Embedding the Myceilski of a Graph and the Amalgamation of Two Graphs in Pages

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Abstract

A *book* consists of a line in the 3-dimensional space, called the spine, and a number of *pages*, each a half-plane with the spine as boundary. A *book embedding* (π, ρ) of a graph consists of a linear ordering of π, of vertices, called the *spine ordering*, along the spine of a book and an assignment ρ, of edges to pages so that edges assigned to the same page can be drawn on that page without crossing. That is, we cannot find vertices u, v, x, y with $\pi(u) < \pi(x) < \pi(v) < \pi(y)$, yet the edges uv and xy are assigned to the same page, that is $\rho(uv) = \rho(xy)$. The *book thickness* or *page number* of a graph G is the minimum number of pages in required to embed G in a book. In this paper the book embedding of the Myceilski of P_n, $n > 2$ and C_n are given. If G is any graph, an upper bound for the pagenumber of the Myceilski of G is given. When G and H are any two graphs with pagenumber k and l it is proved that the amalgamation of G and H can be embedded in a $max(k, l)$ pages. Further we remark that the amalgamation of G with itself requires the same number of pages as G, irrespective of the vertices identified in the two copies of G, to form an amalgamation.

1. Introduction

The growth of the subject 'graph theory' has been very rapid in recent years, particularly since the domain of its application is extremely varied. Graph algorithms play a very important role in design of various computer networks. Among the problems one comes across in graph theory, is the embedding of graphs. A particular way of embedding graphs is in the pages of a book. The book embedding of graphs was first introduced by Bernhart and Kainen [1] and since then, many researchers have actively studied it. Determining the book thickness for general graphs is *NP*-hard. But obtaining the book thickness for particular graphs have been found to be possible. The book embeddings have been studied for many classes of graphs. To name a few, we have: Complete Graphs [1, 2], Complete Bipartite Graphs[10], Trees, Grids and X-trees [3],

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hypercubes [3, 8], incomplete hypercubes [7], iterated line digraphs [5], de Bruijn graphs, Kautz graphs, shuffle-exchange graphs [6], for each of which embedding in books have been studied.

The book embedding problem has many different applications, which include sorting with parallel stacks, single-row routing of printed circuit boards, and the design of fault-tolerant processor arrays [4, 10].

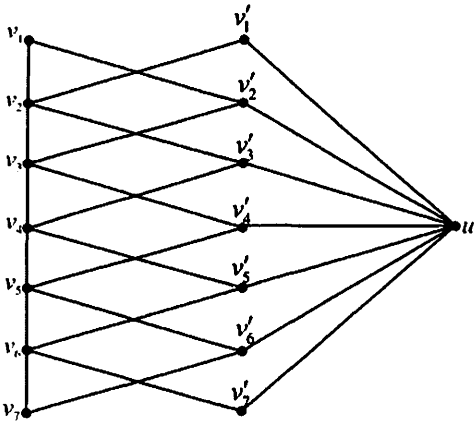
2. Preliminaries

In a search for triangle free graphs, with arbitrarily large chromatic numbers, Myceilski [9] developed a graph transformation that transforms a given graph G into a new graph $\mu(G)$, called the Myceilski graph of G. Another operation on graphs is the graph amalgamation. While the Myceilski graph is obtained starting with a single graph, the amalgamation requires two graphs.

Definition 1. For a graph G with vertex set $V(G) = V$ and edge set $E(G) = E$, the Mycielsky graph of G [9] is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$ where $V' = \{x' : x \in V\}$, and edge set $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of x (and x is also called the twin of x') and the vertex u is called the root of $\mu(G)$.
Figures 1 (a) and 1 (b) show the path P7 and the Myceilski of P7, $\mu(P_7)$

Definition 2. [11] A graph G with a fixed vertex u will be denoted by the ordered pair $(G; u)$. Given two ordered pairs $(G; u)$ and $(H; v)$, one can form a new graph by an operation called amalgamation defined as the disjoint union of G and H and identifying the vertices u and v. The amalgamation of $(G; u)$ and $(H; v)$ is denoted by $(G; u) \quad (H; v)$ or simply by $G \quad H$.
Figure 2 shows the amalgamation graph $(C_8, v_3) \quad (P_9, u_4)$.

(a)



(b)

Figure 1 (a) The graph P_7, (b) The graph $\mu(P_7)$

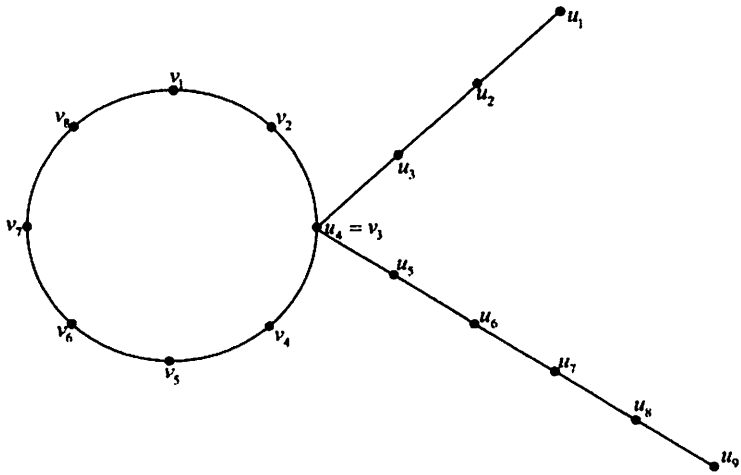


Figure 2 The graph (C_8, V_3) (P_9, U_4)

3. Embedding the Myceilski of a graph in pages

Given a graph G on n vertices, the associated Myceilski graph $\mu(G)$ is a graph on $2n+1$ vertices. Here, we find the pagenumber of $\mu(P_n)$, $n > 2$ and $\mu(C_n)$ where P_n and C_n denote the path on n vertices the cycle on n vertices respectively.

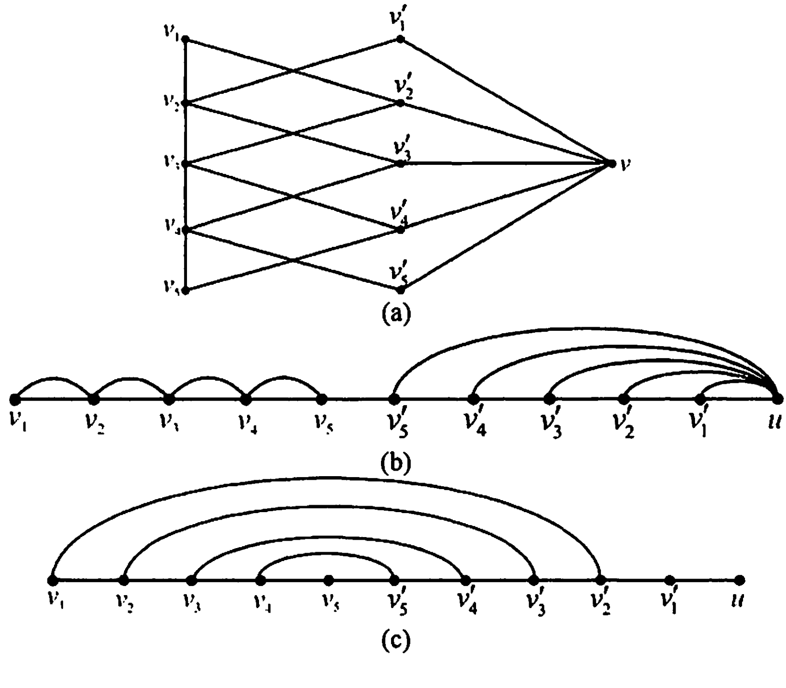
Theorem 3. Let P_n be the path on n vertices, $n \geq 3$. Then the pagenumber of $\mu(P_n) = 3$.

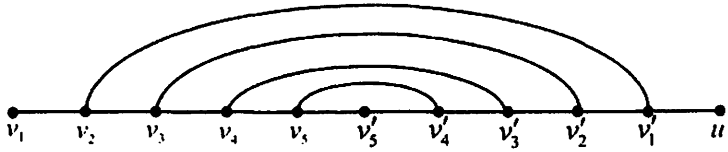
Proof. Let $V = V(P_n) = \{V_1, \ldots, V_n\}$ and $V' = \{V_1', \ldots, V_n'\}$, the set of twin vertices. Let v be the root of $\mu(P_n)$.

For the spine ordering of $\mu(P_n)$, consider the sequence $V_1, \ldots, V_n, V_n', \ldots, V_1', V$. With this sequence marked on the spine, embed the edges of $\mu(P_n)$ as follows.

$$\text{Embed} \begin{cases} (v_i, v_{i+1}); i = 1,2,\ldots,n-1, \text{ in page 1} \\ (v_{i+1}, v_i'); i = 1,2,\ldots n-1, \text{ in page 1} \\ (v_i, V_{i+1}'); i = 1,2,..n-1 \text{ in page 2} \\ (V, V_i'); i = 1,2,\ldots,n \text{ in page 3.} \end{cases}$$

The above scheme embeds all the edges of P_n in 3 pages with page-width of each page of $O(n)$. Figure 3 shows P_5 and its book embedding in three pages.





(d)

Figure 3 (a) The graph $\mu(P_5)$ and (b), (c), (d) Pages 1, 2, 3 in embedding of $\mu(P_5)$

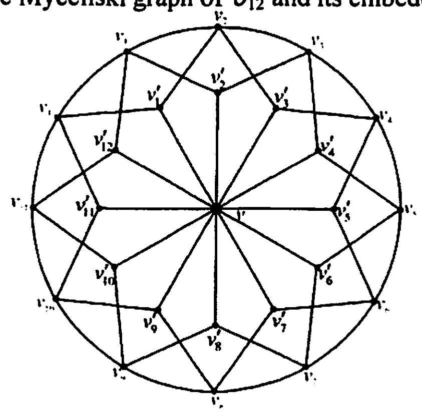
The following theorem gives the embedding of the Myceilski of cycles C_n.

Theorem 4. Let C_n be the cycle on n vertices, $n \geq 3$. Then the pagenumber of $\mu(C_n) = 4$.

Proof. Let $V = V(C_n) = \{V_1, \ldots, V_n\}$ and $V' = \{V_1', \ldots, V_n'\}$, the set of twin vertices. Let V be the root of $\mu(C_n)$. For the spine ordering of $\mu(C_n)$, consider the sequence $V_1, \ldots, V_n, V_n', \ldots, V_1', V$. With this sequence marked on the spine, embed the edges of $\mu(C_n)$ as follows.

$$\text{Embed} \begin{cases} (V_i, V_{i \bmod n+1}); i = 1, 2, \ldots, n, \text{ in page } 1 \\ (V_n, V_1') \text{ in page } 1 \\ (V_{i+1}, V_i'); i = 1, 2, \ldots n-1, \text{ in page } 2 \\ (V_i, V_{i+1}'); i = 1, 2, .. n-1 \text{ in page } 3 \\ (V, V_i'); i = 1, 2, \ldots, n \text{ in page } 4 \\ (V_1, V_n') \text{ in page } 4. \end{cases}$$

The above scheme embeds all the edges of $\mu(C_n)$ in 4 pages. Figure 4 (a), (b), (c), (d), (e) shows the Myceilski graph of C_{12} and its embedding in four pages.



(a)

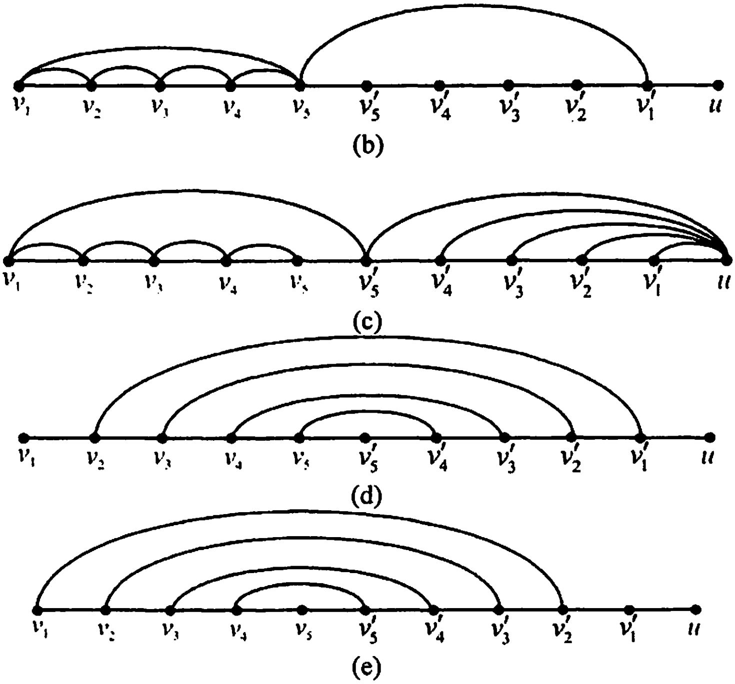


Figure 4 (a), (b), (c), (d), (e) The graph $\mu(C_5)$ and Pages 1, 2, 3, 4 in embedding of $\mu(C_5)$

We shall now prove a general result giving an upper bound for the pagenumber of the Myceilski of an arbitrary graph G.

Theorem 5. If G is any graph with pagenumber k, $\mu(G)$ can be embedded in $3k+1$ pages.

Proof. Let $V = V(G) = \{V_1, \ldots, V_n\}$ and $V' = \{V'_1, \ldots, V'_n\}$, the set of twin vertices. Let V be the root of $\mu(G)$. For the spine ordering of $\mu(G)$, consider the sequence $V_1, \ldots, V_n, V'_n, \ldots, V'_1, V$. Let the edge (V_r, V_l) of G be embedded in page i. Then (V_r, V'_l) can be embedded in page $k+i$ and (V'_r, V_l) can be embedded in page $2k+i$ for $i = 1, 2, \ldots, k$. Hence, all the edges (V, V'_i) ; $i = 1, 2, \ldots, n$, can be embedded in page $3k+1$. Thus all the edges of $\mu(G)$ can be embedded in $3k+1$ pages.

Note 6. It is notable that the embedding schemes given Theorem 3 and Theorem 4 to embed P_n, $n>2$ and C_n in pages reaches the upper bound given in Theorem 5.

4. Embedding the amalgamation of graphs in pages

The amalgamation of two disjoint graphs provide a new graph with the number of vertices equal to 1 less than the total number of vertices of the two participating graphs. Here we obtain the pagenumber of the amalgamation of two graphs whose pagenumbers are known.

Theorem 7. If the pagenumbers of the graphs G and H are k and l, then the pagenumber of $G \quad H$ is the maximum of k and l.

Proof. Let $V(G)=\{v_1,\ldots,v_m\}$ and $V(H)=\{u_1,\ldots,u_n\}$ be the vertex sets of G and H respectively. Let v be any fixed vertex of G and u be any fixed vertex of H.

Let S be the spine ordering for embedding G in k pages and T be the spine ordering for embedding H in l pages. Since any cyclic permutation of the spine ordering would maintain the pagenumber [1], we can cyclically permute S and T without altering the pagenumber of G and H. Let S' be obtained from S by cyclic permutations so that v is the last vertex in S'. Similarly, let T' be obtained from T by cyclic permutations so that u is the first vertex in T'. Then the edges of G and H can be embedded in k and l pages with S' and T' as their spine orderings respectively. For the spine ordering of $G \quad H$, consider $S' \quad T'$ with v and u identified and replaced by a single vertex. With this spine ordering, all the edges of $G \quad H$ can be embedded in a number of pages equal to the maximum of k and l.

Corollary 8. For any graph G and any amalgamation $G \quad G$, the pagenumber of $G \quad G$ is the same as the pagenumber of G.
The result follows immediately from Theorem 6.

5. Conclusion

The operations on graphs can be considered in several ways, on which the book embedding can be studied. Finding the book embedding of graphs formed using various graph operations would pose considerable challenge to estimate them compared to the original graphs considered.

6. References

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