On cycle frames with cycles of length 8

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Abstract

Let $M(b,n)$ be the complete multipartite graph with b parts $B_0, \ldots, B_{b-1}$ of size n. A z-cycle system of $M(b,n)$ is said to be a *cycle-frame* if the z-cycles can be partitioned into sets $S_1, \ldots, S_k$ such that for $1 \leq j \leq k$, S_j induces a 2-factor of $M(b,n) \setminus B_i$ for some $i \in \mathbb{Z}_b$. The existence of a C_z-frame of $M(b,n)$ has been settled when $z \in \{3,4,5,6\}$. Here, we completely settle the case of C_z-frames when z is 8, and we give some solutions for larger values of z.

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1 Introduction

Let $M(b,n)$ be the complete simple multipartite graph with b parts $B_0, \ldots, B_{b-1}$ of size n. The vertex set, $V(M(b,n))$, is always chosen to be $\mathbb{Z}_b \times \mathbb{Z}_n$, with parts $\{j\} \times \mathbb{Z}_n$ for each $j \in \mathbb{Z}_b$. The edge set, $E(M(b,n))$, is $\{\{(i,s),(j,t)\} \mid i,j \in \mathbb{Z}_b, i < j, \text{ and } s,t \in \mathbb{Z}_n\}$. Let C_z denote a cycle of length z.

An H-decomposition of a graph G is a partition of $E(G)$, each element of which induces a copy of H. A *z-cycle system* of a graph G is a set of z-cycles that partition the edges of G. A z-cycle system is a C_z-decompostion of G. There has been considerable interest in 4-cycle systems of bipartite and multipartite graphs. Sotteau has shown in [11] that a complete bipartite graph can be decomposed into cycles of even length under certain conditions. This result has been extended to multipartite graphs in [3] by Billington and Cavenagh. Billington and Hoffman produced a *gregarious* 4-cycle-system of multipartite graphs in [4] (a gregarious 4-cycle has each vertex in a different part).

A *2-factor* of a graph G is a spanning *2-regular* subgraph of G. A *2-factorization* of G is a set of edge-disjoint 2-factors, the edges of which partition $E(G)$. A C_z-factorization is a 2-factorization such that each component of each 2-factor is a cycle of length z; each 2-factor of a C_z-factorization is known as a C_z-factor. C_z-factorizations are also known as *resolvable C_z-decompositions.*

A *frame* of the multipartite graph $M(b,n)$ is a collection of sets of edges, $S_1, \ldots, S_k$, that partition $E(M(b,n))$ such that for $1 \le j \le k$, S_j induces a 2-factor of $M(b,n) \setminus B_i$ for some $i \in \mathbb{Z}_b$. A C_z-frame is a frame such that each component of each 2-factor is a z-cycle. The existence of a C_z-frame of $M(b,n)$ been settled when $z \in \{3,4,5,6\}$ [6, 12, 13]. In this paper, we completely settle the case when z is 8, we give some solutions for larger values of z.

2 Preliminary Results

We begin by finding some necessary conditions for the existence of a C_z-frame of $M(b,n)$.

Lemma 2.1 *If there exists a C_z-frame of $M(b,n)$, then*

1. $b \neq 2$,
2. $|E(M(b,n))| \equiv 0 \ (mod\ z)$,
3. $(b-1)n \equiv 0 \ (mod\ z)$, *and*

4. at least one of b and n is even.

Proof If $b = 0$ or $b = 1$, then there are no edges to partition in $M(b,n)$. There are no edges joining vertices in the same part in $M(b,n)$. So in order to produce 2-factors of $M(b,n) \setminus (\{d\} \times \mathbb{Z}_n)$, it must contain more than one part. If $b = 2$, then $M(b,n) \setminus (\{d\} \times \mathbb{Z}_n)$ contains only one part. So $b \neq 2$.

Since a C_z-frame of $M(b,n)$ is a z-cycle-system, the number of edges in $M(b,n)$, $\binom{b}{2}n^2$, must be divisible by z. Also, in order to produce C_z-factors of $M(b,n)\setminus(\{d\} \times \mathbb{Z}_n)$ for $d \in \mathbb{Z}_b$, the number of vertices of $M(b,n)\setminus(\{d\} \times \mathbb{Z}_n)$ for $d \in \mathbb{Z}_b$ must be divisible by z. So $(b-1)n \equiv 0 \pmod z$.

Each of the C_z-factors consists of $(b-1)n$ edges. In the multipartite graph $M(b,n)$, there are $\binom{b}{2}n^2$ edges. So the number of C_z-factors in a C_z-frame of $M(b,n)$ is

$$\frac{\binom{b}{2}n^2}{(b-1)n} = \frac{1}{2}bn,$$

which implies that at least one of b and n is even. ∎

Lemma 2.2 *[7] Suppose $z \neq 6$ with $z \geq 4$. Then $K_{z,z}$ has a C_z-factorization for all $z \equiv 0 (mod\ 2)$.*

Lemma 2.3 *[2] Suppose $(b-1)n \geq 2$ is even. Then $M(b,n)$ is the union of $\frac{(b-1)n}{2}$ Hamilton cycles.*

Lemma 2.4 *[5] Suppose $b \equiv 1 (mod\ 4)$. Then near C_4-factorizations of λK_b exist for all even λ.*

Lemma 2.5 *[13] There exists a C_4-frame of $M(b,n)$ for all $M(b,n)$ that satisfy Lemma 2.1.*

Lemma 2.6 *[6] There exists a C_6-frame of $M(b,n)$ for all $M(b,n)$ that satisfy Lemma 2.1.*

Lemma 2.7 *[1] (Fundamental Cycle Frame Construction) If there exists a $(K,1,b)$-Pairwise Balanced Design of the complete graph K_b and C_z-frame of $M(m,2)$ for each $m \in K$, there exists a C_z-frame of $M(b,2)$.*

3 Some General Constructions

In this section, we give some general constructions of C_z-frames. Unfortunately, these general constructions often do not completely settle the existence of C_z-frames for a given z. So we must also use additional techniqes for given values of z, b, and n, but they are good beginnings. We begin when b is odd and n is a multiple of $z/2$.

Theorem 3.1 *Let b be odd and $z \equiv 0 (mod\ 4)$. There exists a C_z-frame of $M(b, mz/2)$.*

Proof Let F' be a near 1-factorization on the vertex set $\mathbb{Z}_b$, and for each $d \in \mathbb{Z}_b$ let F'_d be the near 1-factor in F' with deficiency d; so each vertex in $\mathbb{Z}_b \setminus \{d\}$ occurs in exactly one edge in F'_d.

Let F be a 1-factorization on the vertex set $\mathbb{Z}_m \times \mathbb{Z}_m$, and for each $t \in \mathbb{Z}_m$, let F_t be a 1-factor in F. Let $K(B_x, B_y)$ be the complete simple bipartite graph on the parts $B_x = \{x\} \times \mathbb{Z}_{mz/2}$ and $B_y = \{y\} \times \mathbb{Z}_{mz/2}$, $0 \leq x < y \leq b-1$. Let $K(B_{x,k}, B_{y,\ell})$ be the complete simple bipartite graph on parts $B_{x,k} = \{x\} \times \{kz/2, kz/2+1, kz/2+2, ..., kz/2+(z/2-1)\}$ and $B_{y,\ell} = \{y\} \times \{\ell z/2, \ell z/2+1, \ell z/2+2, ..., \ell z/2+(z/2-1)\}$, $0 \leq x < y \leq b-1$, $k, \ell \in \mathbb{Z}_m$.

Notice that

$$K(B_x, B_y) = \bigcup_{\substack{\{k,\ell\} \in E(F_t) \\ t \in \mathbb{Z}_m}} K(B_{x,k}, B_{y,\ell}).$$

Notice also that the size of each part of $K(B_{x,k}, B_{y,\ell})$ is $z/2$, and the graph is an example of $M(2, z/2)$, which can be decomposed into Hamilton cycles of length z by Lemma 2.3. So for each $\{k, \ell\} \in E(F_t)$, define a C_z-factorization of $K(B_{x,k}, B_{y,\ell})$, consisting of $z/4$ C_z-factors, $\pi_{xk,y\ell}(0), \pi_{xk,y\ell}(1), ..., \pi_{xk,y\ell}(z/4-1)$, as prescribed in Lemma 2.3.

For each $d \in \mathbb{Z}_b$, let

$$M_d = \bigcup_{\{x,y\} \in E(F'_d)} K(B_x, B_y),$$

which has a C_z-factorization, P_d, consisting of the $mz/4$ C_z-factors:

$$M_d(j,t) = \bigcup_{\substack{\{x,y\} \in E(F'_d) \\ \{k,\ell\} \in E(F_t)}} \pi_{xk,y\ell}(j) \text{ for each } j \in \mathbb{Z}_{z/4},\ t \in \mathbb{Z}_m.$$

Notice that

$$M(b, mz/2) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which therefore occurs in exactly one cycle in

$$\bigcup_{\substack{d \in \mathbb{Z}_b \\ j \in \mathbb{Z}_{z/4} \\ t \in \mathbb{Z}_m}} M_d(j,t).$$

Notice also that each $M_d(j,t)$ is a C_z-factor of $M(b, mz/2)\setminus(\{d\}\times\mathbb{Z}_{mz/2})$ so the z-cycles in

$$\bigcup_{d\in\mathbb{Z}_b} P_d,$$

form a C_z-frame of $M(b, mz/2)$. ■

If $z = 2p$ where p is a prime greater than 3, then there are several cases depending on the values of b and n. If b is odd, then n must be a multiple of $2p$, and a construction is given here. If b is even, then there are two cases depending on if b is or is not equivalent to 1 modulo p. If $b \equiv 1 (\text{mod } p)$, then n may be any even integer, but it suffices to solve the case when $n = 2$. If $b \not\equiv 1(\text{mod } p)$, then $n = 2mp$.

Theorem 3.2 *Let b be odd and $z = 2p$ where p is a prime greater than 3. There there exists a C_z-frame of $M(b,n)$ for $n = 2mp$.*

Proof Let F' be a near 1-factorization on the vertex set $\mathbb{Z}_b$, and for each $d \in \mathbb{Z}_b$ let F'_d be the near 1-factor in F' with deficiency d; so each vertex in $\mathbb{Z}_b \setminus \{d\}$ occurs in exactly one edge in F'_d.

Let F be a 1-factorization on the vertex set $\mathbb{Z}_m \times \mathbb{Z}_m$, and for each $t \in \mathbb{Z}_m$, let F_t be a 1-factor in F. Let $K(B_x, B_y)$ be the complete simple bipartite graph on the parts $B_x = \{x\} \times \mathbb{Z}_{2mp}$ and $B_y = \{y\} \times \mathbb{Z}_{2mp}$, $0 \le x < y \le b-1$. Let $K(B_{x,k}, B_{y,\ell})$ be the complete simple bipartite graph on parts $B_{x,k} = \{x\} \times \{2kp, 2kp+1, 2kp+2, ..., 2kp+(2p-1)\}$ and $B_{y,\ell} = \{y\} \times \{2\ell p, 2\ell p+1, 2\ell p+2, ..., 2\ell p+(2p-1)\}$, $0 \le x < y \le b-1$, $k, \ell \in \mathbb{Z}_m$.

Notice that

$$K(B_x, B_y) = \bigcup_{\substack{\{k,\ell\}\in E(F_t)\\ t\in\mathbb{Z}_m}} K(B_{x,k}, B_{y,\ell}).$$

For each $\{k,\ell\} \in E(F_t)$, define a C_z-factorization of $K(B_{x,k}, B_{y,\ell})$, consisting of $z/2$ C_z-factors, $\pi_{xk,y\ell}(0), \pi_{xk,y\ell}(1), ..., \pi_{xk,y\ell}(z/2-1)$, as prescribed in Lemma 2.2.

For each $d \in \mathbb{Z}_b$, let

$$M_d = \bigcup_{\{x,y\}\in E(F'_d)} K(B_x, B_y),$$

which has a C_z-factorization, P_d, consisting of the $mz/2$ C_z-factors:

$$M_d(j,t) = \bigcup_{\substack{\{x,y\}\in E(F'_d)\\ \{k,\ell\}\in E(F_t)}} \pi_{xk,y\ell}(j) \text{ for each } j \in \mathbb{Z}_{z/2},\ t \in \mathbb{Z}_m.$$

Notice that

$$M(b, 2mp) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which therefore occurs in exactly one cycle in

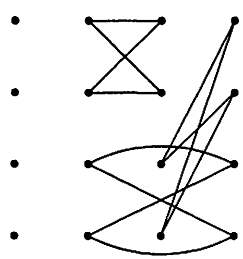
$$\bigcup_{\substack{d \in \mathbb{Z}_b \\ j \in \mathbb{Z}_{z/2} \\ t \in \mathbb{Z}_m}} M_d(j, t).$$

Notice also that each $M_d(j,t)$ is a C_z-factor of $M(b, 2mp) \setminus (\{d\} \times \mathbb{Z}_{2mp})$ so the z-cycles in

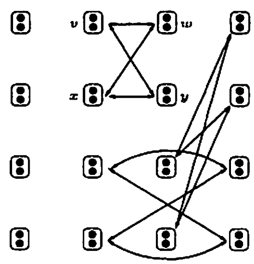
$$\bigcup_{d \in \mathbb{Z}_b} P_d,$$

form a C_z-frame of $M(b, 2mp)$. ■

In the next theorem, we show how a C_4-frame of $M(b,n)$ can be extrapolated to a C_z-frame of $M(b, nz/4)$ where $z \equiv 0 (\text{mod } 4)$; it is particularly useful for producing C_z-frames when b is even. This allows for the construction of infinitely many C_z-frames of $M(b,n)$, but it does not settle every case of each specific cycle length z. Those leftover cases must then be solved. An example of the extrapolation is given in Figures 1 and 2.



(a) A C_4-factor of $M(4,4) \setminus \{B_0\}$

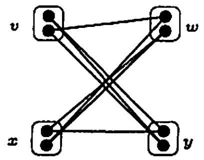


(b) To be 2 C_8-factors of $M(4,8) \setminus \{B_0\}$

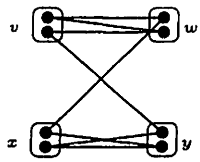
Figure 1

Theorem 3.3 *Let $z \equiv 0(\text{mod } 4)$. If there exists a C_4-frame of $M(b,n)$, then there exists a C_z-frame of $M(b, nz/4)$.*

Proof We assume that we do in fact have a C_4-frame of $M(b,n)$ as produced in [13]. In [13], it is shown that every C_4-frame can be constructed



(a) First C_8-factor of $M(2, 4)$



(b) Second C_8-factor of $M(2, 4)$

Figure 2

such that each cycle of each factor consists of vertices in only two parts. This arrangement is what allows this extrapolation to work.

To begin, replace each vertex u in $M(b, n)$ with $t = z/4$ vertices $u_1, u_2, ..., u_t$. Then for each 4-cycle (v, w, x, y) in a C_4-factor of the C_4-frame, produce a complete simple bipartite graph, $M(2, z/2)$, with parts $\{v_1, v_2, ..., v_t, x_1, x_2, ..., x_t\}$ and $\{w_1, w_2, ..., w_t, y_1, y_2, ..., y_t\}$.

By Lemma 2.3, there exists a C_z-factorization of $M(2, z/2)$ into $z/4$ C_z-factors. If we create the bipartite graphs $M(2, z/2)$ from each 4-cycle in a C_4-factor of the C_4-frame and then decompose those bipartite graphs into C_z-factorizations, we yield $z/4$ C_z-factors of $M(b, nz/4) \setminus \{B_i\}$ for some $i \in \mathbb{Z}_b$.

If we repeat this process for each C_4-factor of the C_4-frame, we have produced a C_z-frame of $M(b, nz/4)$. ■

It is worth repeating that while Theorem 3.3 does allow for the construction of a great many cycle-frames, much of the work is in producing cycle-frames for the graphs it omits. The rest of the paper provides results for some of these small cases. In the next section, we completely settle the remaining cases for $z = 8$.

4 The Main Result - C_8-Frames

First, we must consider the values of b and n for which a C_8-frame of $M(b, n)$ is possible. Since $(b - 1)n \equiv 0 (\text{mod } 8)$, the values of $b \neq 2$ and n that satisfy Lemma 2.1 are the following:

i) if b is even, then n must be a multiple of 8, and

ii) if b is odd, then n must be even.

Notice that if b is either even or odd, we may use Theorem 3.3 to extrapolate a C_4-frame of $M(b, 4)$ to a C_8-frame of $M(b, 8m)$.

Now we must consider $M(b,n)$ when b is odd and $n = 2$ or 4. Notice that we need not consider $n = 6$ since that would imply $b \equiv 1 (\text{mod } 4)$, which is also implied when $n = 2$. We produce a construction for $M(b,2)$ that can be used to form C_8-frames of $M(b,n)$ when $b \equiv 1(\text{mod } 4)$ and $n = 2m$. When b is odd and $n = 4m$, we use Theorem 3.1 to produce the C_8-frame of $M(b,n)$.

The proceeding construction completely settles the existence of C_8-frames of $M(b,n)$.

Theorem 4.1 *Let $b \equiv 1(\text{mod } 4)$. There exists a C_8-frame of $M(b, 2m)$.*

Proof Let N' be a near C_4-factorization of $2K_b$ on the vertex set $\mathbb{Z}_b$ [5], and for each $d \in \mathbb{Z}_b$, let N'_d be the near C_4-factor in N' with deficiency d; so each vertex in $\mathbb{Z}_b \setminus \{d\}$ occurs in exactly one 4-cycle in N'_d. Each N'_d contains $\frac{b-1}{4}$ 4-cycles, (v,w,x,y), with $v < w,x,y$. For $s \in \mathbb{Z}_{\frac{b-1}{4}}$, let $c_d(s)$ be the s^{th} 4-cycle in N'_d. So $N'_d = \{c_d(s) = (v,w,x,y) \mid v,w,x,y \in \mathbb{Z}_b, v < w,x,y,$ and $s \in \mathbb{Z}_{\frac{b-1}{4}}\}$.

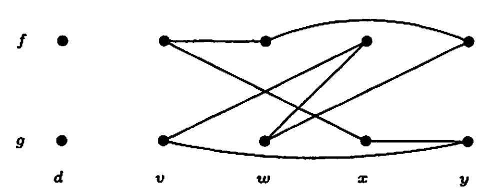
Let F be a 1-factorization on the vertex set $\mathbb{Z}_{2m} \times \mathbb{Z}_{2m}$, and for each $t \in \mathbb{Z}_{2m}$, let F_t be a 1-factor in F.

Given $c_d(s) \in N'_d$ and $\{f,g\} \in E(F_t)$, define an 8-cycle on the vertices $\{v,w,x,y\} \times \{f\}$ and $\{v,w,x,y\} \times \{g\}$ as follows:

$$\pi_d(s,f,g) = \{((v,f),(w,f),(y,f),(w,g),(x,f),(v,g),(y,g),(x,g))$$

$$|(v,w,x,y) \in N'_d, f,g \in \mathbb{Z}_{2m}\}.$$

Figure 3 shows an example of the 8-cycle.



(a) An example 8-cycle

Figure 3

Let $d \in \mathbb{Z}_b$, $s \in \mathbb{Z}_{\frac{b-1}{4}}$, and $t \in \mathbb{Z}_{2m}$. Notice that

$$P_d(s,t) = \bigcup_{\{f,g\} \in E(F_t)} \pi_d(s,f,g)$$

is a C_8-factor on the vertices $\{v, w, x, y\} \times \mathbb{Z}_{2m}$ for $v, w, x, y \in \mathbb{Z}_b$. Notice also that for each $c_d(s) \in N'_d$,

$$P_d(t) = \bigcup_{c_d(s) \in N'_d} P_d(s,t)$$

is a C_8-factor of $M(b, 2m) \setminus (\{d\} \times \mathbb{Z}_{2m})$.

For each $d \in \mathbb{Z}_b$, $c_d(s) = (v, w, x, y) \in N'_d$ and $t \in \mathbb{Z}_{2m}$, let $M_d(s,t)$ be the multipartite graph on vertices $\{v, w, x, y\} \times \mathbb{Z}_{2m}$ for $v, w, x, y \in \mathbb{Z}_b$ induced by the edges of the C_8-factor $P_d(s,t)$.

Let

$$M_d(t) = \bigcup_{s \in \mathbb{Z}_{\frac{b-1}{4}}} M_d(s,t)$$

be the multipartite graph on vertices $(\mathbb{Z}_b \times \mathbb{Z}_{2m}) \setminus (\{d\} \times \mathbb{Z}_{2m})$ induced by the edges of the C_8-factor $P_d(t)$.

Let

$$M_d = \bigcup_{t \in \mathbb{Z}_{2m}} M_d(t),$$

which has a C_8-factorization,

$$P_d = \bigcup_{t \in \mathbb{Z}_{2m}} P_d(t),$$

consisting of t C_8-factors.

Notice that

$$M(b, 2m) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which occurs in exactly one cycle in

$$\bigcup_{\substack{d \in \mathbb{Z}_b \\ s \in \mathbb{Z}_{\frac{b-1}{4}} \\ t \in \mathbb{Z}_{2m}}} P_d(s,t).$$

Notice also that each $P_d(s,t)$ is a C_8-factor of $M(b, 2m) \setminus (\{d\} \times \mathbb{Z}_{2m})$ so the 8-cycles in

$$\bigcup_{d \in \mathbb{Z}_b} P_d$$

form a C_8-frame of $M(b, 2m)$. ∎

5 C_{12}-Frames

In this section, we consider $z = 12$. Again, we consider the values of b and n for which a C_{12}-frame of $M(b,n)$ is possible. The factors of 12 are 1, 2, 3, 4, 6, and 12. We begin by examining the possibilites of the part size with respect to these factors.

i) if n were to be 1, then b must also be odd, which contradicts Lemma 2.1;

ii) if n were to be 2, then $b \equiv 1 (\text{mod } 6)$;

iii) if n were to be 3, then b must again be odd, which is again a contradiction;

iv) if n were to be 4, then all $b \equiv 4 (\text{mod } 6)$ are possible;

v) if n were to be 6, then b must be odd; and

vi) if n were to be 12, then b may be any value other than 2.

Notice that we may use Theorem 3.3 to extrapolate a C_4-frame of $M(b,4)$ to a C_{12}-frame of $M(b,12m)$. When b is odd and $n = 6m$, we use Theorem 3.1 to produce a C_{12}-frame of $M(b,6m)$. So we have two cases left to consider.

We produce a construction to form C_{12}-frames of $M(b,n)$ when $b \equiv 4 (\text{mod } 6)$ and $n = 4m$. For the case when $n = 2$ and $b \equiv 1 (\text{mod } 6)$, we are able to produce frames for several small examples and recursively for very large values of b.

So the existence of C_{12}-frames of $M(b,n)$ is not completely settled, but much progress has been made.

Theorem 5.1 *There exist C_{12}-frames of $M(7,2), M(13,2), M(19,2)$, and $M(25,2)$.*

Proof For each of these graphs, label the vertices $\mathbb{Z}_{2b}$ with parts $\{\{v, v+1\} \mid v \in \mathbb{Z}_{2b} \text{ and } v \text{ is even}\}$. For each of the given base cycles, generate the rest of the cycles in the frame by adding two modulo $2b$ to each vertex.
For a C_{12}-frame of $M(7,2)$, use the base cycle:

$$(2, 12, 3, 7, 8, 6, 9, 11, 4, 10, 5, 13)$$

For a C_{12}-frame of $M(13,2)$, use the base cycles:

$$(2, 24, 3, 13, 14, 12, 15, 7, 20, 6, 21, 25) \text{ and}$$

$$(4, 22, 5, 19, 8, 18, 9, 11, 16, 10, 17, 23).$$

For a C_{12}-frame of $M(19, 2)$, use the base cycles:

$$(2, 36, 3, 19, 20, 18, 21, 11, 28, 10, 29, 37),$$
$$(4, 34, 5, 25, 14, 24, 15, 17, 22, 16, 23, 35), \text{ and}$$
$$(6, 32, 7, 31, 8, 30, 9, 13, 26, 12, 27, 33).$$

For a C_{12}-frame of $M(25, 2)$, use the base cycles:

$$(2, 48, 3, 25, 26, 24, 27, 13, 38, 12, 39, 49),$$
$$(4, 46, 5, 29, 22, 28, 23, 21, 30, 20, 31, 47),$$
$$(6, 44, 7, 37, 14, 36, 15, 19, 32, 18, 33, 45), \text{ and}$$
$$(8, 42, 9, 41, 10, 40, 11, 17, 34, 16, 35, 43).$$

■

By the Fundamental Cycle Frame Construction, if there exists a $(K, 1, b)$-Pairwise Balanced Design of the complete graph K_b and C_z-frame of $M(m, 2)$ for each $m \in K$, there exists a C_z-frame of $M(b, 2)$. So we can produce a C_{12}-frame for $M(b, 2)$ if there exists a $(K, 1, b) - PBD$ of K_b with blocks of size $K = \{7, 13, 19, 25\}$. The values of b for which this is guaranteed begin at quite a large number, and there are quite a few possible exceptions for values less than the guaranteed value. We leave the work of solving the possible exceptions as further research.

Theorem 5.2 *Let $b \equiv 4 (mod\ 6)$. There exists a C_{12}-frame of $M(b, 4m)$.*

Proof By Lemma 2.6, there exists a C_6-frame of $M(6k + 4, 2m)$. We extrapolate it to produce a C_{12}-frame of $M(6k + 4, 4m)$. For each 6-cycle, (t, u, v, w, x, y), of each C_6-factor of the frame, replace each vertex by a pair of vertices. Replace each edge of the cycle with $M(2, 2)$ such that the resulting graph looks like Figure 4.

This graph can be decomposed into the two 12-cycles: $(t_1, u_1, v_1, w_1, x_1, y_1, t_2, u_2, v_2, w_2, x_2, y_2)$ and $(t_1, u_2, v_1, w_2, x_1, y_2, t_2, u_1, v_2, w_1, x_2, y_1)$. Doing so for all 6-cycles of each factor of the C_6-frame produces a C_{12}-frame of $M(6k + 4, 4m)$. ■

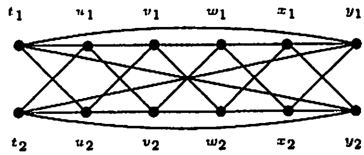


Figure 4

Unfortunately, it seems that constructing C_{12}-frames for $M(b,2)$ is much more difficult than we had hoped. All of the techniques described here have no general extrapolation. If there existed C_{12}-frames of $M(k,2)$ for $k \in \{31,37,43\}$, then we would have something more interesting to say about exactly when a C_{12}-frame of $M(b,2)$ would exist since we could then invoke the Fundamental Cycle Construction.

6 Further Research & Acknowledgments

While the existence of C_z-frames of $M(b,n)$ is settled when z is 8, $z = 12$ is still unfinished. In order to settle the existence of C_{12}-frames, the question of $M(b,n)$ must be answered. At this time, we are still searching for a general construction.

Most of the constructions in this paper concern values of z that are 0 modulo 4. Much work can be done when z is 2 modulo 4.

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