On large sets of $K_{1,p}$-decomposition of complete bipartite graphs*

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Abstract. Let H and G be two simple graphs, where G is a subgraph of H. A G-decomposition of λH, denoted by $(\lambda H, G)$-GD, is a partition of all the edges of λH into subgraphs (G-blocks), each of which is isomorphic to G. A large set of $(\lambda H, G)$-GD, denoted by $(\lambda H, G)$-LGD, is a partition of all subgraphs isomorphic to G of H into $(\lambda H, G)$-GDs (called small sets). In this paper, we investigate the existence of $(\lambda K_{m,n}, K_{1,p})$-$LGD$ and obtain some existence results, where $p \geq 3$ is a prime.

Keywords: large set; $K_{1,p}$-decomposition; complete bipartite graph

1 Introduction

Let $G = (V(G), E(G))$ be a graph, where each edge in $E(G)$ is denoted by an unordered pair $\{u, v\}$, $u, v \in V(G)$. The *degree* $d_G(v)$ of a vertex v in G is $|\{u : \{u, v\} \in E(G)\}|$. A graph G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. A *spanning subgraph* of H is a subgraph G with $V(G) = V(H)$. Let G be a *spanning subgraph* of H. G is called an *F-factor* if each component of G is isomorphic to a given graph F. Let G be a spanning subgraph of H. If G can be partitioned into some subgraphs isomorphic to F, and the number of times each vertex of H appears in subgraphs isomorphic to F is exactly λ, then G is called a λ-fold F-factor of

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H, denoted by $S_\lambda(1, F, H)$. A *large set* of λ*-fold F-factors* of G, denoted by $LS_\lambda(1, F, G)$, is a partition $\{\mathcal{B}_i\}_i$ of all subgraphs of G isomorphic to F, such that each $\mathcal{B}_i$ is a λ*-fold F-factor* of G. For $\lambda = 1$, the index 1 is often omitted. About $LS_\lambda(1, F, G)$, we have the following result.

Lemma 1.1 [1] *There exists an* $LS_\lambda(1, K_k, K_v)$ *if and only if* $k|\lambda v$ *and* $\frac{\lambda v}{k}|\binom{v}{k}$.

Let G be a graph and λ be a positive integer. We use λG to denote the multigraph obtained from G by repeating each edge λ times. In this paper, K_n is the complete graph on n vertices, where any two distinct vertices x and y of K_n are joined by exactly one edge $\{x, y\}$. Also, $K_{m,n}$ is the complete bipartite graph with two parts X and Y of cardinalities m and n, respectively, where any vertex x in X and any vertex y in Y are joined by exactly one edge $\{x, y\}$.

Let H and G be two simple graphs, where G is a subgraph of H. A *G-decomposition* (or *G-design*) of λH, denoted by $(\lambda H, G)$-GD, is a partition of $E(\lambda H)$ into subgraphs (called *G-blocks*), each of which is isomorphic to G. For $H = K_n$ and some simple graphs of G, such as the cycle C_k, path P_k, star S_k, k-cube and some graphs with fewer vertices and fewer edges, the existence of these G-decompositions has been solved (see [2]). A *large set* of $(\lambda H, G)$-GD, denoted by $(\lambda H, G)$-LGD, is a partition of all subgraphs isomorphic to G of H into $(\lambda H, G)$-GDs (called *small sets*). The large set (K_n, C_3)-LGD (that is large sets of Steiner triple systems $LSTS(n)$) has been completely solved (see [6-8]). There are some other results regarding the existence of $(\lambda H, G)$-LGD(see [4],[5],[9]). Not a long time ago, the existence spectrums of $(\lambda K_{m,n}, P_3)$-LGD (that is, large sets of $K_{1,2}$-decompositions of complete bipartite graphs) and $(\lambda K_{m,n}, K_{2,2})$-$LGD$ (that is, large sets of $K_{2,2}$-decompositions of complete bipartite graphs) were obtained (see [10] and [3]).

In this paper, a $K_{1,p}$ which contains p edges $\{a, b_1\}, \{a, b_2\}, \cdots, \{a, b_{p-1}\}$ and $\{a, b_p\}$ is denoted by $[a; b_1, b_2, \cdots, b_p]$. We investigate the existence of $(\lambda K_{m,n}, K_{1,p})$-$LGD$ and obtain some existence results, where $p \geq 3$ is a prime. (Note: In the following content, p is always a prime.)

2 Main Constructions

A $(\lambda K_{m,n}, K_{1,p})$-$GD$ consists of $\frac{\lambda mn}{p}$ $K_{1,p}$-blocks. A $(\lambda K_{m,n}, K_{1,p})$-$LGD$ contains $\frac{\binom{m-1}{p-1}+\binom{n-1}{p-1}}{\lambda}$ pairwise disjoint $(\lambda K_{m,n}, K_{1,p})$-$GD$s (small sets). So we have the following result.

Lemma 2.1 *There exists a $(\lambda K_{m,n}, K_{1,p})$-$LGD$ only if $p|\lambda mn$ and $\lambda|[\binom{m-1}{p-1}+\binom{n-1}{p-1}]$.*

Therefore, in order to determine the existence spectrum of $(\lambda K_{m,n}, K_{1,p})$-$LGD$, it is enough to construct $(K_{pm,pn}, K_{1,p})$-LGD, $(K_{pm,n}, K_{1,p})$-LGD (where $n \not\equiv 0 \pmod p$) and $(pK_{m,n}, K_{1,p})$-LGD (where $m \not\equiv 0 \pmod p$ and $n \not\equiv 0 \pmod p$)). In this paper, we obtain the sufficient and necessary conditions of $(\lambda K_{pm,pn}, K_{1,p})$-$LGD$ and $(\lambda K_{m,n}, K_{1,p})$-$LGD$ (where $m \not\equiv 0 \pmod p$ and $n \not\equiv 0 \pmod p$)).

Lemma 2.2 *There exists a $(K_{pm,pn}, K_{1,p})$-LGD for any positive integers m and n.*

Proof. Let $v = pt$, by Lemma 1.1, there exists an

$$LS(1, K_p, K_{pt}) = \{(Z_{pt}, \mathcal{T}_i) : 1 \le i \le \binom{pt-1}{p-1}\}.$$

Each $\mathcal{T}_i$ consists of t p-subsets of Z_{pt}, which forms a parallel class on Z_{pt}.

Let the vertex set of $K_{pm,pn}$ be $Z_{pm} \cup \bar{Z}_{pn}$. There exist

$$LS(1, K_p, K_{pm}) = \{(Z_{pm}, \mathcal{P}_i) : 1 \le i \le \binom{pm-1}{p-1}\}$$

and

$$LS(1, K_p, K_{pn}) = \{(\bar{Z}_{pn}, \mathcal{Q}_j) : 1 \le j \le \binom{pn-1}{p-1}\}$$

on Z_{pm} and on $\bar{Z}_{pn}$ respectively, where each $\mathcal{P}_i$ consists of m p-subsets of Z_{pm}, which forms a parallel class on Z_{pm}, and each $\mathcal{Q}_j$ consists of n p-subsets of $\bar{Z}_{pn}$, which forms a parallel class on $\bar{Z}_{pn}$.

Define

$\mathcal{A}_i = \{[x; a_1, a_2, \cdots, a_p] : x \in \bar{Z}_{pn}, \{a_1, a_2, \cdots, a_p\} \in \mathcal{P}_i\}, 1 \le i \le \binom{pm-1}{p-1}$.

$\mathcal{B}_j = \{[y; b_1, b_2, \cdots, b_p] : y \in Z_{pm}, \{b_1, b_2, \cdots, b_p\} \in \mathcal{Q}_j\}, 1 \le j \le \binom{pn-1}{p-1}$.

Then each $(Z_{pm} \cup \bar{Z}_{pn}, \mathcal{A}_i)$ is a $(K_{pm,pn}, K_{1,p})$-GD for $1 \le i \le \binom{pm-1}{p-1}$ because each $\mathcal{P}_i$ is a parallel class on Z_{pm}. Similarly, each $(Z_{pm} \cup \bar{Z}_{pn}, \mathcal{B}_j)$ is a $(K_{pm,pn}, K_{1,p})$-GD for $1 \le j \le \binom{pn-1}{p-1}$ because each $\mathcal{Q}_j$ is a parallel class on $\bar{Z}_{pn}$. So we have $\binom{pm-1}{p-1} + \binom{pn-1}{p-1}$ small sets, just as expected.

Furthermore, the family $\{\mathcal{A}_i\}$ just forms a partition of all $K_{1,p}$-blocks in the form $[x; a_1, a_2, \cdots, a_p]$ (where $x \in \bar{Z}_{pn}$, $\{a_1, a_2, \cdots a_p\}$ is a p-subset in Z_{pm}) because $\{(Z_{pm}, \mathcal{P}_i)\}$ forms a $LS(1, K_p, K_{pm})$ on Z_{pm}, and the family $\{\mathcal{B}_j\}$ just forms a partition of all $K_{1,p}$-blocks in the form $[y; b_1, b_2, \cdots, b_p]$ (where $y \in Z_{pm}$, $\{b_1, b_2, \cdots, b_p\}$ is a p-subset in $\bar{Z}_{pn}$) because $\{(\bar{Z}_{pn}, \mathcal{Q}_j)\}$ forms a $LS(1, K_p, K_{pn})$ on $\bar{Z}_{pn}$. So $(\bigcup_i \mathcal{A}_i) \cup (\bigcup_j \mathcal{B}_j)$ forms a $(K_{pm,pn}, K_{1,p})$-LGD on $Z_{pm} \cup \bar{Z}_{pn}$. □

Lemma 2.3 *Let p be a prime, if $n \not\equiv 0 \pmod p$, then $n | \binom{n}{p}$.*

Proof. $\binom{n}{p} = \frac{n}{p}\binom{n-1}{p-1} = \frac{n\binom{n-1}{p-1}}{p}$. $\binom{n}{p}$ is an integer, so $\frac{n\binom{n-1}{p-1}}{p}$ is an integer. p is a prime and $n \not\equiv 0 \pmod p$, so $p | \binom{n-1}{p-1}$, that is, $\frac{\binom{n-1}{p-1}}{p} = t$ is an integer, then $\binom{n}{p} = nt$, so we have $n | \binom{n}{p}$. □

Lemma 2.4 *There exists a $(pK_{m,n}, K_{1,p})$-LGD, where $m \not\equiv 0 \pmod p$ and $n \not\equiv 0 \pmod p$.*

Proof. Let the vertex set of $K_{m,n}$ be $Z_m \cup \bar{Z}_n$. Because $m \not\equiv 0 \pmod p$, by Lemma 1.1 and Lemma 2.3, we know there exist

$$LS_p(1, K_p, K_m) = \{(Z_m, \mathcal{P}_i) : 1 \le i \le \tfrac{\binom{m-1}{p-1}}{p}\},$$

where each $\mathcal{P}_i$ consists of m p-subsets of Z_m, which forms a p-parallel class on Z_m. Similarly, because $n \not\equiv 0 \pmod p$, by Lemma 1.1 and Lemma 2.3, we know there exist

$$LS_p(1, K_p, K_n) = \{(\bar{Z}_n, \mathcal{Q}_j) : 1 \le j \le \tfrac{\binom{n-1}{p-1}}{p}\},$$

where each $\mathcal{Q}_j$ consists of n p-subsets of $\bar{Z}_n$, which forms a p-parallel class on $\bar{Z}_n$. Define

$$\mathcal{A}_i = \{[x; a_1, a_2, \cdots, a_p] : x \in \bar{Z}_n, \{a_1, a_2, \cdots, a_p\} \in \mathcal{P}_i)\}, 1 \le i \le \tfrac{\binom{m-1}{p-1}}{p}.$$

$$\mathcal{B}_j = \{[y; b_1, b_2, \cdots, b_p] : y \in Z_m, \{b_1, b_2, \cdots, b_p\} \in \mathcal{Q}_j)\}, 1 \le j \le \tfrac{\binom{n-1}{p-1}}{p}.$$

Then each $(Z_m \cup \bar{Z}_n, \mathcal{A}_i)$ is a $(pK_{m,n}, K_{1,p})$-GD for $1 \le i \le \frac{\binom{m-1}{p-1}}{p}$ because each $\mathcal{P}_i$ is a p-parallel class on Z_m. Similarly, each $(Z_m \cup \bar{Z}_n, \mathcal{B}_j)$ is a $(pK_{m,n}, K_{1,p})$-GD for $1 \le j \le \frac{\binom{n-1}{p-1}}{p}$ because each $\mathcal{Q}_j$ is a p-parallel class on $\bar{Z}_n$. So we have $\frac{\binom{m-1}{p-1}}{p} + \frac{\binom{n-1}{p-1}}{p}$ small sets, just as expected.

Furthermore, the family $\{\mathcal{A}_i\}$ just forms a partition of all $K_{1,p}$-blocks in the form $[x; a_1, a_2, \cdots, a_p]$ (where $x \in \bar{Z}_n$, $\{a_1, a_2, \cdots, a_p\}$ is a p-subset in Z_m) because $\{(Z_m, \mathcal{P}_i)\}$ forms a $LS_p(1, K_p, K_m)$ on Z_m, and the family $\{\mathcal{B}_j\}$ just forms a partition of all $K_{1,p}$-blocks in the form $[y; b_1, b_2, \cdots, b_p]$ (where $y \in Z_m$, $\{b_1, b_2, \cdots, b_p\}$ is a p-subset in $\bar{Z}_n$) because $\{(\bar{Z}_n, \mathcal{Q}_j)\}$ forms a $LS_p(1, K_p, K_n)$ on $\bar{Z}_n$. So $(\bigcup_i \mathcal{A}_i) \bigcup (\bigcup_j \mathcal{B}_j)$ forms a $(pK_{m,n}, K_{1,p})$-LGD on $Z_m \cup \bar{Z}_n$. □

3 Conclusion

Theorem 3.1 *There exists a $(\lambda K_{pm,pn}, K_{1,p})$-LGD if and only if $\lambda | [\binom{pm-1}{p-1} + \binom{pn-1}{p-1}]$.*

Proof. By Lemma 2.1, we only need to prove the sufficiency.

By Lemma 2.2, there exists a

$(K_{pm,pn}, K_{1,p})$-LGD=$\{(Z_{pm} \cup \bar{Z}_{pn}, \mathcal{C}_i) : 1 \le i \le \binom{pm-1}{p-1} + \binom{pn-1}{p-1}\}$.

Define

$$\mathcal{D}_k = \bigcup_{i=k\lambda+1}^{(k+1)\lambda} \mathcal{C}_i, \ 0 \le k \le \frac{\binom{pm-1}{p-1} + \binom{pn-1}{p-1}}{\lambda} - 1,$$

then $\{(Z_{pm} \cup \bar{Z}_{pn}, \mathcal{D}_k) : 0 \le k \le \frac{\binom{pm-1}{p-1} + \binom{pn-1}{p-1}}{\lambda} - 1\}$ is just a $(\lambda K_{pm,pn}, K_{1,p})$-$LGD$. □

Theorem 3.2 *There exists a $(\lambda K_{m,n}, K_{1,p})$-LGD if and only if $p|\lambda$ and $\lambda | [\binom{m-1}{p-1} + \binom{n-1}{p-1}]$, where $m \not\equiv 0 \pmod p$ and $n \not\equiv 0 \pmod p$.*

Proof. By Lemma 2.1, we only need to prove the sufficiency.

By Lemma 2.4, there exists a

$$(pK_{m,n}, K_{1,p})\text{-}LGD = \{(Z_m \cup \bar{Z}_n, \mathcal{C}_i) : 1 \le i \le \frac{\binom{m-1}{p-1}}{p} + \frac{\binom{n-1}{p-1}}{p}\}.$$

Define

$$\mathcal{D}_k = \bigcup_{i=k\frac{\lambda}{p}+1}^{(k+1)\frac{\lambda}{p}} \mathcal{C}_i, \ 0 \le k \le \frac{\binom{m-1}{p-1} + \binom{n-1}{p-1}}{\lambda} - 1,$$

then $\{(Z_m \cup \bar{Z}_n, \mathcal{D}_k) : 0 \le k \le \frac{\binom{m-1}{p-1} + \binom{n-1}{p-1}}{\lambda} - 1\}$ is just a $(\lambda K_{m,n}, K_{1,p})$-$LGD$. □

In order to obtain the existence spectrum of $(\lambda K_{m,n}, K_{1,p})$-$LGD$, we only need to solve the existence problem of $(K_{pm,n}, K_{1,p})$-LGD (where $n \not\equiv 0 \pmod p$). About $(K_{pm,n}, K_{1,p})$-LGD (where $n \not\equiv 0 \pmod p$), we obtain the following result.

Lemma 3.3 *There exists a $(K_{pm,p+1}, K_{1,p})$-LGD for any positive integer m.*

Proof. Let the vertex set of $K_{pm,p+1}$ be $Z_{pm} \cup \bar{Z}_{p+1}$. There exist

$$LS(1, K_p, K_{pm}) = \{(Z_{pm}, \mathcal{P}_i) : 1 \leq i \leq \binom{pm-1}{p-1}\}$$

on Z_{pm}, where each $\mathcal{P}_i$ consists of m p-subsets of Z_{pm}, which forms a parallel class on Z_{pm}.

Let

$$\mathcal{Q}_j = \bar{Z}_{p+1} \setminus \{\bar{j}\},\ 0 \leq j \leq p,$$

then the family $\{\mathcal{Q}_j\}$ just forms a partition of all the p-subsets in $\bar{Z}_{p+1}$.

Define

$\mathcal{A}_i = \{[x; a_1, a_2, \cdots, a_p] : x \in \bar{Z}_{p+1}, \{a_1, a_2, \cdots, a_p\} \in \mathcal{P}_i\}, 2 \leq i \leq \binom{pm-1}{p-1}$.

$\mathcal{B}_j^1 = \{[y; b_1, b_2, \cdots, b_p] : y \in Z_{pm}, \{b_1, b_2, \cdots, b_p\} = \mathcal{Q}_j\}$,

$\mathcal{B}_j^2 = \{[\bar{j}; a_1, a_2, \cdots, a_p] : \{a_1, a_2, \cdots, a_p\} \in \mathcal{P}_1\}$,

Let

$$\mathcal{B}_j = \mathcal{B}_j^1 \cup \mathcal{B}_j^2,\ 0 \leq j \leq p.$$

Then it is easy to verify that each of $(Z_{pm} \cup \bar{Z}_{p+1}, \mathcal{A}_i)$ and $(Z_{pm} \cup \bar{Z}_{p+1}, \mathcal{B}_j)$ is a $(K_{pm,p+1}, K_{1,p})$-*GD* for $2 \leq i \leq \binom{pm-1}{p-1}$ and $0 \leq j \leq p$. So we have $\binom{pm-1}{p-1} - 1 + (p+1) = \binom{pm-1}{p-1} + p$ small sets, just as expected.

Furthermore, the family $\{\mathcal{A}_i : 2 \leq i \leq \binom{pm-1}{p-1}\} \cup \{\mathcal{B}_j^2 : 0 \leq j \leq p\}$ just forms a partition of all $K_{1,p}$-blocks in the form $[x; a_1, a_2, \cdots, a_p]$ (where $x \in \bar{Z}_{p+1}$, $\{a_1, a_2, \cdots a_p\}$ is a p-subset in Z_{pm}), and the family $\{\mathcal{B}_j^1 : 0 \leq j \leq p\}$ just forms a partition of all $K_{1,p}$-blocks in the form $[y; b_1, b_2, \cdots, b_p]$ (where $y \in Z_{pm}$, $\{b_1, b_2, \cdots, b_p\}$ is a p-subset in $\bar{Z}_{p+1}$). So $(\bigcup_i \mathcal{A}_i) \cup (\bigcup_j \mathcal{B}_j)$ forms a $(K_{pm,p+1}, K_{1,p})$-*LGD* on $Z_{pm} \cup \bar{Z}_{p+1}$. □

Theorem 3.4 *There exists a $(\lambda K_{pm,p+1}, K_{1,p})$-LGD if and only if $\lambda | [\binom{pm-1}{p-1} + p]$.*

Proof. By Lemma 2.1, we only need to prove the sufficiency.

By Lemma 3.3, there exists a

$(K_{pm,p+1}, K_{1,p})$-*LGD*$=\{(Z_{pm} \cup \bar{Z}_{p+1}, \mathcal{C}_i) : 1 \leq i \leq \binom{pm-1}{p-1} + p\}$.

Define

$$\mathcal{D}_k = \bigcup_{i=k\lambda+1}^{(k+1)\lambda} \mathcal{C}_i,\ 0 \leq k \leq \frac{\binom{pm-1}{p-1}+p}{\lambda} - 1,$$

then $\{(Z_{pm}\cup\bar{Z}_{p+1},\mathcal{D}_k):\ 0\le k\le \frac{\binom{pm-1}{p-1}+p}{\lambda}-1\}$ is just a $(\lambda K_{pm,p+1},K_{1,p})$-*LGD*. □

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