Double shells with two pendant edges at the apex are k-graceful

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Abstract

A double shell is defined to be two edge disjoint shells with a common apex. In this paper we prove that double shells (where the shell orders are m and $(2m+1)$) with exactly two pendant edges at the apex are $k-$ graceful when $k = 2$. We extend this result to double shells of any order m and ℓ $(m \geq 3, \ell \geq 3)$ with exactly two pendant edges at the apex.

1 INTRODUCTION

In 1967, Rosa [13] introduced the labeling method called $\beta-$valuation as a tool for decomposing the complete graph into isomorphic sub graphs. Later on, this $\beta-$valuation was renamed as graceful labeling by Golomb [5]. A *graceful labeling* of a graph G with q edges and vertex set V is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a *graceful graph.* Graceful graphs have several applications in coding theory, $x-$ray crystallography, radar communication network and radio astronomy. Various kinds of graphs are shown to be graceful. In particular, cycle - related graphs have been a major focus of attention for nearly five decades. Rosa [13] showed that the $n-$cycle C_n is graceful if and only if $n \equiv 0$ or $3(mod 4)$. Frucht [4] has shown that the Wheels $W_n = C_n + K_1$ are graceful. Helms H_n (graph obtained from a wheel by attaching a pendant edge at

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each vertex of the $n-$ cycle) are shown to be graceful by Ayel and Favaron [1]. The *web graph* (graph obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle) was proved to be graceful by Kang , Liang, Gao and Yang [6].

Delorme, Koh Maheo, Teo, Thuillier [3] showed that any cycle with a chord is graceful. In 1985 Koh and Yap [9] defined a cycle with a P_k-chord to be a cycle with the path P_k joining two non consecutive vertices of the cycle and proved that these graphs are graceful when $k = 3$. Liu [11] has shown that the $n-$cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the chords v_1v_k and v_1v_{k+2} $(2 \leq k \leq n-3)$ are adjoined is graceful.

A natural generalization of graceful graphs is the notion of $k-$graceful graphs introduced independently by Slater [14] in 1982 and then by Maheo and Thuillier [12] also in 1982. A graph G with q edges is $k-$ graceful if there is a labeling $f : V(G) \to \{0, 1, 2, \ldots, (q + k - 1)\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k+1, \ldots, (q+k-1)\}$. Obviously, a 1-graceful graph is a graceful graph. Various kinds of graphs are shown to be $k-$graceful. Results of Maheo and Thuillier [12] together with those of Slater [14] show that C_n is $k-$graceful if and only if either $n \equiv 0$ or $1(\text{mod } 4)$ with $k$even and $k \leq (n^2 - 1)/2$ or $n \equiv 3(mod4)$ with k odd and $k \leq (n^2 - 1)/2$. Maheo and Thuillier [12] also proved that the wheel W_{2k+1} is $k-$graceful and conjectured that W_{2k} is $k-$graceful when $k \neq 3$ or $k \neq 4$. This conjecture was proved by Liang, Sun, and Xu [10]. For an exhaustive survey, refer to the dynamic survey by Gallian [8].

Deb and Limaye [2] have defined a shell graph as a cycle C_n with $(n-3)$ chords sharing a common end point called the *apex*. Shell graphs are denoted as $C(n, n-3)$ (See Figure 1). Note that the shell $C(n, n-3)$ is the same as the fan $F_{n-1} = P_{n-1} + K_1$. A *multiple shell* is defined to be a collection of edge disjoint shells that have their apex in common. Hence a *double shell* consists of two edge disjoint shells with a common apex.

In this paper we prove that double shells (where the shell orders are m and $(2m + 1)$) with exactly two pendant edges at the apex are $k-$ graceful when $k = 2$ (shell order excludes the apex). We extend this result to double shells of any order m and $\ell(m \geq 3, \ell \geq 3)$ with exactly two pendant edges at the apex.

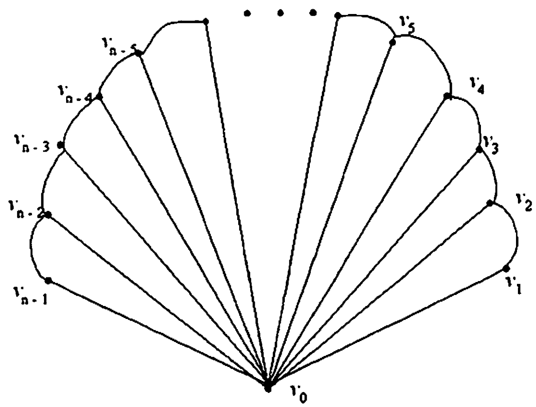

Figure 1: Shell Graph $C(n, n-3)$

2 MAIN RESULT

In this section we prove that double shell (where the shell orders are m and $(2m+1)$) with two pendant edges at the apex are $k-$ graceful when $k=2$ (shell order excludes the apex) in the first theorem. In the second theorem we prove that double shells of any order m and $\ell(m \geq 3, \ell \geq 3)$ with two pendant edges at the apex are $k-$ graceful when $k=2$.

Theorem 1: Double shells of order m and $(2m+1)$ with two pendant edges at the apex (shell order excludes the apex) are $k-$ graceful when $k=2$.

Proof: Let G be a a double shell of orders m and $(2m+1)$ with two pendant edges at the apex (shell order excludes the apex). Let n be the number of vertices and q be the number of edges in G. Note that $n = (3m+4)$ and $q = (6m+2)$. We describe the graph G as follows : In G, the shell that is present to the left of the apex is called as the left wing and the shell that is present to the right of the apex is considered as the right wing. The apex of G is denoted as v_0. Denote the vertices in the right wing of G from bottom to top as $v_1, v_2, \ldots, v_m$. The vertices in the left wing are denoted from top to bottom as $v_{m+1}, v_{m+2}, \ldots, v_{m+(m-1)}, v_{2m}, v_{2m+1}, v_{2m+2}, \ldots, v_{2m+(m-1)}, v_{2m+m} = v_{3m}, v_{(3m+1)}$. The two pendant vertices of G are denoted as $v_{(3m+2)}, v_{(3m+3)}$ (See Figure 2).

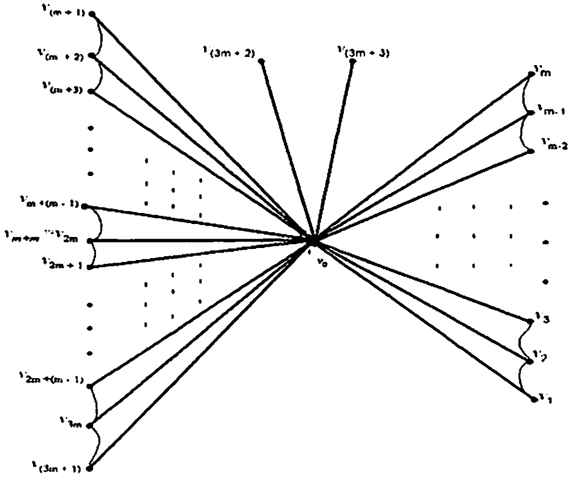

Figure 2: Double shell of orders m and $(2m+1)$ with two pendant edges

We label the vertices of the graph G as follows.

Case 1: When m is odd.
Here $m = (2j+3)$ where $j = 0, 1, 2, 3, \ldots$
Define $f(v_0) = 0$

$$f(v_{2i-1}) = \begin{cases} 4m+2j+2i+2, & \text{for } 1 \le i \le (m-j-1) \\ 6m+2j-2i+7, & \text{for } (m-j) \le i \le (m+j+2) \\ q+1, & \text{for } i = (m+j+3) \end{cases}$$

$$f(v_{2i}) = \begin{cases} 4m+2j-2i+4, & \text{for } 1 \le i \le (m-j-2) \\ 2m-2j+2i-4, & \text{for } (m-j-1) \le i \le (m+j+2) \\ q, & \text{for } i = (m+j+3) \end{cases} \quad (1)$$

From the above definition given in (1) we see that the vertices have distinct labels. We compute the edge labels as follows.

$$|f(v_o) - f(v_{2i-1})| = \begin{cases} 4m+2j+2i+2, & \text{for } 1 \le i \le (m-j-1) \\ 6m+2j+2i+7, & \text{for } (m-j) \le i \le (m+j+2) \\ q+1, & \text{for } i = (m+j+3) \end{cases}$$

$$|f(v_0) - f(v_{2i})| = \begin{cases} 4m+2j-2i+4, & \text{for } 1 \le i \le (m-j-2) \\ 2m-2j+2i-4, & \text{for } (m-j-1) \le i \le (m+j+2) \\ q, & \text{for } i = (m+j+3) \end{cases}$$

$$|f(v_{2i-1}) - f(v_{2i})| = \begin{cases} 4i-2, & \text{for } 1 \le i \le (m-j-2) \\ 4m+4j-4i+11, & \text{for } (m-j) \le i \le (m+j+2) \end{cases}$$

$$|f(v_{2i}) - f(v_{2i+1})| = \begin{cases} 4i, & \text{for } 1 \leq i \leq (m-j-2) \\ 4m+4j-4i+9, & \\ \quad \text{for } (m-j-1) \leq i \leq (m+j+1) & \end{cases} \tag{2}$$

From the computations given in (2) we can see that the edge labels are distinct.

Case 2 : When m is even.

Here $m = 2j + 4$, where $j = 0, 1, 2, 3, \ldots$

Define $f(v_0) = 0$

$$f(v_{2i-1}) = \begin{cases} 4m+2j+2i+6, & \text{for } 1 \leq i \leq (m-j-2) \\ 2m-2j+2i-6, & \text{for } (m-j-1) \leq i \leq (m+j+3) \\ q, & \text{for } i = (m+j+4) \end{cases}$$

$$f(v_{2i}) = \begin{cases} 4m+2j+2i+4, & \text{for } 1 \leq i \leq (m-j-2) \\ 6m+2j-2i+7, & \text{for } (m-j-1) \leq i \leq (m+j+2) \\ q+1, & \text{for } i = (m+j+3) \end{cases} \tag{3}$$

From the above definition given in (3) we see that the vertices have distinct labels.

We compute the edge labels as follows.

$$|f(v_o) - f(v_{2i-1})| = \begin{cases} 4m+2j-2i+6, & \text{for } 1 \leq i \leq (m-j-2) \\ 2m-2j+2i-6, & \\ \quad \text{for } (m-j-1) \leq i \leq (m+j+3) & \\ q, & \text{for } i = (m+j+4) \end{cases}$$

$$|f(v_o)-f(v_{2i})| = \begin{cases} 4m+2j+2i+4, & \text{for } 1 \leq i \leq (m-j-2) \\ 6m+2j-2i+7, & \text{for } (m-j-1) \leq i \leq (m+j+2) \\ q+1, & \text{for } i = (m+j+3) \end{cases}$$

$$|f(v_{2i-1}) - f(v_{2i})| = \begin{cases} 4i-2, & \text{for } 1 \leq i \leq (m-j-2) \\ 4m+4j-4i+13, & \\ \quad \text{for } (m-j-1) \leq i \leq (m+j+2) & \end{cases}$$

$$|f(v_{2i}) - f(v_{2i+1})| = \begin{cases} 4i, & \text{for } 1 \leq i \leq (m-j-3) \\ 4m+4j-4i+11, & \\ \quad \text{for } (m-j-1) \leq i \leq (m+j+2) & \end{cases} \tag{4}$$

From the above definition as in (4) it is clear that the vertex labels and the edge labels are distinct. Hence double shell of order m and $(2m+1)$ with two pendant edges at the apex (shell order excludes the apex) is $k-$graceful when $k = 2$.

Theorem 2: All double shells of any order m and ℓ (where $m \geq 3, \ell \geq 3$) with exactly two pendant edges at the apex are $k-$graceful when $k = 2$.

Proof: Let G be a double shell of order m and ℓ with two pendant edges at the apex as described in Theorem 1. Denote the vertices in the right wing of G from bottom to top as $v_1, v_2 \ldots, v_m$. The vertices in the left wing are denoted from top to bottom as $v_{m+1}, v_{m+2}, \ldots, v_{m+(m-1)}, v_{2m}, v_{2m+1}, v_{2m+2}, v_{2m+3}, \ldots, v_{(m+\ell-2)}, v_{(m+\ell-1)}, v_{(m+\ell)}$. The two pendant vertices of G are denoted as $v_{(m+\ell+1)}, v_{(m+\ell+2)}$.

Note that G has $n = m + \ell + 3$ vertices and $q = 2m + 2\ell$ edges. To prove that G is $k-$graceful when $k = 2$, we consider the following cases.

Case 1: When ℓ is odd, $\ell = 2t + 1\,(t \geq 1)$.

Case 1.1: When $t \equiv m(mod3)$. In this case $\ell = 2m + 1 + 6p$, where p is any integer.

We label the vertices of the graph G as follows

Case 1.1.1 : When m is odd.
Define $f(v_0) = 0$

$$f(v_{2i-1}) = \begin{cases} 4p + 2m + 2i - 2, & \text{for } 1 \leq i \leq (m+1)/2 \\ 12p + 6m - 2i + 7, & \text{for } (m+3)/2 \leq i \leq (m + 2p + 1) \\ 16p + 8m - 2i + 4, & \\ \qquad \text{for } (m + 2p + 2) \leq i \leq (6p + 3m + 1)/2 & \\ 12p + 6m + 2, & \text{for } i = (6p + 3m + 3)/2 \end{cases}$$

$$f(v_{2i}) = \begin{cases} 12p + 6m - 2i + 5, & \text{for } 1 \leq i \leq (m-1)/2 \\ 4p + 2m + 2i, & \\ \qquad \text{for } (m+1)/2 \leq i \leq (6p + 3m + 1)/2 & \\ 8p + 2m + 5, & \text{for } i = (6p + 3m + 3)/2 \end{cases}$$

From the above definition we see that the vertices have distinct labels.

The edge labels are computed as follows.

$$|f(v_o)-f(v_{2i-1})| = \begin{cases} |4p+2m+2i-2|, & \text{for } 1 \le i \le (m+1)/2 \\ |12p+6m-2i+7|, & \\ & \text{for } (m+3)/2 \le i \le (m+2p+1) \\ |16p+8m-2i+4|, & \\ & \text{for } (m+2p+2) \le i \le (6p+3m+1)/2 \\ |12p+6m+2|, & \text{for } i = (6p+3m+3)/2 \end{cases}$$

$$|f(v_o)-f(v_{2i})| = \begin{cases} |12p+6m-2i+5|, & \text{for } 1 \le i \le (m-1)/2 \\ |4p+2m+2i|, & \\ & \text{for } (m+1)/2 \le i \le (6p+3m+1)/2 \\ 8p+2m+5, & \text{for } i = (6p+3m+3)/2 \end{cases}$$

$$|f(v_{2i-1})-f(v_{2i})| = \begin{cases} |8p+4m-4i+7|, & \text{for } 1 \le i \le (m-1)/2 \\ & \quad \& \ (m+3)/2 \le i \le (m+2p+1) \\ |12p+6m-4i+4|, & \\ & \text{for } (m+2p+2) \le i \le (6p+3m+1)/2 \end{cases}$$

$$|f(v_{2i})-f(v_{2i+1})| = \begin{cases} |8p+4m-4i+5|, & \text{for } 1 \le i \le (m+2p) \\ |12p+6m-4i+2|, & \\ & \text{for } (m+2p+1) \le i \le (6p+3m-1)/2 \end{cases}$$

Case 1.1.2: When m is even.

Define $f(v_0) = 0$

$$f(v_{2i-1}) = \begin{cases} 4p+2m+2i, & \text{for } 1 \le i \le (6p+3m+2)/2 \\ 8p+2m+3, & \text{for } i = (6p+3m+4)/2 \end{cases}$$

$$f(v_{2i}) = \begin{cases} 12p+6m-2i+5, & \text{for } 1 \le i \le (m+2p) \\ 16p+8m-2i+4, & \text{for } (m+2p+1) \le i \le (6p+3m)/2 \\ 8p+4m+3, & \text{for } i = (6p+3m+2)/2 \end{cases}$$

From the above definition it is clear that the vertices have distinct labels.

As in previous case we can calculate the edge labeling . Hence it is obvious that all vertex labels are distinct and all edge labels are also distinct.

Case 1.2: When $t \equiv m+1(mod3)$.

In this case $\ell = 2m+3+6p$, where p is any integer. We label the vertices of the graph G as follows.

Case 1.2.1: When m is odd.

Define $f(v_0) = 0$

$$f(v_{2i-1}) = \begin{cases} 4p+2m+2i, & \text{for } 1 \le i \le (6p+3m+2)/2 \\ q, & \text{for } i = (6p+3m+5)/2 \end{cases}$$

$$f(v_{2i}) = \begin{cases} 12p+6m-2i+9, & \text{for } 1 \le i \le (m+2p+1) \\ 16p+8m-2i+8, & \text{for } (m+2p+2) \le i \le (6p+3m+3)/2 \\ 8p+2m+7, & \text{for } i = (6p+3m+5)/2 \end{cases}$$

From the above definition it is clear that the vertices have distinct labels. As in case 1.1.1 we can calculate the edge labels which are distinct.

Case 1.2.2: When m is even.
Define $f(v_0) = 0$

$$f(v_{2i-1}) = \begin{cases} 4p+2m+2i, & \text{for } 1 \le i \le (6p+3m+2)/2 \\ 8p+2m+7, & \text{for } i = (6p+3m+6)/2 \end{cases}$$

$$f(v_{2i}) = \begin{cases} 12p+6m-2i+9, & \text{for } 1 \le i \le (m+2p+1) \\ 16p+8m-2i+8, & \text{for } (m+2p+2) \le i \le (6p+3m+3)/2 \\ q, & \text{for } i = (6p+3m+4)/2 \end{cases}$$

From the above definition it is clear that the vertices have distinct labels. As in case 1.1.1 we can calculate the edge labels which are distinct.

Case 1.3: When $t \equiv m + 2(mod3)$.

In this case $\ell = 2m+5+6p$, where p is any integer. We label the vertices of the graph G as follows.

Case1.3.1: When m is odd.
Define $f(v_0) = 0$

$$f(v_{2i-1}) = \begin{cases} 4p+2m+2i+2, & \text{for } 1 \le i \le (m+1)/2, \\ 12p+6m-2i+15, & \text{for } (m+3)/2 \le i \le (m+2p+2), \\ 16p+8m-2i+16, & \\ \quad \text{for } (m+2p+3) \le i \le (6p+3m+5)/2 & \\ 8p+4m+9, & \text{for } i = (6p+3m+7)/2 \end{cases}$$

$$f(v_{2i}) = \begin{cases} 12p+6m-2i+13, & \text{for } 1 \le i \le (m-1)/2 \\ 4p+2m+2i+4, & \text{for } (m+1/2) \le i \le (6p+3m+5)/2 \\ 8p+2m+9, & \text{for } i = (6p+3m+7)/2 \end{cases}$$

The above definition shows that the vertices have distinct labels. The edge labels are also distinct as we have shown in case 1.1.1.

Case1.3.2: When m is even.
Define $f(v_0) = 0$

$$f(v_{2i-1}) = \begin{cases} 4p+2m+2i+2, & \text{for } 1 \le i \le (m/2), \\ 12p+6m-2i+13, & \text{for } (m/2)+1 \le i \le (m+2p+2), \\ 16p+8m-2i+14, & \\ \quad \text{for } (m+2p+3) \le i \le (6p+3m+6)/2 & \\ 8p+4m+9, & \text{for } i = (6p+3m+6)/2 \end{cases}$$

$$f(v_{2i}) = \begin{cases} 12p+6m-2i+13, & \text{for } 1 \le i \le (m/2), \\ 4p+2m+2i+2, & \text{for } (m/2)+1 \le i \le (6p+3m+4)/2, \\ q, & \text{for } i = (6p+3m+6)/2 \end{cases}$$

The above definition shows that the vertices have distinct labels. Edge labels can be calculated similar to the case 1.1.1.

Case 2: When ℓ is even, $\ell = 2t (t \ge 2)$

The proof for l = 2t is similar to case 1. Hence all Double shells of any order m and ℓ (where $m \ge 3$ and $\ell \ge 3$) with exactly two pendant edges at the apex are k−graceful when $k = 2$.
Illustrations for the above theorems are presented in the appendix.

Conclusion: In this paper we have proved that double shells (where the shell orders are m and $(2m+1)$) with exactly two pendant edges at the apex are k− graceful when $k = 2$ where a double shell is defined to be two edge disjoint shells with a common apex in theorem 1. In theorem 2 we have proved that double shells of any order m and ℓ $(m \ge 3, \ell \ge 3)$ with exactly two pendant edges at the apex are k− graceful when $k = 2$. In future we intend to prove that double shells of any order m and ℓ $(m \ge 3, \ell \ge 3)$ with exactly two pendant edges at the apex are k− graceful for any k.

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APPENDIX

Theorem: 2 Case 1.1.1 , $m = 7, p = 0, \ell = 15$.

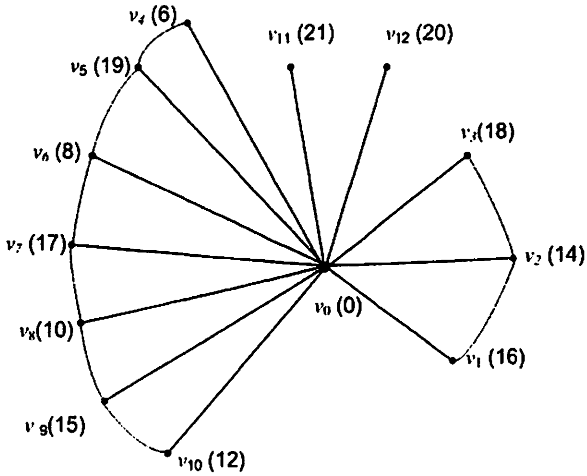

Figure 3: A Theorem 1: Case 1, A k−graceful double shell with pendant edges at the apex When $m = 3, n = 13, q = 20, j = 0$

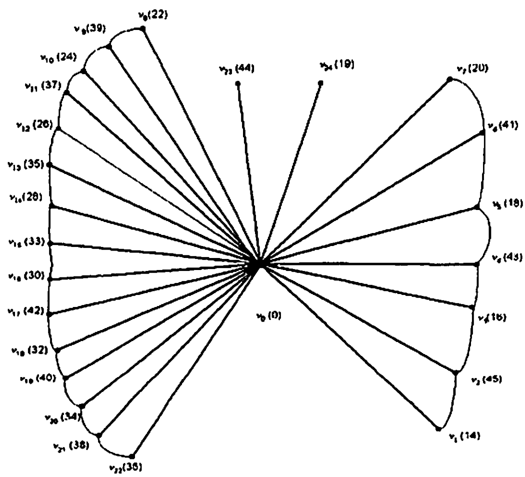

Figure 4: A Theorem 2: Case 1.1.1, A k−graceful double shell with pendant edges at the apex When $m = 7, p = 0, \ell = 15$