hence $\sigma_m(T) = t + 1$ which proves the theorem.

Now in order to find the value of $\sigma_m(T)$ when T is a caterpillar with at least one non-support vertex on the spine, we define a collection $\mathcal{T}$ of caterpillars as follows: A caterpillar $T \in \mathcal{T}$, if every vertex on the spine is a support with at least one strong support. Consider a caterpillar T. Let $H_1, H_2, \ldots, H_s$ be subgraphs of T which are in $\mathcal{T}$ and n_i be the number of vertices on the spine of H_i, $1 \leq i \leq s$. Define $S_1 = \bigcup_{i=1}^{s} V(H_i)$ and $S_2 = \{x : x \notin S_1 and x$ is a weak support or x is a leaf adjacent to a weak support$\}$. Now let $T_1 = V(T) \backslash (N[S_1] \cup S_2)$ and $Q_1, Q_2, \ldots, Q_r$ be the components of T_1 which are paths. We now consider a weak support z on T to be an *artificial strong support*, if z is adjacent to vertices of Q_a and Q_b, $1 \leq a, b \leq r$ of T_1 and both Q_a and Q_b are of even length (Refer Figure 11).

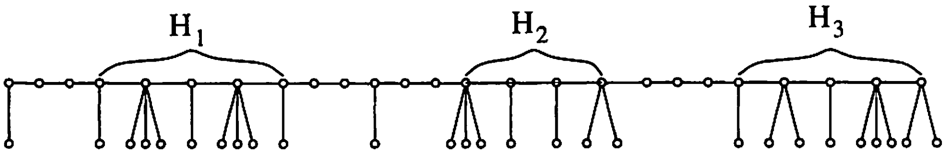

Figure 11: A caterpillar with subgraphs $H_i \in \mathcal{T}, 1 \leq i \leq 3$

Theorem 6.3. *Let T be a caterpillar with at least one vertex on the spine of degree 2. Then*

$$\sigma_m(T) = \sum_{i=1}^{s}(n_i + 1) + \sum_{j=1}^{r} \left\lceil \frac{\ell(Q_j) + 1}{2} \right\rceil + |W|.$$

where W is the set of all weak supports which are not artificial and H_i, n_i, $1 \le i \le s$, S_1, S_2, Q_j, $1 \le j \le r$ are as defined earlier.

Proof. Let $S = \bigcup_{i=1}^{s} V(H_i)$. By Theorem 6.2, exactly $\sum_{i=1}^{s}(n_i + 1)$ guards are needed to safeguard vertices in $N[S]$. Further these $\sum_{i=1}^{s}(n_i+1)$ guards cannot move to the vertices in $V(T)\backslash N[S]$. Now consider the path components Q_j, $1 \le j \le r$. By Theorem 6.1,$\sigma_m(Q_j) = \left\lceil \frac{\ell(Q_j)+1}{2} \right\rceil$, $1 \le j \le r$. If z is an artificial strong support of T, then by definition, the path components say Q_a, Q_b, $1 \le a, b \le r$, which are adjacent to z are of even length. Hence the vertices of Q_a and Q_b which are adjacent to z will receive one guard each and these two guards will safeguard the vertices of $N[z]$. The rest of the weak supports which are not artificial clearly need one guard each.

Hence

$$\sigma_m(T) = \sum_{i=1}^{s}(n_i + 1) + \sum_{j=1}^{r} \left\lceil \frac{\ell(Q_j) + 1}{2} \right\rceil + |W|.$$

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