Balance, partial balance and balanced-type spectra in graph-designs *

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Abstract

For a given graph G, the set of positive integers v for which a G-design exists is usually called the 'spectrum' for G and the determination of the spectrum is sometimes called the 'spectrum problem'.

We consider the spectrum problem for G-designs satisfying additional conditions of 'balance', in the case where G is a member of one of the following infinite families of trees: caterpillars, stars, comets, lobsters and trees of diameter at most 5. We determine the existence spectrum for balanced G-designs, degree-balanced and partially degree-balanced G-designs, orbit-balanced G-designs.

We also address the existence question for non-balanced G-designs, for G-designs which are either balanced or partially degree-balanced but not degree-balanced, for G-designs which are degree-balanced but not orbit-balanced.

Key words: graph-decomposition; G-design; replication number; balanced G-design

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1 Introduction: balance in graph designs

Throughout the paper G denotes a simple graph with at least two vertices, none of which is isolated. Our notation is $G = (V(G), E(G))$ and we set $k = |V(G)|$, $m = |E(G)|$.

The question of 'balance' in graph-designs can be approximately described as the additional assumption that some 'local' parameter is constant, see [8]. So, for instance, if we assume that the replication number $r(x)$ of each point x in a G-design is equal to a constant r, then we get the notion of a *balanced* G-design introduced in [11].

1.1 Orbit-balance and degree-balance

The following variations have been suggested.

Let V_1, V_2, ..., V_h be the vertex-orbits of G under its automorphism group. A G-design is said to be *orbit-balanced* if, for $i = 1, 2, \ldots, h$, there exists a constant R_i such that, for each point x, the number of blocks of the G-design in which x occurs as an element in the orbit V_i is equal to R_i.

A G-design is said to be *degree-balanced* if, for each degree d occurring in the graph G, there exists a constant r_d such that, for each point x, the number of blocks containing x as a vertex of degree d is equal to r_d.

The notion of an orbit-balanced G-design was formulated in [12] (under the name *strongly balanced*), while that of a degree-balanced G-design was proposed in [3]. In the same paper [3] it was observed that the definitions immediately imply that an orbit-balanced G-design is degree-balanced and that a degree-balanced G-design is balanced.

1.2 Partial degree-balance

In this subsection we slightly generalize the notion of a degree-balanced graph-design. Define $D(G)$ to be the set of all degrees of the vertices of G. We write D for $D(G)$ if G is fixed once for all. Let D' be a designated subset of D. A G-design is said to be *degree-balanced with respect to* D' if for each degree $d \in D'$ there exists a constant r_d such that, for each point x, the number of blocks containing x as a vertex of degree d is equal to r_d. In case D' coincides with the whole set D then we shall simply speak of a *degree-balanced* G-design. In case D' is a proper subset of D that we do not want to mention explicitly, then we shall occasionally speak of a *partially* degree-balanced G-design. In other words, a partially degree-balanced G-design is a G-design which is balanced with respect to some – possibly not all – of the degrees occurring in G.

A similar approach can naturally lead to an idea of partial balance with respect to orbits rather than degrees: we shall omit any attempt in this direction here.

While it is immediately clear that a degree-balanced graph-design is necessarily balanced, partially degree-balanced graph-designs which are not balanced do exist as the example in Figure 1 shows.

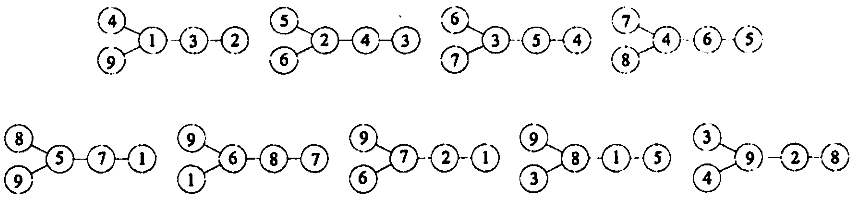

Figure 1: A partially degree-balanced graph-design.

It is not hard to imagine that partial degree-balance may well imply degree-balance in same circumstances, as we shall see in Section 5.

1.3 A social network problem

The description of round robin tournaments involving $2n$ teams in terms of one-factorizations of the complete graph K_{2n} is rather known, see [16, Ch.5]. Similar scheduling problems in which participants play some special role may well be modeled by graph-designs. Here is an example.

Tutor Dox of XY High School organizes support activities for freshmen who revealed language difficulties during the Fall Term. Mr. Dox selects seven students to form a self-study group. Each student is assigned a reading on a different subject, which may be of potential interest to the whole group. For each selected subject a discussion group is organized in which the student who prepared the reading serves as a discussion leader. He/she is expected to actually read a selection of the text that was assigned to him/her and to introduce his/her point of view on the subject in a further talk of at most ten minutes. The leader should then open and organize a discussion among the participants of the forum on the given subject. As a final assignment, the discussion leader should briefly summarize the main contents of the discussion and present a written report to Mr. Dox in a week's time. In order to make people feel easier within each discussion group, the Tutor decides that each such forum should be limited to four group members, including the discussion leader.

Improving acquaintanceship is another goal that Mr. Dox has in mind. So any two group members should sit once together in a forum, with either one as a discussion leader. On the other hand, in order to avoid work-load complaints, each group member should attend the same number of forums.

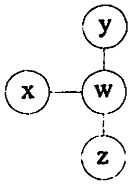

Figure 2: A discussion-forum

Each forum can be modeled as the graph of Figure 2 – which is a star S_3 in our later notation– where the vertex w of degree 3 identifies the discussion leader, while the remaining three vertices x, y, z of degree 1 are the participants with no special role. If 1, 2, 3, 4, 5, 6, 7 are the group members, then an adequate schedule for the discussion groups can be obtained from a degree-balanced S_3-decomposition of K_7, see Figure 3 (note that degree-balanced is equivalent to balanced in this case, since only two degrees occur in S_3).

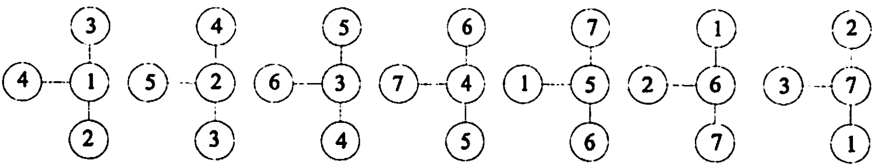

Figure 3: A degree-balanced S_3-decomposition of K_7.

Consider now the following slight modification of the previous situation. Suppose the support group consists of only six persons 1, 2, 3, 4, 5, 6. Mr. Dox has noticed that person n.6 is extremely shy and is not in the position to lead any forum, but would greatly benefit from attending as many as possible as a simple participant. Mr. Dox comes up with the schedule illustrated in Figure 4 which forms an S_3-decomposition of K_6.

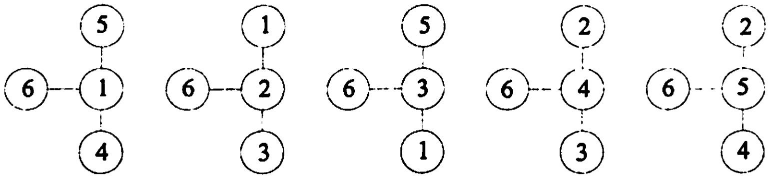

Figure 4: A S_3-decomposition of K_6.

The condition that member n.6 is not allowed to be a forum-leader implies that he/she cannot occur as a vertex of degree 3 in any block. Hence the decomposition cannot be degree-balanced and in fact the proposed scheme is not.

A further slight modification shows an instance in which partial degree-balance plays a role. Assume the support group consists of nine people. The written report to be handed in to Mr. Dox, rather than being an assignment of the discussion leader, is in charge of another participant, who is therefore the secretary of the forum. The secretary is in turn supported by another participant who records the forum on an MP3 recorder. Each forum is now modeled as the graph of Figure 5 – $T_2(4)$ in our later notation – where the vertex w of degree 3 identifies the discussion leader, the vertex s of degree 2 is the secretary, while the vertex r of degree 1, that is adjacent to the secretary, is the recording person.

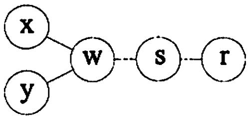

Figure 5: Another model for a discussion-forum.

Mr. Dox insists on having each one of the group members to serve once as a discussion leader, while he is flexible on the other roles. Any two group members should sit together in a forum, with either one as a discussion leader or a secretary. Mr. Dox comes up with the schedule of Figure 1, forming a graph-design which is balanced with respect to the degree 3, but not with respect to the degrees 1 and 2.

1.4 Spectra of balanced-type

The spectrum problem for a given graph G consists in the determination of the set of values v for which a G-design on v vertices exists. In the case where G is a tree, full or partial solutions of the spectrum problem were generally found assuming either that the number of vertices of G is small [14] or that G belongs to some specified infinite family, such as that of caterpillars [15].

The spectrum problem for a given graph G, in particular for a tree, can be formulated equally well with respect to G-designs satisfying some "balanced-type" condition like the ones we just illustrated. The *balanced* spectrum for a graph G can thus be defined as the set of all values v for which a balanced G-design on v vertices exists. The *orbit-balanced* spectrum is defined in the same way, and so is the *degree-balanced* spectrum or, more generally, the degree-balanced spectrum with respect to any designated subset D' of $D(G)$.

The problem of determining the balanced-type spectra for trees with at most six vertices has already been addressed in [2], [4], [5]. In subsequent sections we assume G to be a member of one of the following infinite families of trees: caterpillars, stars, comets, lobsters and trees of diameter at most 5. For each such choice of G, under additional assumptions involving, for instance, the number of vertices of the graph, we determine the balanced spectrum, the degree-balanced spectrum and the orbit-balanced spectrum. Partially degree-balanced spectra are also determined in some cases.

As we shall see in Section 4, it may well happen that for some choice of G the class of degree-balanced G-designs is strictly larger than the class of orbit-balanced G-designs even though the corresponding spectra coincide. In practice, such phenomena occur when it is possible to construct G-designs satisfying the weaker balanced-type condition but not the stronger one.

2 Labelings and spectra

We begin this section with two simple necessary conditions for the existence of either a degree-balanced G-design or a balanced G-design, respectively.

In our notation the number of blocks in a G-design on v vertices is denoted by b and we have $b = v(v-1)/(2m)$, with $m = |E(G)|$. We also recall that we denote by k the cardinality of the vertex-set $V(G)$.

Proposition 2.1. *Let G be a graph with a unique vertex of degree d. If there exists a degree-balanced G-design on v vertices, then $v \equiv 1 \pmod{2m}$.*

Proof. Given a degree-balanced G-design on v vertices, define Λ to be the set of all point-block pairs (x, B) such that x occurs in B as a vertex of degree d. Since G has precisely one vertex of degree d, we have $|\Lambda| = b \cdot 1$. On the other hand, for each vertex x of K_v we have $r_d(x) = r_d$, yielding $|\Lambda| = v \cdot r_d$. We obtain $v(v-1)/(2m) = v \cdot r_d$, whence the assertion. □

Proposition 2.2. *Assume k is odd and $|m - k| = 1$. If there exists a balanced G-design on v vertices, then $v \equiv 1 \pmod{2m}$.*

Proof. In a balanced G-design on v vertices the replication number $r = k \cdot (v-1)/2m$ is an integer. Since $\gcd(k, m) = \gcd(k, 2) = 1$, the statement follows. □

Note that the condition $|m-k| = 1$ holds for all trees and all cycles with a chord (the latter ones are generally known as the Theta graphs $\Theta(1, b, c)$ [1]).

In the remainder of this section we shall encounter various types of graph labelings. Generally speaking, a labeling of a graph G is an assignment of integers to the vertices of G subject to certain conditions. In particular, we refer to [6] for the definition of an α-labeling and of a ρ^+-labeling, and to [9] for the definition of a near α-labeling.

We recall that a G-design is said to be cyclic if it admits a cyclic automorphism group acting transitively on vertices, see for instance [6, Def. 24.4].

Proposition 2.3. *Assume k is odd and $|m - k| = 1$. If G has an α-labeling, then the set $\{v : v \equiv 1 \pmod{2m}, v > 1\}$ is simultaneously the balanced, the degree-balanced and the orbit-balanced spectrum for G. The same conclusion holds if G is assumed to be a bipartite graph with either a ρ^+-labeling or a near α-labeling.*

Proof. If G has an α-labeling then Theorem 8 in [15] shows that a cyclic G-design on v vertexes exists for all values of v under consideration. The same conclusion is obtained from either [10, Thm.5] or [9, Thm.5] for bipartite graph with either ρ^+-labeling or near-α-labeling. Each cyclic G-design is orbit-balanced [3, Prop.3]. Each orbit-balanced G-design is degree-balanced and each degree-balanced G-design is balanced. Hence the assertion follows from Proposition 2.2. □

The following result does not depend on the parity of k.

Proposition 2.4. *Let G be a graph with a unique vertex of degree d. If G has an α-labeling, then the set $\{v : v \equiv 1 \pmod{2m}, v > 1\}$ is the degree-balanced spectrum for G. The same conclusion holds if G is assumed to be a bipartite graph with either a ρ^+-labeling or a near α-labeling.*

Proof. Each cyclic G-design is degree-balanced. Again one of [15, Thm.8], [10, Thm.5] and [9, Thm.5] together with Proposition 2.1 imply the assertion. □

3 Balanced-type spectra for certain trees

In this section we apply the above results on graph labelings to the determination of the the balanced-type spectra for a tree G in case G is a member of one of the following infinite families of graphs: certain caterpillars in particular stars, comets, lobsters, trees of diameter at most 5.

If G is a tree with at least three vertices, then the graph obtained from G by removing all of its end-vertices (which are the vertices of degree 1) is still a tree and is called the *base* of G. A *path* is a tree with exactly two end-vertices or the trivial tree with a unique vertex. A tree is said to be a *caterpillar* if its base is a path. A *m-star*, $m \geq 2$, denoted by S_m, is the complete bipartite graph $K_{1,m}$. A *comet* $S_{t,s}$, with $t \geq 3$ and $s \geq 2$, is a tree obtained from a star of t edges by replacing each edge with a path of length s. A *lobster* G is a tree whose base is a caterpillar.

Proposition 3.1. *Assume k is odd and G is a caterpillar. Then the set $\{v : v \equiv 1 \pmod{2(k-1)}, v > 1\}$ is simultaneously the balanced, the degree-balanced and the orbit-balanced spectrum for G. The same conclusion holds if G is a lobster or a tree of diameter at most 5 or a comet.*

Proof. If G is a caterpillar, then it has a α-labeling, see [15, Thm.2]. If G is a comet or a lobster or a tree of diameter at most 5, then it has a ρ^+-labeling, see [10, Thm.6]. Therefore, the statement follows from Proposition 2.3. For a lobster G the assertion can alternatively be obtained from [14, Lemma 2.7] and Proposition 2.3. □

Proposition 3.2. *Let G be a graph with a unique vertex of degree d. If G is a caterpillar , then the degree-balanced spectrum for G is the set $\{v : v \equiv 1 \pmod{2m}, v > 1\}$. The same conclusion holds if G is a comet or a lobster or a tree of diameter at most 5.*

Proof. If G is a caterpillar the assertion follows from [15, Thm.2] and Proposition 2.4. If G is a comet or a lobster or a tree of diameter at most 5 the statement is obtained from [10, Thm.6] and Proposition 2.4. □

Since a star S_m is a caterpillar admitting a unique vertex of degree m we have the following.

Corollary 3.3. *The set* $\{v : v \equiv 1 \pmod{2m}, v > 1\}$ *is the degree-balanced spectrum for the star* S_m.

Proposition 3.4. *The following statements are equivalent.*

(1) *an* S_m*-design is balanced;*

(2) *an* S_m*-design is degree-balanced;*

(3) *an* S_m*-design is orbit-balanced.*

Proof. In the graph S_m only two distinct degrees occur and the number of vertex-orbits is also two. It follows thus from [3, Prop.1] that a balanced S_m-design is also orbit-balanced, hence degree-balanced as well. □

Corollary 3.5. *The set* $\{v : v \equiv 1 \pmod{2m}, v > 1\}$ *is simultaneously the balanced, the degree-balanced and the orbit-balanced spectrum for the star* S_m.

Note that the balanced spectrum for S_m was already known from [13, Thm.3.5]

Given a value v in the balanced spectrum of S_m, does there exist a non balanced S_m-design on v vertices? The next statement answers this question affirmatively.

Proposition 3.6. *For every* $v \equiv 1 \pmod{2m}$, $v > 1$, *there exists*

(1) *a balanced* S_m*-design on* v *vertices;*

(2) *a non-balanced* S_m*-design on* v *vertices.*

Proof. (1) The first statement follows from Corollary 3.5.
(2) For $h \geq 1$ an S_m-design on $2mh+1$ vertices is constructed in Lemma 5 of [7]. We recall the construction here, so as to emphasize the fact that this S_m-design is not degree-balanced. The complete graph K_{2mh+1} is the join graph $K_{2mh} + K_1$ of the complete graph on $2mh$ vertices and the trivial tree K_1. There exists an S_m-design $\mathcal{D}$ on $2mh$ vertices, see [7, Lemma 2] and [7, Thm.1]. The relation $2mh \equiv 0 \pmod{2m}$ implies that $\mathcal{D}$ is not balanced, and so it is not degree-balanced either. In other words, if $r_d(x)$ denotes the number of blocks containing x as a vertex of degree d, with $d \in D(S_m) = \{1, m\}$, there exist two vertices x_1, x_2 in $\mathcal{D}$ such that either $r_1(x_1) = a_1$ and $r_1(x_2) = a_2$ with $a_1 \neq a_2$, or $r_m(x_1) = b_1$ and $r_m(x_2) = b_2$ with $b_1 \neq b_2$. The graph $(K_{2mh} + K_1) \setminus K_{2mh}$ is the star $K_{1,2mh}$ which admits a S_m-decomposition, say $\mathcal{D}'$, with $2h$ blocks. The S_m-design $\mathcal{D} \cup \mathcal{D}'$ has $2mh+1$ vertices, among which x_1, x_2 are such that

either $r_1(x_1) = a_1 + 1 \neq a_2 + 1 = r_1(x_2)$, or $r_m(x_1) = b_1 \neq b_2 = r_m(x_2)$. We conclude that the constructed S_m-design is not degree-balanced and so it is not balanced either. □

We conclude this section with some notes on comet-designs. Let us consider the comet $S_{t,2}$, $t \geq 3$ (see Figure 6 where $m = 2t$). The balanced-type spectra for $S_{t,2}$ are determined in Proposition 3.1 and they all coincide with the set $\{v : v \equiv 1 \pmod{4t}, v > 1\}$.

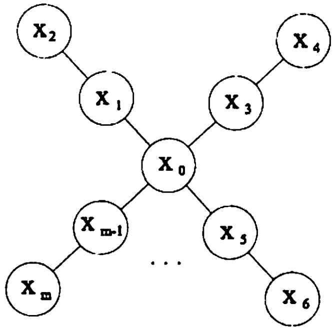

Figure 6: The tree $[x_0; x_1, x_2; x_3, x_4; \ldots; x_{m-1}, x_m]$

Corollary 3.7. *The class of degree-balanced $S_{t,2}$-designs, $t \geq 3$, coincides with the class of orbit-balanced $S_{t,2}$-designs.*

Proof. Assume $v \equiv 1 \pmod{4t}$, $v > 1$. Since exactly three degrees and three vertex-orbits occur in the comet $S_{t,2}$, then each $S_{t,2}$-design on v vertices is degree-balanced if and only if it is orbit-balanced. □

Denote the comet $S_{t,2}$ of Figure 6 by $[x_0; x_1, x_2; x_3, x_4; \ldots; x_{m-1}, x_m]$, with $m = 2t$. A degree-balanced $S_{t,2}$-design on v vertices, $v = 4th + 1$, can be obtained from [14, Lemma 2.7]: the vertex-set is $\mathbb{Z}_{4th+1}$ and the block-set is

$$\{B^{(p)} + i : i = 0, 1, \ldots, 4th,\ p = 1, 2, \ldots, h\},$$

where $B^{(p)}$ is the base block

$$[(t+2)p - 1; 0, (t+2)p; 1, 2 + (t+2)h + (p-1)(t-2); \ldots$$
$$\ldots; j, 2j + (t+2)h + (p-1)(t-2); \ldots$$

$$\ldots; t-2, 2(t-2)+(t+2)h+(p-1)(t-2); t-1, (t+2)p-2].$$

Furthermore, a degree-balanced $S_{t,2}$-design on v vertices with $v \equiv 1 \pmod{4t}$ can also be obtained from a near α-labeling of $S_{t,2}$ and the orbit-balanced $S_{t,2}$-design arising from it, according to [9, Theorems 5, 8].

4 Some caterpillar-designs

In this section an infinite family of caterpillars is studied. For the remainder of this section, G will denote the tree in Figure 7 which is also described by the short notation $[{}^{x_1}_{x_2}\, x_3, x_4, \ldots, x_m, x_{m+1}]$. It is a caterpillar with m edges, $m > 4$, which is denoted by $T_2(m)$ in [14], where the complete spectrum is determined for $m < 10$. The balanced-type spectra for $T_2(5)$ are studied in [4]. The tree G possesses a unique vertex of degree 3, consequently the degree-balanced spectrum for G is determined by Proposition 3.2. If m is even, then the three balanced-type spectra coincide, see Proposition 3.1.

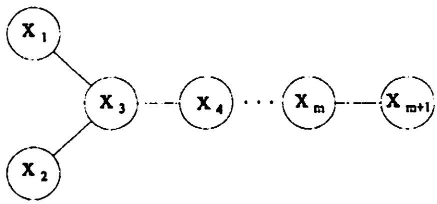

Figure 7: The tree $[{}^{x_1}_{x_2}\, x_3, x_4, \ldots, x_m, x_{m+1}]$

The next construction shows that for all m the orbit-balanced spectrum for G coincides with its degree-balanced spectrum.

Proposition 4.1. *For every* $v \equiv 1 \pmod{2m}$, $v > 1$, *there exists an orbit-balanced* G*-design on* v *vertices.*

Proof. A cyclic G-design, which is thus orbit-balanced by [3, Prop.3], can be obtained from Theorems 2 and 8 of [15] with vertex-set $\mathbb{Z}_{2mh+1}$ and with block-set

$$\mathcal{B}_h^m = \{G^{(p)} + i : i = 0, 1, \ldots, 2mh, p = 1, 2, \ldots, h\},$$

where $G^{(p)}$ is the base-block

$$[{}^{mp}_{(m-1)+m(p-1)}\, 0, (m-2)+m(p-1), 1, (m-3)+m(p-1), 2, \ldots$$

$$\dots, m'-2, (m'+1)+m(p-1), m'-1, m'+m(p-1)]$$

in case $m = 2m'+1$ (odd), while $G^{(p)}$ is the base-block

$$[{}^{mp}_{(m-1)+m(p-1)}\, 0, (m-2)+m(p-1), 1, (m-3)+m(p-1), \dots$$

$$\dots, m'-3, (m'+1)+m(p-1), m'-2, m'+m(p-1), m'-1]$$

in case $m = 2m'$ (even). □

The next three constructions follow basically the same idea. The G-design of the previous proposition is orbit-balanced because it has enough symmetry, namely the symmetry provided by an automorphism group acting vertex-transitively. We destroy this symmetry to some extent, so as to come just one step down in the hierarchy of balance.

In the notation of Figure 7, we denote by X_{m+1} the orbit of the vertex x_{m+1} under $\mathrm{Aut}(G)$.

Proposition 4.2. *For every $v \equiv 1 \pmod{2m}$, $v > 1$, there exists a degree-balanced G-design on v vertices which is not orbit-balanced.*

Proof. Let us consider the orbit-balanced G-design with vertex-set $\mathbb{Z}_{2mh+1}$ and block-set $\mathcal{B}_h^m$ occurring in the proof of Proposition 4.1.

The idea of the first two cases below is to find three vertices a, b, c and two G-blocks

$$[{}^{z_1}_{z_2}\, z_3, z_4, \dots, z_m, z_{m+1}], \quad [{}^{y_1}_{y_2}\, y_3, y_4, \dots, y_m, y_{m+1}]$$

where $a = z_1$ does not occur in the second block, while $b = z_3 = y_m$ and $c = y_{m+1}$ do not occur in the first block.

We exchange the edges $[a, b]$ and $[c, b]$. The vertices involved in this exchange of edges maintain their degree in the new blocks, while, for instance, the vertex a occurs in a different orbit in the new block.

In the third case below the idea is similar: we find four vertices a, b, c, d and four G-blocks

$$[{}^{a}_{q_2}\, b, q_4, \dots, q_m, q_{m+1}], \quad [{}^{d}_{y_2}\, c, y_4, \dots, y_m, y_{m+1}],$$

$$[{}^{b}_{z_2}\, d, z_4, \dots, z_m, z_{m+1}], \quad [{}^{w_1}_{w_2}\, w_3, w_4, \dots, w_{m-1}, b, c]$$

where d does not occur in the first block, b does not occur in the second block, c does not occur in the third block and a does not occur in the fourth block. We substitute the above four blocks with the following once:

$$[{}^{d}_{q_2}\, b, q_4, \dots, q_m, q_{m+1}], \quad [{}^{b}_{y_2}\, c, y_4, \dots, y_m, y_{m+1}],$$

$$[{}^{c}_{z_2}\, d, z_4, \dots, z_m, z_{m+1}], \quad [{}^{w_1}_{w_2}\, w_3, w_4, \dots, w_{m-1}, b, a]$$

Also in this case the involved vertices maintain their degree in the new blocks, while $r_{X_{m+1}}(a) = r_{X_{m+1}}(q_{m+1}) + 1$.

<u>Case $m = 2m'$</u>. The G-design with vertex-set $\mathbb{Z}_{2mh+1}$ and block-set

$$(\mathcal{B}_h^m \setminus \{G^{(1)}, G^{(1)}+m'\}) \cup$$

$$\{[^{m}_{(m-1)}\, 0, m-2, 1, m-3, 2, m-4, 3, \dots$$
$$\dots, m'-3, m'+1, m'-2, m', m+m'],$$

$$[^{m'-1}_{m+m'-1}\, m', m+m'-2, m'+1, m+m'-3, m'+2, \dots,$$
$$\dots, m+2, m-3, m+1, m-2, m, m-1]\}$$

is such that $r_{X_{m+1}}(m+m') = h+1$ while $r_{X_{m+1}}(m'-1) = h-1$; hence it is not orbit-balanced but it remains degree-balanced.

<u>Case $m = 2m'+1$ and $h \geq 2$</u>. The G-design with vertex-set $\mathbb{Z}_{2mh+1}$ and block-set

$$(\mathcal{B}_h^m \setminus \{G^{(1)}+(m'-1), G^{(2)}\}) \cup$$

$$\{[^{m+m'}_{m+m'-2}\, m'-1, m+m'-3, m', m+m'-4, m'+1, \dots$$
$$\cdots, 2m'+1, 2m'-3, 2m', 2m'-2, 2m'-1],$$

$$[^{2m}_{2m-1}\, 0, 2m-2, 1, 2m-3, \dots, m'-2, m'+m+1, m'-1, m+m'-1]\}$$

has $r_{X_{m+1}}(m+m'-1) = h+1$ while $r_{X_{m+1}}(m'+m) = h-1$; hence, it is not orbit-balanced but it is degree-balanced.

<u>Case $m = 2m'+1$ and $h = 1$</u>. The G-design with vertex-set $\mathbb{Z}_{2m+1}$ and block-set

$$(\mathcal{B}_h^m \setminus \{G^{(1)}+1, G^{(1)}+2, G^{(1)}+(m+2), G^{(1)}+(2m-m'+3)\}) \cup$$

$$\{[^{m+2}_{m}\, 1, m-1, 2, m-2, 3, \dots, m'-2, m'+3, m'-1, m'+2, m', m'+1],$$

$$[^{1}_{m+1}\, 2, m, 3, m-1, 4, \dots, m'-1, m'+4, m', m'+3, m'+1, m'+2],$$

$$[^{2}_{0}\, m+2, 2m, m+3, 2m-1, m+4, \dots$$
$$\dots, m+m'-1, m+m'+4, m+m', m+m'+3, m+m'+1, m+m'+2],$$

$$[^{m-m'+2}_{m-m'+1}\, 2m-m'+3, m-m', 2m-m'+4, m-m'-1, 2m-m'+5, \dots$$
$$\dots, 2m, 4, 0, 3, 1, m+1]\}$$

has $r_{X_{m+1}}(m+1) = 2$ while $r_{X_{m+1}}(m'+1) = 1$, hence the design is not orbit-balanced but it is degree-balanced. □

Corollary 4.3. *For $m \geq 5$ the set $\{v : v \equiv 1 \pmod{2m}, v > 1\}$ is the degree-balanced spectrum as well as the orbit-balanced spectrum for the graph G. However, the class of degree-balanced G-designs is strictly larger than the class of orbit-balanced G-designs.*

Proof. The statement follows from Propositions 4.1, 2.1 and 4.2. □

Proposition 4.4. *Assume m is odd. For every $v \equiv 1 \pmod{2m}$, $v > 1$, there exists a balanced G-design on v vertices which is not degree-balanced.*

Proof. In this case we still start from a cyclic G-design and the idea is to find six vertices a, b, c, d, e, f, and two G-blocks

$$\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix} z_3, z_4, \ldots, z_m, z_{m+1}\right], \quad \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix} y_3, y_4, \ldots, y_m, y_{m+1}\right]$$

with $a = z_1 = y_{m-1}$, $b = z_2 = y_{m+1}$, $c = z_4 = y_m$, $d = z_3$, $e = z_{m+1} = y_3$, $f = y_1$.

We break the symmetry of the G-design substituting the above two G-blocks with the following G-blocks:

$$\left[\begin{smallmatrix} b \\ a \end{smallmatrix} c, z_5, z_6, \ldots, z_m, e, f\right], \quad \left[\begin{smallmatrix} c \\ b \end{smallmatrix} d, a, y_{m-2}, y_{m-3}, \ldots, y_4, e, y_2\right].$$

The replication number of each vertex does not change. On the other hand, the vertex e appears one time less than the vertex d as a vertex of degree 3 in the blocks of the new design.

Let us consider the orbit-balanced G-design with vertex-set $\mathbb{Z}_{2mh+1}$ and block-set $\mathcal{B}_h^m$ occurring in the proof of Proposition 4.1. Assume $m = 2m'+1$. The G-design with vertex-set $\mathbb{Z}_{2mh+1}$ and block-set

$$(\mathcal{B}_h^m \setminus \{G^{(1)}, G^{(1)}+(2mh+1-m'),\})\cup$$

$$\{\left[\begin{smallmatrix} m'+1 \\ m' \end{smallmatrix} m'-1, 2mh+2-m', m'-2, \ldots, 2mh-2, 2, 2mh-1, 1, 2mh, 0, m\right]$$

$$\left[\begin{smallmatrix} m' \\ m'-1 \end{smallmatrix} 2hm+1-m', m'+1, m'-2, m'+2, m'-3, \ldots, m-3, 1, m-2, 0, m-1\right]\}.$$

is not degree-balanced but it is balanced. In fact the vertex 0 appears one time less than the vertex $2hm + 1 - m'$ as a vertex of degree 3. □

Proposition 4.5. *Assume m is even. For every $v \equiv 1 \pmod{2m}$, $v > 1$, there exists a non-balanced G-design on v vertices.*

Proof. Again we start from the cyclic G-design with vertex-set $\mathbb{Z}_{2mh+1}$ and block-set $\mathcal{B}_h^m$ occurring in the proof of Proposition 4.1. We find five vertices a, b, c, d, e, and two G-blocks

$$\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix} z_3, z_4, \ldots, z_m, z_{m+1}\right], \quad \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix} y_3, y_4, \ldots, y_m, y_{m+1}\right]$$

with $a = z_3 = y_m$, $b = z_2$, $c = z_m = y_1$, $d = z_{m+1}$, $e = y_3$ and e does not occur in the first G-block.

We break the symmetry of the G-design substituting the above two G-blocks with the following G-blocks:

$$\left[{}^{d}_{e}\, c, z_{m-1}, z_{m-2}, \ldots, z_4, a, z_1\right], \quad \left[{}^{y_{m+1}}_{b}\, a, y_{m-1}, y_{m-2}, \ldots, y_4, e, y_2\right].$$

The replication number of e increases by 1 while the replication number of a remains unaltered.

Assume $m = 2m'$. The G-design with vertex-set $\mathbb{Z}_{2mh+1}$ and block-set

$$(\mathcal{B}_h^m \setminus \{G^{(1)}, G^{(1)}+(2mh+1-m')\}) \cup$$

$$\{\left[{}^{m'-1}_{2mh+1-m'}\, m', m'-2, m'+1, m'-3, \ldots, m-4, 2, m-3, 1, m-2, 0, m\right]$$

$$\left[{}^{2mh}_{m-1}\, 0, 2hm-1, 1, 2mh-2, 2, \ldots\right.$$

$$\left.\ldots, m'-3, 2mh+2-m', m'-2, 2mh+1-m', m'-1\right]\},$$

is not balanced since $r(2mh+1-m') = (m+1)h+1$, $r(0) = (m+1)h$. □

Remark 4.6. If m is even, $m > 4$, then $G_1 = \left[{}^{x_1}_{x_2}\, x_3, x_4, \ldots, x_m, x_{m+1}\right]$ and $G_2 = S_m$ are non-isomorphic trees with equally many vertices. Despite the fact that for each one of G_1, G_2 the three spectra of balanced-type coincide (and are actually the same for the two graphs) the situation for the corresponding designs is quite different. As a matter of fact, the three classes of G_2-designs of balanced-type coincide, while the class of orbit-balanced G_1-designs is strictly contained in the class of degree-balanced G_1-designs.

5 Some partially degree-balanced spectra

In this section some properties related to partially degree-balanced G-designs are outlined and finally some partially degree-balanced spectra are also determined for the caterpillar $T_2(m)$ (see Section 4).

Proposition 5.1. *Assume $|D(G)| = j$ with $j \geq 3$. If a balanced G-design is degree-balanced with respect to a subset D' of $D(G)$ of cardinality $j-2$, then it is degree-balanced.*

Proof. Set $D(G) = \{d_1, d_2, \ldots, d_j\}$ and $D' = \{d_1, d_2, \ldots, d_{j-2}\}$. For each point x we have

$$\begin{cases} r_{d_1}(x)\, d_1 + \ldots + r_{d_{j-2}}(x)\, d_{j-2} + r_{d_{j-1}}(x)\, d_{j-1} + r_{d_j}(x)\, d_j = v-1 \\ r_{d_1}(x) + \ldots + r_{d_{j-2}}(x) + r_{d_{j-1}}(x) + r_{d_j}(x) = r \end{cases}$$

Since there exist integers c_1, c_2 satisfying the relations

$$\sum_{i=1}^{j-2} r_{d_i}(x) = c_1, \quad \sum_{i=1}^{j-2} r_{d_i}(x)\, d_i = c_2,$$

the linear system is equivalent to

$$\begin{cases} r_{d_{j-1}}(x)\, d_{j-1} + r_{d_j}(x)\, d_j &= \overline{v} \\ r_{d_{j-1}}(x) + r_{d_j}(x) &= \overline{r} \end{cases}$$

where $\overline{v} = v - 1 - c_2$ and $\overline{r} = r - c_1$. The above linear system has a unique solution for $\left(r_{d_{j-1}}(x), r_{d_j}(x)\right)$ given by

$$r_{d_{j-1}}(x) = \frac{\overline{r} d_j - \overline{v}}{d_j - d_{j-1}}, \qquad r_{d_j}(x) = \frac{\overline{v} - \overline{r} d_{j-1}}{d_j - d_{j-1}},$$

showing that both $r_{d_{j-1}}(x)$ and $r_{d_j}(x)$ are also constant. □

We shall call a graph G a *j-degree graph* if $|D(G)| = j$. We shall call G a *j-orbit graph* if it has exactly j vertex-orbits under its automorphism group. It is an immediate consequence of these definitions that if G is a j-degree graph which is also a j-orbit graph then a G-design is degree-balanced if and only if it is orbit-balanced. Hence the following properties hold.

Proposition 5.2. *Let G be a j-degree graph which is also a j-orbit graph, $j > 2$. Then each balanced G-design which is degree-balanced with respect to a subset D' of $D(G)$ of cardinality $j-2$ is also orbit-balanced.*

Proposition 5.3. *Let G be a j-degree graph which is also a j-orbit graph, $j > 2$. Each G-design which is degree-balanced with respect to a subset D' of $D(G)$ of cardinality $j-1$ is also orbit-balanced.*

In analogy with Propositions 2.1 and 2.4 we get the following:

Proposition 5.4. *Assume G is a graph with a unique vertex of degree d. If there exists a G-design on v vertices which is degree-balanced with respect to $\{d\}$, then $v \equiv 1 \pmod{2m}$.*

Proposition 5.5. *Let G be a graph with a unique vertex of degree d. If G has an α-labeling, then the set $\{v : v \equiv 1 \pmod{2m}, v > 1\}$ is the degree-balanced spectrum with respect to $\{d\}$ for G. The same conclusion holds if G is assumed to be a bipartite graph with either a ρ^+-labeling or a near α-labeling.*

Corollary 5.6. *The set $\{v : v \equiv 1 \pmod{2m}, v > 1\}$ is the degree-balanced spectrum with respect to $\{3\}$ for $T_2(m)$.*

Proposition 5.7. *Let G be the caterpillar $T_2(5)$. For each $v \equiv 1 \pmod{10}$ there exists a G-design on v vertices, which is degree-balanced with respect to $\{3\}$ but is not degree-balanced.*

Proof. Again the idea of the proof is to start from a cyclic G-design with vertex-set $\mathbb{Z}_{10h+1}$. Firstly, we find six vertices a, b, c, d, e, f and four G-blocks

$$[{}^{z_1}_{z_2}\, a, b, c, z_6], [{}^{w_1}_{w_2}\, w_3, w_4, d, b], [{}^{y_1}_{y_2}\, y_3, a, e, f], [{}^{b}_{d}\, e, c, f, a]$$

where the vertex f does not appear in the first block, the vertex e does not appear in the second block and the vertex b does not appear in the third block.

Secondly, we break the symmetry of the G-design substituting the above four G-blocks with the following G-blocks:

$$[{}^{z_1}_{z_2}\, a, f, c, z_6], [{}^{w_1}_{w_2}\, w_3, w_4, d, e], [{}^{y_1}_{y_2}\, y_3, a, b, e], [{}^{f}_{a}\, e, c, b, d].$$

In this new design the number of blocks containing a point as a vertex of degree 3 is unaltered, while the number of blocks containing the point e as a vertex of degree 1 increases.

Explicitly, let us consider the cyclic G-design with vertex-set $\mathbb{Z}_{2mh+1}$ and block-set $\mathcal{B}_h^m$ with $m = 5$ occurring in the proof of Proposition 4.1. The G-design with vertex-set $\mathbb{Z}_{10h+1}$ and block-set

$$(\mathcal{B}_h^5 \setminus \{G^{(1)}, G^{(1)}+1, G^{(1)}+(10h-2), G^{(1)}+(10h-1)\}) \cup \{[{}^{5}_{4}\, 0, 10h, 1, 2],$$
$$[{}^{6}_{5}\, 1, 4, 2, 10h-1], [{}^{2}_{1}\, 10h-2, 0, 3, 10h-1], [{}^{10h}_{0}\, 10h-1, 1, 3, 2]\}$$

is such that each point occurs exactly h times as a vertex of degree 3. It is not degree-balanced: the point 1 appears $3h$ times as a vertex of degree 1 while the point $10h-1$ appears $3h+2$ times as a vertex of degree 1. Hence it is partially degree-balanced. Note that this design is not balanced since the replication number of the point $10h-1$ is $6h+1$, while the replication number of the point $10h-2$ is $6h$. □

Proposition 5.8. *Let G be the caterpillar $T_2(7)$. For each $v \equiv 1 \pmod{14}$ there exists a G-design on v vertices which is degree-balanced with respect to $\{3\}$ but is not degree-balanced.*

Proof. Again, we start from a cyclic G-design with vertex-set $\mathbb{Z}_{14h+1}$. Firstly, we find sixteen points $a, b, c, d, e, f, g, , i, l, o, p, q, s, t, u, z$ and seven G-blocks

$$[{}^{a}_{c}\, b, d, e, f, g, z], [{}^{i}_{a}\, e, c, g, d, z, f], [{}^{s}_{i}\, g, a, z, c, f, d], [{}^{z}_{g}\, p, e, q, b, o, l],$$

$$[{}^{e}_{b}\, t, l, u, o, p, q], [{}^{c}_{d}\, l, f, b, z, e, g], [{}^{f}_{z}\, q, g, o, e, l, b].$$

Secondly, we break the symmetry of the G-design substituting the above G-blocks with the following G-blocks:

$$[{}^{q}_{c}\, b, o, l, d, e, i], [{}^{l}_{q}\, e, c, g, d, z, f], [{}^{s}_{i}\, g, a, z, c, f, e], [{}^{e}_{q}\, p, z, g, f, d, b]$$

$$[{}^{e}_{b}\, t, l, u, o, p, g], [{}^{c}_{f}\, l, b, z, e, g, o], [{}^{g}_{z}\, q, f, b, a, e, o].$$

We obtain a design in which the number of blocks passing through each point as a vertex of maximum degree is the same for all points, while the number of blocks passing through the point z as a vertex of degree 1 decreases by two with respect to the original design. On the other hand, the number of blocks passing through the point b as a vertex of degree 1 in this new design coincides with the same parameter in the original design.

Now we describe the construction in detail. We consider the cyclic G-design with point-set $\mathbb{Z}_{14h+1}$ and block-set $\mathcal{B}_h^m$ with $m = 7$ occurring in the proof of Proposition 4.1. The G-design with vertex-set $\mathbb{Z}_{14h+1}$ and block-set

$$(\mathcal{B}_h^7 \smallsetminus \{G^{(1)}, G^{(1)}+1, G^{(1)}+2, G^{(1)}+(14h-3), G^{(1)}+(14h-5), G^{(1)}+14h,$$

$$G^{(1)}+(14h-2)\}) \cup \{[{}^{14h-2}_{6}\, 0, 14h-1, 14h, 5, 1, 8], [{}^{14h}_{14h-2}\, 1, 6, 2, 5, 3, 4],$$

$$[{}^{9}_{8}\, 2, 7, 3, 6, 4, 1], [{}^{1}_{14h-2}\, 14h-3, 3, 2, 4, 5, 0],$$

$$[{}^{1}_{0}\, 14h-5, 14h, 14h-4, 14h-1, 14h-3, 2],$$

$$[{}^{6}_{4}\, 14h, 0, 3, 1, 2, 14h-1], [{}^{2}_{3}\, 14h-2, 4, 0, 7, 1, 14h-1]\}$$

is such that each point appears exactly h times as a vertex of degree 3. It is not degree-balanced: the point 3 appears $3h-2$ times as a vertex of degree 1, while the point 0 appears $3h$ times as a vertex of degree 1. Hence it is partially degree-balanced. □

From the above two propositions we get the following:

Corollary 5.9. *Let G be $T_2(m)$ with $m = 5, 7$. The set $\{v : v \equiv 1 \pmod{2m}, v > 1\}$ is simultaneously the degree-balanced spectrum and the degree-balanced spectrum with respect to $\{3\}$ for the graph G. However, the class of G-designs which are degree-balanced with respect to $\{3\}$ is strictly larger than the class of degree-balanced G-designs.*

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