Long–Run Behavior of Intuitionistic Markov Chain

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Abstract

Unlike an ordinary fuzzy set, the concept of intuitionistic fuzzy set (IFS), characterized both by a membership degree and by a non-membership degree, is a more flexible way to capture uncertainty. In this paper we have classified the states of intuitionistic Markov chain (IMC) [1] and analyzed the long–run behavior of the system.

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1 Introduction

Fuzzy Markov model has been defined and is being widely used in [2]–[4]. It has been asserted by many authors that there are a large number of life problems for which intuitionistic fuzzy set (IFS) theory is a more suitable tool than fuzzy set theory for searching solution. For example, in decision making problems, particularly in the case of medical diagnosis, etc. Intuitionistic fuzzy set theory (IFS theory) introduced by K.Atanassov [5], is a significant extension of fuzzy set theory. Fuzzy sets can be viewed as intuitionistic fuzzy sets but the converse is not true. Intuitionistic Markov Chain (IMC) is proposed on intuitionistic possibility space in [1]. To know more on IMC, we refer the readers to [1].

In this paper we have classified the states of IMC and analyzed the long–run behavior of IMC.

Definition 1. Let X be an universal set. For an intuitionistic fuzzy sets $A_i = \{(x, \mu_{A_i}(x), \nu_{A_i}(x)) | x \in X\} = (\mu_{A_i}, \nu_{A_i}), i \in I$ the meet $\wedge$ and join $\vee$ operators [6] are defined by

$$\begin{aligned}(\mu_{A_i}, \nu_{A_i}) \wedge (\mu_{A_j}, \nu_{A_j}) &= (\min(\mu_{A_i}, \mu_{A_j}), \max(\nu_{A_i}, \nu_{A_j})) \\ (\mu_{A_i}, \nu_{A_i}) \vee (\mu_{A_j}, \nu_{A_j}) &= (\max(\mu_{A_i}, \mu_{A_j}), \min(\nu_{A_i}, \nu_{A_j}))\end{aligned}$$

In the following section we have classified the states of an IMC.

2 Classification of states of an IMC

Consider an IMC with n states and let us see some notations to classify the states of an system.

Definition 2. Intuitionistic possibility for the first time visit to state j from state i (initial state) at the n^{th} step is denoted by $\widetilde{\mathfrak{p}}_{ij}(n) = (\mu_{\widetilde{\mathfrak{p}}_{ij}(n)}, \nu_{\widetilde{\mathfrak{p}}_{ij}(n)})$. Then, the intuitionistic possibility of ever reaching the state j from state i is given by $\widetilde{\mathfrak{P}}_{ij} = \sup_n [\widetilde{\mathfrak{p}}_{ij}(n)]$

Definition 3. A state i is said to be recurrent iff starting from i, the process eventually returns to state i. i.e., $\widetilde{\mathfrak{p}}_{ii} = (1, 0)$.

Example 1: Consider a IMC with the transition diagram as

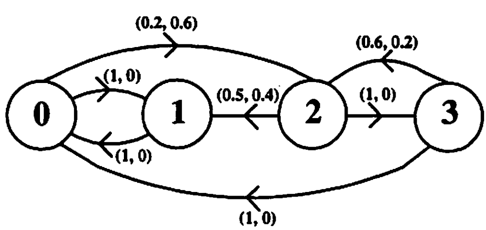

Figure 1: Transition Diagram

For $n = 1$, $\widetilde{\mathfrak{p}}_{00}(1) = (0, 1)$
For $n = 2$, there exist a path $0 \to 1 \to 0$. Then

$$\begin{aligned}\widetilde{\mathfrak{p}}_{00}(2) &= \min(\widetilde{p}_{01}, \widetilde{p}_{10}) \\ &= \min((1,0), (1,0)) \\ &= (1,0)\end{aligned}$$

For $n = 3$, the path is $0 \to 2 \to 1 \to 0$ (or) $0 \to 2 \to 3 \to 0$. Hence

$$
\begin{aligned}
\widetilde{p}_{00}(3) &= \min(\widetilde{p}_{02},\widetilde{p}_{21},\widetilde{p}_{10}) \textit{ or } \min(\widetilde{p}_{02},\widetilde{p}_{23},\widetilde{p}_{30}) \\
&= \max\left\{ \begin{array}{c} \min[\ (0.2,0.6) \quad (0.5,0.4) \quad (1,0)\ ] \\ \min[\ (0.2,0.6) \quad (1,0) \quad (1,0)\ ] \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} (0.2,0.6) \\ (0.2,0.6) \end{array} \right\} \\
&= (0.2,0.6)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\widetilde{\mathfrak{P}}_{00} &= \max_n(\widetilde{\mathfrak{p}}_{00}(n)) \\
&= \max((0,1),(1,0),(0.2,0.6),\cdots) \\
&= (1,0)
\end{aligned}
$$

Hence the state '0' is a recurrent state. Similarly state '1' is also recurrent. It is observed that to get $\widetilde{\mathfrak{P}}_{ii}$ =(1, 0) for at least one n, $\widetilde{\mathfrak{p}}_{ii}(n)$ = (1, 0). $\widetilde{\mathfrak{p}}_{ii}(n)$ = (1, 0) is possible only when all the edge values in its corresponding path are equal to (1, 0). For $n \geq 1$, $\widetilde{\mathfrak{p}}_{22}(n)$, $\widetilde{\mathfrak{p}}_{33}(n)$ do not have such paths and therefore $\widetilde{\mathfrak{P}}_{22}(n)$, $\widetilde{\mathfrak{P}}_{33}(n)$ are not equal to (1, 0). Hence the states '2' and '3' are not recurrent.

In the probability space, positive recurrent is defined in terms of mean recurrence time whereas it is not in the case in IMC. If we define the mean recurrence time for a recurrent state i as

$$\widetilde{m}_i = \sup_n \left[\inf(n,\ \widetilde{\mathfrak{p}}_{ii}(n)) \right]$$

since the n values are positive integers and clearly we get

$$\inf\left[n,\ \widetilde{\mathfrak{p}}_{ii}(n)\right] = \widetilde{\mathfrak{p}}_{ii}(n)$$

provided $n \neq 0$. Hence, $\widetilde{m}_i = \sup_n\left[\widetilde{\mathfrak{p}}_{ii}(n)\right] = \widetilde{\mathfrak{P}}_{ii} = (1,0)$ which is always finite. Hence the positive recurrence for a state of IMC is defined as follows.

Definition 4. A recurrent state i is said to be positive recurrent if the process returns to state i in a finite number of transitions (n).

In Example 1, the process returns to state '0' starting from '0' and returns to state '1' starting from '1' in a finite number of transitions, n=2. Hence the recurrent states 0, 1 are positive recurrent.

Definition 5. A recurrent state is null recurrent if the number of transitions (n) that the process takes to return to i is infinite.

Definition 6. A state i is transient iff there is an intuitionistic possibility for the process that are not returning to state i, i.e.,$\widetilde{\mathfrak{P}}_{ii} <$ $(1, 0)$.

From the Example 1, it is clear that $\widetilde{\mathfrak{P}}_{22}$, $\widetilde{\mathfrak{P}}_{33}$ are less than (1, 0). Hence the states '2', '3' are transient states.

The theorem proved in [7] states that "the powers of the fuzzy matrix $\widetilde{A}$ either converge to idempotent $\widetilde{A}^m$ where m is a finite number or oscillate with a finite period v starting from some finite power". Since the intuitionistic transition possibility matrix $\widetilde{P}$ is a fuzzy matrix using composition of intuitionistic fuzzy relations we can find its higher powers therefore this theorem holds for $\widetilde{P}$. Aperiodic IMC is defined as follows.

Definition 7. If the powers of the intuitionistic transition possibility matrix $\widetilde{P}$ converge in n steps to a non-periodic solution, then the associated IMC is said to be aperiodic.

Example 2: Consider the IMC with state space $\{A, B, C\}$ as follows.

$$\widetilde{P} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \left[\begin{array}{ccc} (0.3, 0.4) & (1.0, 0.0) & (0.2, 0.6) \\ (0.0, 1.0) & (0.4, 0.3) & (1.0, 0.0) \\ (0.2, 0.6) & (0.0, 1.0) & (1.0, 0.0) \end{array}\right] \end{array}$$

$$\widetilde{P}^2 = \widetilde{P}^3 = \ldots = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \left[\begin{array}{ccc} (0.3, 0.4) & (0.4, 0.3) & (1.0, 0.0) \\ (0.2, 0.6) & (0.4, 0.3) & (1.0, 0.0) \\ (0.2, 0.6) & (0.2, 0.3) & (1.0, 0.0) \end{array}\right] \end{array}$$

Hence the corresponding IMC is aperiodic.

Definition 8. A state i is said to be an absorbing state iff $\widetilde{p}_{ii} = (1, 0)$ and $\widetilde{p}_{ij} = (0, 1)$ for $i \neq j$.

Example 3: Let the transition possibility matrix of a IMC with state space {1,2,3} be

$$\widetilde{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} (1.0, 0.0) & (0.5, 0.3) & (0.4, 0.4) \\ (0.0, 1.0) & (1.0, 0.0) & (0.0, 1.0) \\ (1.0, 0.0) & (0.1, 0.4) & (0.3, 0.2) \end{array}\right] \end{array}$$

Since $\widetilde{p}_{22}$ =(1.0, 0.0) and $\widetilde{p}_{21} = \widetilde{p}_{23}$ =(0.0, 1.0), thus the state '2' is an absorbing state.

Definition 9. An IMC is said to be irreducible, if for every pair i, j, there exist an integer $n \geq 1$, such that $\widetilde{p}_{ij}^{n} > 0$.

Consider the IMC in Example 2 in which the state space is {A, B, C}. From the entries of $\widetilde{P}$, it is observed that there is link between every pair of states except BA and CB. But in $\widetilde{P}^2$, we get BA=(0.2, 0.6) and CB=(0.2, 0.6). Hence every state is reached from every other state and the corresponding IMC is irreducible.

To know the performance of the system we need to know its long–run behavior. Hence in the following section we have analyzed the system's behavior during long–run.

3 Long–Run Behavior of IMC

Consider the intuitionistic transition possibility matrix $\widetilde{P}$ and let $\widetilde{\pi}$ be the long–run behavior of states of the system. If the composition of $\widetilde{P}$ and $\widetilde{\pi}$ gives $\widetilde{P}'$ and when $\widetilde{P}'$ equals $\widetilde{\pi}$, then we say that $\widetilde{\pi}$ is an eigen fuzzy set associated with the given transition matrix $\widetilde{P}$. By using composition of intuitionistic fuzzy relation we can compute the greatest and least eigen fuzzy set associated with $\widetilde{P}$ for membership and non–membership degree respectively.

Algorithm:

Consider $\widetilde{P} \subseteq X$ x X with membership function $\mu_{\widetilde{P}}(x, x')$ and non–membership function $\nu_{\widetilde{P}}(x, x')$ are given.

1. Find the set A_1, B_1 defined by

$$\mu_{A_1}(x') = \max_{x \in X} \mu_{\widetilde{P}}(x, x'), \forall\ x' \in X$$

$$\nu_{B_1}(x') = \min_{x \in X} \nu_{\widetilde{P}}(x, x'), \forall\ x' \in X$$

2. Set the index $n = 1$

3. Calculate $A_{n+1} = A_n \circ \widetilde{P}$ and $B_{n+1} = B_n \circ \widetilde{P}$

4. $A_{n+1} = A_n$ $\rightarrow$ No $n = n + 1$, go to step 3; $\rightarrow$ Yes $\mathrm{A_{n+1} = A_n}$ Similarly,

$B_{n+1} = B_n$ $\rightarrow$ No $n = n + 1$, go to step 3; $\rightarrow$ Yes $\mathrm{B_{n+1} = B_n}$

The above greatest and least eigen vectors for membership and non–membership degree function are independent of the initial vector and these vectors help us to know the steady state behavior of the system after long–run.

Example 1:

$$\widetilde{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} (0.3,0.1) & (0.4,0.2) & (0.0,0.9) & (0.4,0.3) \\ (0.6,0.3) & (0.7,0.3) & (0.5,0.2) & (0.5,0.3) \\ (0.5,0.2) & (0.6,0.2) & (0.4,0.4) & (0.8,0.1) \\ (0.8,0.1) & (0.2,0.2) & (0.6,0.3) & (0.3,0.0) \end{array} \right] \end{array}.$$

The membership degree matrix $\mu_{\widetilde{P}}$ and non-membership degree matrix $\nu_{\widetilde{P}}$ is given below

$$\mu_{\widetilde{P}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0.3 & 0.4 & 0.0 & 0.4 \\ 0.6 & 0.7 & 0.5 & 0.5 \\ 0.5 & 0.6 & 0.4 & 0.8 \\ 0.8 & 0.2 & 0.6 & 0.3 \end{array} \right] \end{array},$$

$$\nu_{\widetilde{P}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0.1 & 0.2 & 0.9 & 0.3 \\ 0.3 & 0.3 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.3 & 0.0 \end{array} \right] \end{array}.$$

For $n = 1$,

$$\begin{aligned} A_1 &= [\ 0.8 \quad 0.7 \quad 0.6 \quad 0.8\ ] \\ A_2 = A_1 \circ \widetilde{P} &= [\ 0.8 \quad 0.7 \quad 0.6 \quad 0.6\ ] \\ A_3 = A_2 \circ \widetilde{P} &= [\ 0.6 \quad 0.7 \quad 0.6 \quad 0.6\ ] \\ A_4 = A_3 \circ \widetilde{P} &= [\ 0.6 \quad 0.7 \quad 0.6 \quad 0.6\ ] \end{aligned}$$

$$\begin{aligned} B_1 &= [\ 0.1 \quad 0.2 \quad 0.2 \quad 0.0\ ] \\ B_2 = B_1 \circ \widetilde{P} &= [\ 0.1 \quad 0.2 \quad 0.2 \quad 0.0\ ] \end{aligned}$$

which implies

$$\begin{aligned} A_4 &= A_3 \\ B_2 &= B_1 \end{aligned}$$

Combining these two eigen vectors we get the steady state vector of IMC to be

$$\widetilde{\pi} = ((0.6, 0.1) \quad (0.7, 0.2) \quad (0.6, 0.2) \quad (0.6, 0.0))$$

4 Conclusion

In this paper, we have classified the states of an IMC and presented an algorithm to find greatest and least eigen vector for membership and non–membership degree matrix to analyze long–run behavior of the system.

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