Strong Kernel in Certain Oriented Networks

Joice Punitha M.
Department of Mathematics, L. N. Government College, Ponneri, India
joicepunitha@gmail.com

Abstract

A kernel in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in V / S there is a vertex v in S, such that (u, v) is an arc of D. The problem of existence of a kernel itself is *NP*-complete for a general digraph. But in this paper we solve the strong kernel problem of certain oriented networks in polynomial time.

1 Introduction

The concept of kernel is widespread and appears in diverse fields such as logic, computational complexity, artificial intelligence, graph theory, game theory, combinatorics and coding theory [3, 4]. Efficient routing among a set of mobile hosts is one of the most important functions in ad hoc wireless networks. Dominating-set-based routing to networks with unidirectional links is proposed in [1, 14]. A few years ago a new interest for these studies arose due to their applications in finite model theory. Indeed variants of kernel are the best properties to provide counter examples of $0 - 1$ laws in fragments of monadic second order logic [13].

A kernel [8] in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that (u, v) is an arc of D. The minimum cardinality of all possible kernels in a directed graph D is denoted by $\kappa(D)$ and is called the kernel number. The concept of kernels in digraphs was introduced in different ways [15]. Von Neumann and Morgenstern [21] were the first to introduce kernels when describing winning positions in 2 person games. They proved that any directed acyclic graph has a unique kernel. Not every digraph has a kernel and if a digraph has a kernel, this kernel is not necessarily unique. For example, the directed 3-cycle (with vertices x, y, z and arcs (x, y), (y, z), (z, x)) has no kernel. All odd length directed cycles and most tournaments have no kernels [3, 4]. If D is finite, the decision problem of the existence of a kernel is *NP*-complete for a general digraph [7, 16, 20] and for a planar digraph with indegrees ≤ 2, outdegrees ≤ 2 and degrees ≤ 3 [9]. Finding kernels in special classes of digraphs seems to be an open field of study. It has been shown [5] that the kernel problem is solvable in polynomial time for locally semicomplete digraphs, in which the out-neighbours (in-neighbours) of every vertex are adjacent. Kernels in some classes of planar digraphs were investigated in [6]. It is further known that a finite digraph all of whose cycles have even length has a kernel [18], and that the question of the number of kernels is NP-complete even for this restricted class of digraphs [19]. It is somehow related to finding a maximum clique in graphs [11, 12], which is known to be difficult for random dense graphs. This sufficient condition for a digraph to have a kernel has been generalized by several authors [4, 5].

In this paper we view the kernel problem from a different perspective. In the literature, only the existence of kernel of a digraph G and its applications are extensively studied. Our aim in this paper is to investigate all strong orientations of a graph G and to determine the strong kernel number of G. This number is different from the independent domination number γ_i for undirected graphs where γ_i is the cardinality of a minimum independent dominating set [2]. For the graph in Figure 1 (a) $\Gamma = \{3, 4\}$ is an independent dominating set. Thus $\gamma_i = 2$ where as it is easy to verify that the kernel number is 3.

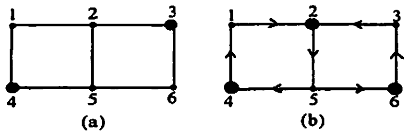

Figure 1: (a) $\gamma_i = 2$; (b): Kernel number = 3

2 Kernel in Oriented Graphs

An orientation of an undirected graph G is an assignment of exactly one direction to each of the edges of G. There are $2^{|E|}$ orientations for G. An orientation O of an undirected graph G is said to be strong if for any two vertices x, y of $G(O)$, there are both (x, y)-path and (y, x)-path in $G(O)$ [22].

Let G be an undirected graph. Let $O_x(G)$ denote all possible orientations of a graph G and $O_s(G)$ denote the set of all strong orientations of G. For an orientation $O \in O_x$, let $G(O)$ denote the directed graph with orientation O and whose underlying graph is G. The kernel number of $G(O)$ is denoted by $\kappa(G(O))$. For convenience we write as $\kappa(O)$.

Definition 2.1[8]: A kernel in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that (u, v) is an arc of D. The minimum cardinality of all possible kernels in a directed graph D is called the kernel number and is denoted by $\kappa(D)$.

Definition 2.2: The kernel number κ_x of G is defined as $\kappa_x(G) = \min\{\kappa(O): O \in O_x(G)\}$ where $\kappa(O) = \min\{|K|, K \text{ is a kernel of } G(O)\}$.

Definition 2.3: The strong kernel number κ_s of G is defined as $\kappa_s(G) = \min\{\kappa(O): O \in O_s(G)\}$ where $\kappa(O) = \min\{|K|, K \text{ is a kernel of } G(O)\}$.

The strong kernel problem: The strong kernel problem of an undirected graph G is to find a kernel K of $G(O)$ for some strong orientation O of G such that $|K| = \kappa_s$.

An optimal lower bound for $\kappa_s(G)$ when G is a regular graph has been obtained in [17] and it is the key result used throughout this paper.

Theorem 2.1[17]: Let G be an r – regular graph on n vertices. Then $\kappa_s \geq \lceil n/r \rceil$.

3 Strong Kernel in Oriented Mobius Ladder M_n

In this section we determine the strong kernel number of Mobius Ladder M_n..

Definition 3.1[10]: The Mobius Ladder M_n is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of P_n.

Remark 3.1: For convenience, we label the vertices of one copy of P_n as 1, 2, ..., n and the other copy of P_n as n+1, n+2, ..., $2n$. $|V| = 2n$. See Figure 2.

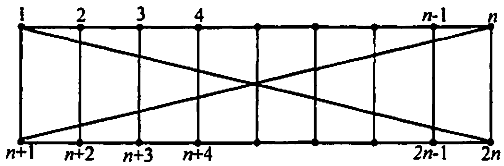

Figure 2: Mobius Ladder M_n

Lemma 3.1: Let G be the Mobius Ladder M_n. Let the cycle C = 1, 2, 3, ..., n-1, n, $2n$, $2n$-1, ..., n+2, n+1, 1 be oriented in the clockwise direction. All the other edges are oriented arbitrarily. Then G is strongly connected.

Proof: For $u, v \in V(G)$, $1 \leq u, v \leq 2n$, we claim that G is strongly connected. Since C is oriented in the clockwise direction, there exist directed paths from u to v and v to u. Thus G is strongly connected.

Theorem 3.1: Let G be M_n, $n \geq 4$. For $n \equiv 1(\text{mod}3)$, $\kappa_s = \left\lceil \frac{2n}{3} \right\rceil$.

Proof: Consider an orientation O of G satisfying the following condition.

1. Orient the cycle C in the clockwise direction.
2. Let $K = \left\{2,5,8,11,...,3\left\lfloor \frac{n}{3} \right\rfloor - 1\right\} \cup \left\{n, n+3, n+6, ..., n+3\left\lfloor \frac{n}{3} \right\rfloor\right\}$

By Lemma 3.1, the orientation O is a strong orientation of G. Since G is strongly connected, there is at least one incoming and one outgoing edge at every vertex of K. Thus to prove that K is a kernel, it is enough to prove that K is independent. Two vertices r and s of G are adjacent if and only if

(i) $|r - s| = 1$, $(r, s) \in (i, i+1)$, $1 \leq i \leq 2n-1$

(ii) $|r - s| = 2n-1$, $(r, s) \in (1, 2n)$

(iii) $|r - s| = n$, $(r, s) \in (i, n+i)$, $1 \leq i \leq n$

Choose any two vertices from K. Suppose we choose $u = 3i - 1$ and $v = n+3j$, $1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor$, $0 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor$. Since $|(n+3j)-(3i-1)| = |n+3(j-i)+1| \neq n$, u and v are not adjacent.

We next claim that $|K| = \left\lceil \frac{2n}{3} \right\rceil$. We verify the claim when K is as in 2. Since $n \equiv 1$ (mod 3), $n = 3k + 1$ for some k.

Let $K = K_1 \cup K_2$ where $K_1 = \left\{2,5,8,11,\ldots,3\left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$ and

$K_2 = \left\{n, n+3, n+6, \ldots, n+3\left\lfloor \frac{n}{3} \right\rfloor\right\}$. The cardinality of K_1 is $\left\lfloor \frac{n}{3} \right\rfloor$.

Similarly $|K_2| = \left\lceil \frac{n}{3} \right\rceil$. See Figure 3.

Hence $|K| = |K_1| + |K_2| = \left\lfloor \frac{n}{3} \right\rfloor + \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil$.

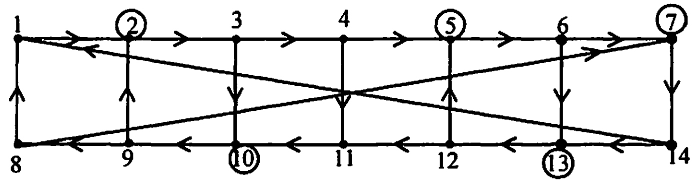

Figure 3: Encircled vertices form a kernel

Theorem 3.2: Let G be M_n, $n \geq 4$. For $n \not\equiv 1 \pmod 3$, there exist a kernel K of M_n such that $|K| = \kappa_s(M_n) + 1 = \left\lceil \frac{2n}{3} \right\rceil + 1$.

Proof: Let M_n be oriented as in Lemma 3.1. Since G is strongly connected regular graph, by theorem 2.1, $\kappa_s \geq \left\lceil \frac{2n}{3} \right\rceil$.

Case 1: $n \equiv 0 \pmod 3$

The independent set $K = \{1,4,7,\ldots,n-2\} \cup \{n+2, n+5, n+8, \ldots, 2n-1\} \cup \{n\}$ forms a kernel for M_n. Thus $|K| = \frac{n}{3} + \frac{n}{3} + 1 = \frac{2n}{3} + 1$.

Case 2: $n \equiv 2 \pmod 3$

The independent set $K = \left\{2,5,8,11,\ldots, 3\left\lfloor \frac{n}{3} \right\rfloor - 1\right\} \cup \{n-1, n+1, 2n\}$ $\cup \left\{n+3, n+6, \ldots, n+3\left\lfloor \frac{n}{3} \right\rfloor\right\}$ forms a kernel for M_n. Thus $|K| = \left\lceil \frac{2n}{3} \right\rceil + 1$.

The proofs for both the cases are similar to that of Theorem 3.1.

4 Strong Kernel in Oriented Circular Ladder $CL(n)$

In this section we determine the strong kernel number of Circular Ladder $CL(n)$.

Definition 4.1[10]: A circular ladder $CL(n)$ is the union of an outer cycle $\Gamma_O = \{u_1, u_2, ..., u_n, u_1\}$ and an inner cycle $\Gamma_I = \{v_1, v_2, ..., v_n, v_1\}$ with additional edges (u_i, v_i), $i = 1, 2, ..., n$ called spokes.

Remark 4.1: For convenience $u_1, u_2, ..., u_n$ are represented by $1, 2, ..., n$ and $v_1, v_2, ..., v_n$ by $n+1, n+2, ..., 2n$ respectively. For $1 \leq i, j \leq n$, we call the oriented spoke $(\overrightarrow{i, n+i})$, an inward spoke and the oriented spoke $(\overrightarrow{n+j, j})$ an outward spoke. See Figure 4.

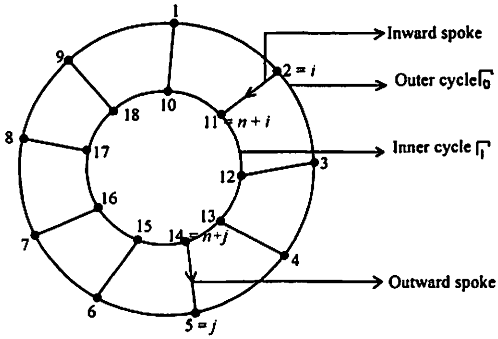

Figure 4: $CL(9)$, (2,11) and (14,5) represent inward spoke and outward spoke respectively

Lemma 4.1[10]: Let G be the circular ladder $CL(n)$, $n \geq 4$ with an inward spoke and an outward spoke. Let the outer cycle Γ_O and the inner cycle Γ_I be oriented in the clockwise and anticlockwise direction respectively. All other spokes are oriented arbitrarily. Then G is strongly connected.

Theorem 4.1: Let G be $CL(n)$, $n \geq 4$. For $n \not\equiv 1(\text{mod}\,3)$, $\kappa_s = \left\lceil \frac{2n}{3} \right\rceil$.

Proof: Consider an orientation O of G satisfying the following condition.
1. Orient the outer cycle Γ_O and the inner cycle Γ_I in the clockwise and anticlockwise direction respectively.
2.(a) Let $K = \{1,4,7,...,n-2\} \cup \{n+2, n+5, n+8,...,2n-1\}$ when $n \equiv 0(\text{mod}\,3)$.
(b) Let $K = \{1, 4, 7, .., n-1\} \cup \{n+2, n+5, n+8, ..., 2n\}$ when $n \equiv 2(\text{mod}\,3)$.

By Lemma 4.1, the orientation O is a strong orientation of G. Since G is strongly connected, there is at least one incoming and one outgoing edge at every vertex of K. Thus to prove that K is a kernel, it is enough to prove that K is independent. Two vertices r and s of G are adjacent if and only if
(i) $|r - s| = 1$, $1 \leq r, s \leq n$, $n+1 \leq r, s \leq 2n$
(ii) $|r - s| = n-1$, $(r, s) \in (n, 1)$ & $(2n, n+1)$
(iii) $|r - s| = n$, $(r, s) \in (i, n+i)$, $1 \leq i \leq n$

Choose any two vertices, one vertex on the outer cycle Γ_O and another vertex on the inner cycle Γ_I. Suppose we choose $u = 3i - 2$ and $v = n + 3j - 1$, $1 \le i, j \le \left\lceil \frac{n}{3} \right\rceil$ when $n \equiv 2 (\text{mod} 3)$. Since $|(n+3j-1)-(3i-2)| = |n+3(j-i)+1| \ne n$, u and v are not adjacent. Similar arguments hold for the case 2(a). See Figure 5.

We next claim that $|K| = \left\lceil \frac{2n}{3} \right\rceil$. We verify the claim when K is as in 2(b). Since $n \equiv 2$ (mod 3), $n = 3k + 2$ for some k. Let $K = K_1 \cup K_2$ where $K_1 = \{1, 4, 7, .., n - 1\}$ and $K_2 = \{n + 2, n + 5, n + 8, \ldots, 2n\}$.

The cardinality of K_1 is given by $|K_1| = \left\lceil \frac{n}{3} \right\rceil$. Similarly, the cardinality of K_2 is given by $|K_2| = \left\lceil \frac{n}{3} \right\rceil$. Hence $|K| = |K_1| + |K_2| = \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil$.

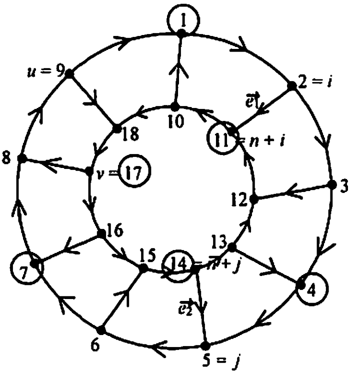

Figure 5: Encircled vertices form a kernel in $CL(9)$

Theorem 4.2: Let G be $CL(n)$, $n \ge 4$. For $n \equiv 1(\text{mod} 3)$, there exist a kernel K of $CL(n)$ such that $|K| = \kappa_s(CL(n)) + 1 = \left\lceil \frac{2n}{3} \right\rceil + 1$.

Proof: Let $CL(n)$ be oriented as in Lemma 4.1. Since G is strongly connected regular graph, by theorem 2.1, $\kappa_s \ge \left\lceil \frac{2n}{3} \right\rceil$.

The independent set $K = \{1,4,7,\ldots,n-3\} \cup \{n+2, n+5, n+8, \ldots, 2n-2\} \cup \{n-1, 2n\}$ forms a kernel for $CL(n)$. Thus $|K| = \left\lceil \frac{2n}{3} \right\rceil + 1$.

The proof is similar to that of Theorem 4.1.

5 Strong Kernel in Oriented Chordal Graphs *CH*(n)

In this section we obtain the strong kernel number of chordal graphs *CH*(n).

Definition 5.1: A chordal graph *CH*(n), $n \geq 2$ is a graph with the vertex set $V(CH(n)) = \{v_1, v_2, ..., v_n\}$ and $E(CH(n)) = \{(v_i, v_{i+1}), 1 \leq i \leq n-1\} \cup \{(v_1, v_n)\} \cup \left\{(v_i, v_{n-i+1}), 1 \leq i \leq \frac{n}{2}\right\}$, n being an even integer. We name the parallel edges as chords. See Figure 6.

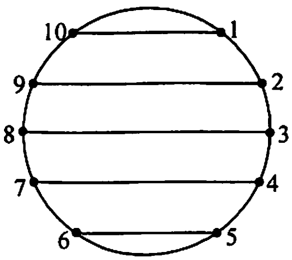

Figure 6: Chordal Graph *CH*10)

Lemma 5.1: Let G be *CH*(n), $n \geq 2$. Let the cycle $C = 1, 2, ..., n-1, n, 1$ in G on n vertices be oriented in the clockwise direction. The chords are oriented arbitrarily. Then G is strongly connected.

Proof: For $u, v \in V$, we claim that G is strongly connected. Since C is oriented in the clockwise direction, our claim is true.

Theorem 5.1: Let G be *CH*(n), $n \geq 2$. Then $k_s = \left\lceil \frac{n}{3} \right\rceil$.

Proof: Consider an orientation O of G satisfying the following condition.

1. Orient the cycle C in the clockwise direction.

2(a). Let $K = \{3, 6, 9, ..., n\}$ when $n \equiv 0 (\text{mod} 3)$.

(b). Let $K = \left\{2, 5, 8, 11, ..., 3\left\lfloor \frac{n}{3} \right\rfloor - 1\right\} \cup \{n\}$ when $n \equiv 1 (\text{mod} 3)$

(c). Let $K = \{1\} \cup \left\{4, 7, 10, ..., 3\left\lfloor \frac{n}{3} \right\rfloor + 1\right\}$ when $n \equiv 2 (\text{mod} 3)$

By Lemma 5.1, the orientation O is a strong orientation of G. Conditions 2(a), (b) and (c) imply that there are at least two incoming edges and one outgoing edge at every vertex of K. Thus to prove that K is a kernel, it is enough to prove that K is independent. Two vertices r and s of G are adjacent if and only if

(i) $|r - s| = 1$, $1 \leq r, s \leq n$

(ii) $|r - s| = n - 2i + 1$, $(r, s) \in (i, n - i + 1)$, $i = 1, 2, ..., \frac{n}{2}$

(iii) $|r - s| = n - 1$, $(r, s) \in (n, 1)$

Choose any two vertices from K. Suppose we choose $u = 3i - 1$ and $v = 3j - 1$, $i \neq j$, $1 \leq i, j \leq \left\lfloor \frac{n}{3} \right\rfloor$, when $n \equiv 1 \pmod 3$. Clearly no two vertices are adjacent to each other. Similar arguments hold in the remaining cases. See Figure 7.

We next claim that $|K| = \left\lceil \frac{n}{3} \right\rceil$. We verify the claim when K is as in 2(b). Since $n \equiv 1 (\text{mod } 3)$, $n = 3k + 1$ for some k. Let $K = K_1 \cup K_2$ where $K_1 = \left\{2, 5, 8, \ldots, 3\left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$ and $K_2 = \{n\}$.

The cardinality of K_1 is given by $3\left\lfloor \frac{n}{3} \right\rfloor - 1 = 2 + 3(|K_1| - 1) = \left\lfloor \frac{n}{3} \right\rfloor$. Similarly $|K_2| = 1$. Hence $|K| = |K_1| + |K_2| = \left\lfloor \frac{n}{3} \right\rfloor + 1 = \left\lceil \frac{n}{3} \right\rceil$.

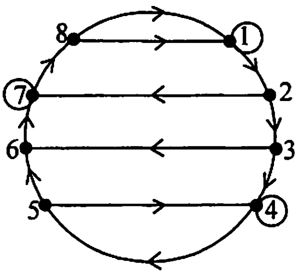

Figure 7: Encircled vertices form a kernel

6 Strong Kernel in Oriented $t\,H$ – Graph

Definition 6.1: A $t\ H$ – graph is a graph with vertex set $\{(i, j) : 1 \leq i \leq 3, 1 \leq j \leq n\}$ and edge set $\{((i, j), (i, j + 1)), i = 1 \text{ and } 3\} \cup \{((2, j), (2, j+1)) : j$ odd, $1 \leq j \leq n - 1\} \cup \{((1, 1), (1, n)), ((3, 1), (3, n))\} \cup \{((i, j), (i+1, j)), i = 1$ and $2, 1 \leq j \leq n\}$ and is denoted by $H_n(t)$ where t denotes the number of copies of H.

Remark 6.1: For convenience (1, 1), (1, 2), …, (1, n) are represented by 1, 2, …, n; (2, 1), (2, 2), …, (2, n) are represented by n+1, n+2, …, 2n and (3, 1), (3, 2), …, (3, n) are represented by 2n+1, 2n+2, …, 3n. The number of vertices of $t\ H$ – graph is 3n.

Lemma 6.1: Let G be $H_n(t)$, $t \geq 2$. Let the cycles 1, 2, …, n, 2n, 3n, 3n – 1,…, 2n + 1, n + 1 1 and 2i – 1, 2i, n + 2i, 2n + 2i, 2n + 2i – 1, n + 2i – 1, 2i – 1, $1 \leq i \leq n/2$, of $H_n(t)$ on 3n vertices be oriented in the clockwise direction. All the remaining edges are oriented arbitrarily. Then G is strongly connected.

Proof: For $u, v \in V$, we claim that G is strongly connected. Since cycles of $H_n(t)$ are oriented in the clockwise direction, our claim is true.

Theorem 6.1: Let G be the t H - Graph on $3n$ vertices. Then $k_s = n$.

Proof: Consider an orientation O of G satisfying the following condition.

1. Orient the cycles $1, 2, \ldots, n, 2n, 3n, 3n-1, \ldots, 2n+1, n+1$ 1 and $2i-1, 2i, n+2i, 2n+2i, 2n+2i-1, n+2i-1, 2i-1$, $1 \le i \le n/2$, in the clockwise direction.
2. Let $K = \{1, 3, 5, \ldots, n-1\} \cup \{2n+2, 2n+4, 2n+6, \ldots, 3n\}$.

By Lemma 6.1, the orientation O is a strong orientation of G. Since G is strongly connected, there is at least one incoming and one outgoing edge at every vertex of K. Thus to prove that K is a kernel, it is enough to prove that K is independent. Two vertices r and s of G are adjacent if and only if

(i) $|r-s| = 1$, $1 \le r, s \le n$, $2n+1 \le r, s \le 3n$, $(r, s) \in (n+2i-1, n+2i)$, $1 \le i \le n/2$

(ii) $|r-s| = n-1$, $(r, s) \in (n, 1)$ & $(3n, 2n+1)$

(iii) $|r-s| = n$, $(r, s) \in (i, n+i)$ & $(n+i, 2n+i)$ $1 \le i \le n$

Choose any two vertices. Suppose we choose $u = 2i-1$ and $v = 2n+2j$, $1 \le i, j \le n/2$. Since $|(2n+2j)-(2i-1)| \neq n$, u and v are not adjacent. See Figure 8.

We next claim that $|K| = n$. We verify the claim when K is as in 2. Let $K = K_1 \cup K_2$ where $K_1 = \{1, 3, 5, \ldots, n-1\}$ and $K_2 = \{2n+2, 2n+4, 2n+6, \ldots, 3n\}$. The cardinality of K_1 is given by $|K_1| = \frac{n}{2}$. Similarly, the cardinality of K_2 is given by $|K_2| = \frac{n}{2}$. Hence $|K| = |K_1| + |K_2| = \frac{n}{2} + \frac{n}{2} = n$.

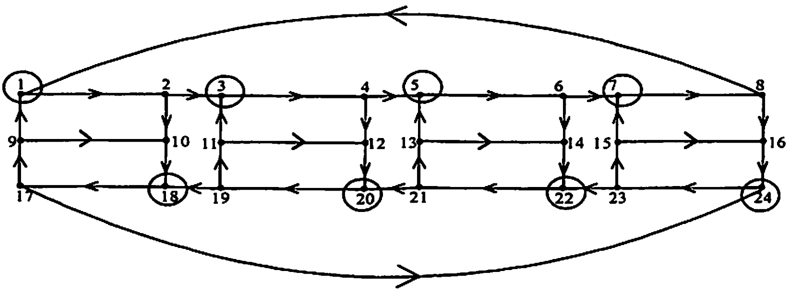

Figure 7: Encircled vertices form a kernel.

Theorem 6.2: The strong kernel problem is polynomially solvable for certain oriented networks such as Mobius Ladder, Circular Ladder, Chordal graphs and t H – graph.

7 Conclusion

In this paper we have determined the strong kernel number for oriented Mobius Ladder $M(n)$, Circular Ladder $CL(n)$, chordal graphs $CH(n)$ and t H – graph $H_n(t)$. And also proved that the strong kernel problem is polynomially solvable for $M(n)$, $CL(n)$, $CH(n)$ and $H_n(t)$. It would be interesting to characterize regular graphs for which the kernel problem is polynomially solvable.

References

[1] K. M. Alzoubi, P. J. Wan and O. Frieder, New Distributed Algorithm for Connected Dominating Set in Wireless Ad Hoc Networks, *Proc. 35th Hawaii Int. Conf. System Sciences*, (2002), 01-07.

[2] S. Armugam and S. Velammal, Maximum size of a connected graph with given domination parameters, *Ars Combin.*, 52(1992), 221-227.

[3] C. Berge and P. Duchet, Recent problems and results about kernels in directed graphs, *Discrete Mathematics*, 86(1990), 27-31.

[4] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorthims and Applications* (Springer, London, 2000).

[5] J. Bang-Jensen, Y.Guo, G. Gutin, L.Volkmann, A classification of locally semicomplete digraphs, *Discrete Math.*, 167/168(1997), 101-114.

[6] C. Banderier, J-M. Le Bars and V. Ravelomana, Generating functions for kernels of digraphs, *Proc. 16th Annual International Conference on Formal Power Series and Algebraic combinatorics*, (2004), 91-105.

[7] V. Chvàtal, On the computational complexity of finding a kernel, *Report No. CRM-300, Centre de Recherches Mathematiques*, Universite de Montreal, 1973.

[8] Claude Berge, *Graph* (North Holland Publishing Co., Amsterdam, Vol. 6, 1985).

[9] A. S. Fraenkel, Planar kernel and Grundy with $d \leq 3$, $d^{+} \leq 2$, $d^{-} \leq 2$ are *NP*-complete, *Discrete Applied Mathematics*, 3(1981), 257-262.

[10] Indra Rajasingh, Bharati Rajan, Joice Punitha, Total-Kernel in Oriented Circular Ladder and Mobius Ladder, *Ars Combinatoria*, Oct. 2011.

[11] Immanuel M. Bomze, Marco Budinich, Panos M. Pardalos and Marcello Pelillo, 1999. The maximum clique problem, *In Handbook of combinatorial optimization, Supplement,* Vol. A, Kluwer Acad. Publ., Dordrecht, .

[12] Jean-Marie Le Bars, Fragments of existential second - order logic without 0-1 laws, *In Thirteenth Anual IEEE Symposium on Logic in Computer Science.* (Indianapolis, IN, 1998), 525–536.

[13] Jean-Marie Le Bars, Counterexample of the 0-1 Law for fragments of existential second-order logic: an overview, *The Bulletin of Symbolic Logic*, 6(1)(2000), 67-82.

[14] Jie Wu, Extended Dominating-Set-Based Routing in Ad Hoc Wireless Networks with Unidirectional Links, *IEEE Transactions on Parallel and Distributed Systems*, 13(2002), 866-881.

[15] M. Kswásnik, The generalisation of Richardson theorem, *Discussiones Math*, 4(1981), 11-14.

[16] Nadia Creignou, The class of problems that are linearly equivalent to satisfiability or an uniform method for proving *NP* – completeness, *Theoretical Computer Science*, 145(1-2)(1995), 111–145.

[17] Paul Manuel, Indra Rajasingh, Bharati Rajan, Joice Punitha, Kernel in Oriented Circulant Graphs, *Combinatorial Algorithms,* LNCS, Springer Berlin / Heidelberg, 5874 (2009).

[18] M. Richardson, Solutions of irreflexive relations, *Ann. of Math.* 58(1953), 573-590.

[19] J. L. Szwarcfiter and G. Chaty, Enumerating the kernels of a directed graph with no odd circuits, *Inform. Process. Lett.,* 51(1994), 149-153.

[20] J. Van Leeuwen, Having a Grundy-numbering is NP-complete, *Report No. 207,* Computer Science Dept., Pennsylvania State University, University Park, PA, (1976).

[21] J. Von Neumann and O. Morgenstern, *Theory of Games and Economic Behaviour* (Princeton University Press, Princeton, 1944).

[22]J.Xu, *Topological Structure and Analysis of Interconnection Networks,* (Kluwer Academic Publishers, 2001).