On Color Frames of Claws and Matchings

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Abstract

A red-blue coloring of a graph G is an edge coloring of G in which every edge of G is colored red or blue. Let F be a connected graph of size 2 or more with a red-blue coloring, at least one edge of each color, where some blue edge of F is designated as the root of F. Such an edge-colored graph F is called a color frame. An F-coloring of a graph G is a red-blue coloring of G in which every blue edge of G is the root edge of a copy of F in G. The F-chromatic index $\chi'_F(G)$ of G is the minimum number of red edges in an F-coloring of G. A minimal F-coloring of G is an F-coloring with the property that if any red edge of G is re-colored blue, then the resulting red-blue coloring of G is not an F-coloring of G. The maximum number of red edges in a minimal F-coloring of G is the upper F-chromatic index $\chi''_F(G)$ of G. In this paper, we study the two color frames Y_1 and Y_2 that result from the claw $K_{1,3}$, where Y_1 has exactly one red edge and Y_2 has exactly two red edges. For a graph G, let $\alpha'(G)$ and $\alpha''(G)$ denote the matching number and lower matching number of G, respectively. It is shown that if T is a tree of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(T) = \alpha''(T)$ while $\chi'_{Y_2}(T) \leq 3\alpha''(T)$ and this upper bound is sharp. For a color frame F of a claw, sharp bounds are established for $\chi''_F(G)$ in terms of the matching number and a generalized matching parameter of a graph G. Other results and questions are also presented.

1 Introduction

An area of graph theory that has received increased attention during recent decades is that of domination. Two books [8, 9] by Haynes, Hedetniemi and Slater are devoted to this subject. In 1999 a new way of looking at domination was introduced by Chartrand, Haynes, Henning and Zhang [2] that encompassed several of the best known domination parameters in the literature. This new view of domination was based on a concept introduced by Rashidi [20] in 1994. A graph G whose vertex set $V(G)$ is partitioned is a *stratified graph.* If $V(G)$ is partitioned into k subsets, then G is k-*stratified.* In particular, the vertex set of a 2-stratified graph is partitioned into two subsets. Typically, the vertices of one subset in a 2-stratified graph are considered to be colored red and those in the other subset are colored blue. A *red-blue coloring* of a graph G is an assignment of colors to the vertices of G, where each vertex is colored either red or blue. In a red-blue coloring, all vertices of G may be colored the same. A red-blue coloring in which at least one vertex is colored red and at least one vertex is colored blue thereby produces a 2-stratification of G. Let F be a 2-stratified graph in which some blue vertex ρ is designated as the root of F. The graph F is then said to *be rooted at* ρ. Since F is 2-stratified, F contains at least two vertices, at least one of each color. There may be blue vertices in F in addition to the root. By an F-*coloring* of a graph G, we mean a red-blue coloring of G such that for every blue vertex u of G, there is a copy of F in G with ρ at u. Therefore, every blue vertex u of G belongs to a copy F' of F rooted at u. A red vertex v in G is said to F-*dominate* a vertex u if $u = v$ or there exists a copy F' of F rooted at u and containing the red vertex v. The set S of red vertices in a red-blue coloring of G is an F-*dominating set* of G if every vertex of G is F-dominated by some vertex of S, that is, this red-blue coloring of G is an F-coloring. The minimum number of red vertices in an F-dominating set is called the F-*domination number* $\gamma_F(G)$ of G. An F-dominating set with $\gamma_F(G)$ vertices is a *minimum F-dominating set.* The F-domination number of every graph G is defined since $V(G)$ is an F-dominating set. This concept provides a generalization of domination and has been studied in many articles (see [6, 7] and [10] - [14] for example).

An edge version of this concept was introduced by Chartrand in 2011 and studied in [15, 16]. In this context, we refer to a red-blue coloring of a nonempty graph G as an edge coloring of G in which every edge is colored red or blue. Let F be a connected graph of size 2 or more with a red-blue coloring, at least one edge of each color. One of the blue edges of F is designated as the *root edge* of F. The *underlying graph* of F is the graph H obtained by removing the colors assigned to the edges of F. In this case, F is called a *color frame* of H. The simplest example of this is the unique color frame F_0 of the path P_3 in which one edge is red, the other is blue and the blue edge is its root edge shown in Figure 1, where a red edge is labeled r and a blue edge is labeled b. The five (distinct) color frames $F_1, F_2, \ldots, F_5$ of the path P_4 of size 3 are also shown in Figure 1, where each root edge is indicated by a double-line edge.

For a color frame F, an F-*coloring* of a graph G is a red-blue coloring of G in which every blue edge of G is the root edge of a copy of F in G. The F-*chromatic index* $\chi'_F(G)$ of G is the minimum number of red edges in an F-coloring of G. An F-coloring of G having exactly $\chi'_F(G)$ red edges is called a *minimum F-coloring* of G. Although these concepts are related

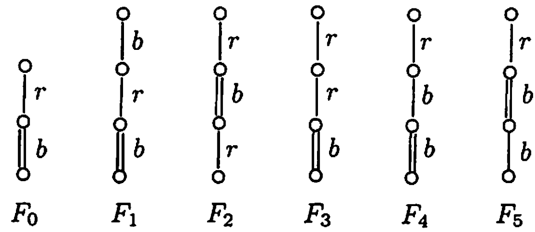

Figure 1: Color frames of P_3 and P_4

to the vertex concepts discussed earlier through the line graph of a graph, this fact, as with proper colorings, has provided no benefit in the study of F-colorings. It was shown in [16] that F-colorings and the F-chromatic indexes of graphs, where F is one of color frames $F_0, F_1, \ldots, F_5$ shown in Figure 1, provide a new framework for studying both edge independence (or matchings) and edge domination in graphs.

The graph $K_{1,3}$ is often referred to as a *claw*. There are two color frames of a claw, which are denoted by Y_1 and Y_2 and shown in Figure 2. The color frame Y_1 of a claw has exactly one red edge while Y_2 has exactly two red edges. In Y_1, there are therefore two blue edges and in Y_2 only one blue edge. By symmetry, we can choose either of the two blue edges in Y_1 as the root edge, while in Y_2, the only blue edge is the root edge of Y_2. The F-colorings where F is a color frame of a claw were studied by Chartrand, Johnston and Zhang in the paper [3]. A vertex version of F-colorings, where F is a 2-stratified graph of a claw were studied by Chartrand, Haynes, Henning and Zhang in the paper [1].

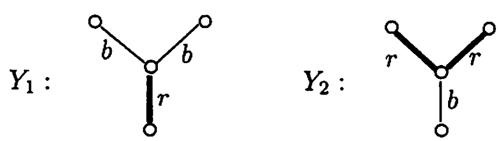

Figure 2: The two color frames of the claw $K_{1,3}$

It was observed in [3] that if G is a nonempty graph of size m, then

$$\chi'_{Y_1}(G) = \chi'_{Y_2}(G) = m \text{ if and only if } \Delta(G) \leq 2.$$

An edge e in a graph G is referred to as a *non-claw edge* if e belongs to no claw in G. Thus, if $e = uv$ is a non-claw edge, then $\max\{\deg u, \deg v\} \leq 2$. Necessarily, every non-claw edge must be colored red in every Y_i-coloring of G for $i = 1, 2$. The relationship among the Y_1-chromatic index, the Y_2-chromatic index and the number of non-claw edges in a graph was established in [3].

Theorem 1.1 *If G is a nontrivial connected graph containing ℓ non-claw edges, then*

$$\chi'_{Y_1}(G) \leq \chi'_{Y_2}(G) \leq 3\chi'_{Y_1}(G) - 2\ell.$$

By Theorem 1.1, if G is a connected graph of order at least 4 with $\chi'_{Y_1}(G) = a$ and $\chi'_{Y_2}(G) = b$, then $a \leq b \leq 3a$ and $b \geq 2$. It was shown in [3] that every pair a, b of positive integers with $a \leq b \leq 3a$ and $b \geq 2$ can be realized as $\chi'_{Y_1}(G)$ and $\chi'_{Y_2}(G)$, respectively, for some connected graph G of order at least 4.

Theorem 1.2 *For a pair a, b of positive integers, there exists a connected graph G of order at least 4 such that $\chi'_{Y_1}(G) = a$ and $\chi'_{Y_2}(G) = b$ if and only if with $a \leq b \leq 3a$ and $b \geq 2$.*

Among the concepts that are fundamental in graph theory is that of matchings. Lovász and Plummer have written a book [18] devoted to the theory of matchings. A set of edges in a graph G is *independent* if no two edges in the set are adjacent in G. The edges in an independent set of edges of G form a *matching* in G. A matching of maximum size in G is a *maximum matching.* The *matching number* $\alpha'(G)$ of G is the number of edges in a maximum matching of G. The number $\alpha'(G)$ is also referred to as the *edge independence number* of G. A matching M in a graph G is a *maximal matching* of G if M is not a proper subset of any other matching in G. While every maximum matching is maximal, a maximal matching need not be a maximum matching. The minimum number of edges in a maximal matching of G is called the *lower matching number* (or *lower edge independence number*) of G and is denoted by $\alpha''(G)$. Necessarily, $\alpha''(G) \leq \alpha'(G)$ for every graph G.

The concepts of matching number and lower matching number can be generalized as follows. For a positive integer k, a set X of edges of a graph G is a *Δ_k-set* if $\Delta(G[X]) = k$, where $G[X]$ is the subgraph of G induced by X. A maximum Δ_k-set in G is a Δ_k-set of maximum size and this size is denoted by $\alpha'_k(G)$. A Δ_k-set is *maximal* if for every edge $e \in E(G) - X$, $\Delta(G[X \cup \{e\}]) > k$. A maximal Δ_k-set of minimum size in G is denoted by $\alpha''_k(G)$. In particular, $\alpha'_1(G) = \alpha'(G)$ is the matching number of G and $\alpha''_1(G) = \alpha''(G)$ is the lower matching number of G. Since every maximum Δ_k-set is maximal, $\alpha''_k(G) \leq \alpha'_k(G)$. As indicated in [3], the concept of F-colorings, where F is a color frame of a claw, provides a new frame work of studying matchings in graphs. Among the results obtained in [3] is the following.

Theorem 1.3 *If G is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(G) \leq \alpha''(G)$ and $\chi'_{Y_2}(G) = \alpha''_2(G)$.*

It is conjectured that if G is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(G) = \alpha''(G)$. In Section 2, we verify this conjecture for trees. In Section 3, sharp lower bounds for $\chi''_{Y_i}(G)$ are established in terms of $\alpha'_i(G)$ for $i = 1, 2$ for a connected graph G of order at least 4 having no vertex of degree 2 and open questions are presented in Section 4. Before beginning this study, it is useful to establish some additional definitions and notation. For an F-coloring c of a graph G, let $E_{c,r}$ denote the set of red edges of G and $E_{c,b}$ the set of blue edges of G. (We also use E_r and E_b for $E_{c,r}$ and $E_{c,b}$, respectively, when the coloring c under consideration is clear.) Thus $\{E_r, E_b\}$ is a partition of the edge set $E(G)$ of G. Furthermore, let $G_r = G[E_r]$ denote the *red subgraph* induced by E_r and $G_b = G[E_b]$ the *blue subgraph* induced by E_b. Thus $\{G_r, G_b\}$ is a decomposition of G. If G is a disconnected graph with components G_1, G_2, $\ldots$, G_k where $k \geq 2$, then $\chi'_F(G) = \chi'_F(G_1) + \chi'_F(G_2) + \cdots + \chi'_F(G_k)$. Thus, it suffices to consider only connected graphs. We refer to the books [4, 5] for graph theory notation and terminology not described in this paper.

2 Color Frames of Claws in Trees

In this section, we first study Y_1-colorings in trees and show that if T is a tree of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(T) = \alpha''(T)$. In order to show this, we first present an additional definition and a lemma. Let C be a caterpillar of order at least 4 and let $(x_1, x_2, \ldots, x_d)$ be the spine of C. For each i with $1 \leq i \leq d$, let X_i be the set of end-vertices that are adjacent to x_i. Suppose that $|X_i| \geq 1$ for $1 \leq i \leq d$ and $d \geq 3$. Define a red-blue coloring c of C such that

(1) $c(x_i x_{i+1})$ is red for an odd integer i with $1 \leq i \leq d-1$ and $c(x_i x_{i+1})$ is blue for an even integer i with $2 \leq i \leq d-1$ and

(2) $c(x_i x)$ is red for all $x \in X_i$ if i is odd and $1 \leq i \leq d-1$ and $c(x_i x)$ is blue for all $x \in X_i$ if i is even and $2 \leq i \leq d-1$.

This edge-colored caterpillar C is then called a *red-blue caterpillar rooted at* x_1.

Lemma 2.1 *Let T be a tree of order at least 4 having no vertex of degree 2. If c is a minimum Y_1-coloring of T such that the red subgraph $G_{c,r}$ has the largest edge independence number among all minimum Y_1-colorings of T, then every non-end-vertex of T is incident with at least two blue edges.*

Proof. Assume, to the contrary, that there is a non-end-vertex u such that u is incident at most one blue edge. Suppose that $N(u) = \{u_1, u_2, \ldots, u_a\}$ where $a \geq 3$. We consider two cases.

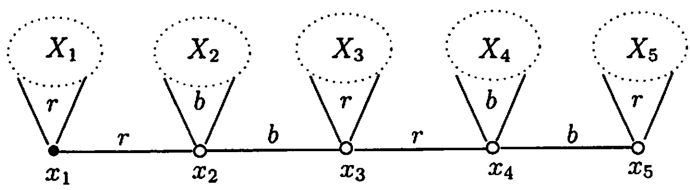

Figure 3: A red-blue caterpillar rooted at x_1 for $d = 5$

Case 1. *u is incident with exactly one blue edge.* First, suppose that there is u_i $(1 \le i \le a)$ such that u_i is an end-vertex. Note that uu_i cannot be colored blue, for otherwise, this blue edge does not belong to any copy of Y_1. Thus uu_i must be red. However then, since u is incident with a blue edge, we can change the color of uu_i to blue and the resulting coloring is also a Y_1-coloring. This is impossible since c is a minimum Y_1-coloring. Therefore, no vertex u_i $(1 \le i \le a)$ is an end-vertex of T.

For each i with $1 \le i \le a$, let $N(u_i) = \{u, u_{i,1}, u_{i,2}, \dots, u_{i,a_i}\}$ where then $a_i \ge 2$. We may assume, without loss of generality, that uu_a is blue and uu_1 and uu_2 are red. If there is p $(1 \le p \le a_1)$ such that $u_1u_{1,p}$ is red, then we can change the color of uu_1 to blue and the resulting coloring is also a Y_1-coloring. This is impossible since c is a minimum Y_1-coloring. Thus $u_1u_{1,p}$ is blue for all p with $1 \le p \le a_1$. Similarly, $u_2u_{2,q}$ is blue for all q with $1 \le q \le a_2$. If there is some p $(1 \le p \le a_1)$ such that $u_{1,p}$ is incident with no red edge, then we can interchange the colors of uu_1 and $u_1u_{1,p}$ to obtain a minimum Y_1-coloring whose red subgraph has a larger edge independence number, a contradiction. Thus each $u_{1,p}$ $(1 \le p \le a_1)$ must be incident with at least one red edge. (Similarly each $u_{2,q}$ $(1 \le q \le a_2)$ must be incident with at least one red edge.)

If every vertex $u_{1,p}$ $(1 \le p \le a_1)$ is incident with two or more blue edges, then we can change the color of uu_1 to blue and the resulting coloring is also a Y_1-coloring with fewer red edges, which is impossible since c is a minimum Y_1-coloring. Thus, there is some $u_{1,p}$ (say $u_{1,1}$) such that $u_{1,1}$ is incident with exactly one blue edge (namely, the blue edge $u_1u_{1,1}$). We now have a red-blue caterpillar C rooted at u with the spine $(x_1 = u, x_2 = u_1, x_3 = u_{1,1})$ such that

(i) $X_1 = \{u_2, \dots, u_{a-1}\}$ where $a \ge 3$ is the set of end-vertices adjacent to x_1 in C, $X_2 = \{u_{1,2}, \dots, u_{1,a_1}\}$ where $a_1 \ge 2$ is the set of end-vertices adjacent to x_2 in C and $X_3 = N(u_{1,1}) - \{u_1\}$ is the set of end-vertices adjacent to x_3 in C;

(ii) each vertex in $X_1 \cup X_2$ is not an end-vertex of T, that is, each end-vertex of C that is adjacent to x_i $(i = 1, 2)$ is not an end-vertex of T;

(iii) each edge incident with $x_1 = u$ in C is red, each edge incident with $x_2 = u_1$ is blue (except for x_1x_2) and each edge incident with $x_3 = u_{1,1}$ is red (except for x_2x_3).

If there is an end-vertex of C adjacent to x_3 that is an end-vertex of T, then this procedure stops. Otherwise, we consider $u_{1,1}$ in the same way as we consider u, that is, let $u_{1,1} = v$ and let $N(v) = \{u_1, v_1, v_2, \ldots, v_b\}$ where $b \geq 2$. We may assume that vv_1 and vv_2 are red.

For each i with $1 \leq i \leq b$, let $N(v_i) = \{v, v_{i,1}, v_{i,2}, \ldots, v_{i,b_i}\}$ where $b_i \geq 2$. Repeating the procedure as above, we may assume that $v_1v_{1,p}$ is blue for all p with $1 \leq p \leq b_1$ and $v_2v_{2,q}$ is blue for all q with $1 \leq q \leq v_2$. Furthermore, there is a vertex $v_{1,p}$ (say $v_{1,1}$) that is incident with exactly one blue edge. We now have a red-blue caterpillar rooted at u with the spine $(u = x_1, u_1 = x_2, u_{1,1} = x_3, v_1 = x_4, v_{1,1} = x_5)$, which is also denoted by C, such that no end-vertex of C adjacent to x_i $(1 \leq i \leq 4)$ is an end-vertex of T. If there is an end-vertex of C adjacent to x_5 that is an end-vertex of T, then this procedure stops. Otherwise, we continue and until we obtain a red-blue caterpillar C rooted at u with the spine $(x_1, x_2, \ldots, x_d)$ (where $x_1 = u$, $x_2 = u_1$, $x_3 = u_{1,1}$ and so on) and $d \geq 3$ is odd. For each i with $1 \leq i \leq d$, let X_i be the set of end-vertices of C that are adjacent to x_i. Thus X_i contains no end-vertex of T for $1 \leq i \leq d-1$ and X_d contains at least one end-vertex of T, say $x \in X_d$ is an end-vertex of T. Since the edge x_dx is red, we can change the color of x_dx to blue and the resulting coloring is also a Y_1-coloring with fewer red edges, a contradiction.

Case 2. *u is incident with no blue edge.* If u is adjacent to at least two end-vertices, say u_1 and u_2, then the coloring obtained from c by changing the color of uu_1 and uu_2 to blue is a Y_1-coloring with fewer red edges than c, which is impossible. Thus u is adjacent to at most one end-vertex. First, suppose that there is a vertex u_i $(1 \leq i \leq a)$ such that $u_iu_{i,s}$ is red and $u_iu_{i,t}$ is blue for some s, t with $1 \leq s, t \leq a_i$. Then the coloring obtained from c by changing the color of uu_i to blue is a Y_1-coloring with fewer red edges than c. This is impossible since c is a minimum Y_1-coloring. Thus for all i with $1 \leq i \leq a$, if u_i is not an end-vertex, then either all edges $u_iu_{i,j}$ are red for $1 \leq i \leq a_i$ or all edges $u_iu_{i,j}$ are blue for $1 \leq i \leq a_i$.

First, suppose that there are two vertices u_i and u_j $(1 \leq i \neq j \leq a)$ such that all edges $u_iu_{i,p}$ and $u_ju_{j,q}$ are red for $1 \leq p \leq a_i$ and $1 \leq q \leq a_j$, say $i = 1$ and $j = 2$. Then the coloring obtained by changing the colors of uu_1 and uu_2 to blue is a Y_1-coloring with fewer red edges, which is impossible. Hence, there is at most one vertex u_i $(1 \leq i \leq a)$ such that all edges $u_iu_{i,p}$ are red for $1 \leq p \leq a_i$. Thus, there is at least one vertex u_i $(1 \leq i \neq j \leq a)$ such that all edges $u_iu_{i,p}$ are blue for $1 \leq p \leq a_i$.

We claim, in fact, that there are two vertices u_i and u_j $(1 \leq i \neq j \leq a)$ such that all edges $u_iu_{i,p}$ and all edges $u_ju_{j,q}$ are blue for $1 \leq p \leq a_i$ and

$1 \leq q \leq a_j$. This is certainly the case if u is adjacent to no end-vertex. Thus, we may assume that u is adjacent to exactly one end-vertex, say u_a is an end-vertex. Thus u_1 and u_2 are not end-vertices and uu_1 and uu_2 are red. If there is p with $(1 \leq i \leq a_1)$ such that $u_1u_{1,p}$ is red, then we can change the colors of uu_1 and uu_a to blue and the resulting coloring is a Y_1-coloring with fewer red edges. Since this is impossible, all edges $u_1u_{1,p}$ are blue for all p with $1 \leq p \leq a_1$. Similarly, all edges $u_2u_{2,q}$ are blue for all q with $1 \leq q \leq a_2$. Therefore, as claimed, all edges $u_1u_{1,p}$ and $u_2u_{2,q}$ are blue for $1 \leq p \leq a_1$ and $1 \leq q \leq a_2$.

With an argument similar to the one used in Case 1, we obtain a red-blue caterpillar C rooted at u such that the spine of C is $(x_1, x_2, \ldots, x_d)$ (where $x_1 = u$, $x_2 = u_1$ and $x_3 = u_{1,1}$ and so on) and $d \geq 3$ is odd. For each i with $1 \leq i \leq d$, let X_i be the set of end-vertices of C that are adjacent to x_i. Thus X_i contains no end-vertex of T for $1 \leq i \leq d-1$ and X_d contains at least one end-vertex of T, say $x \in X_d$ is an end-vertex of T. Since x_dx is red, we can change the color of x_dx to blue and the resulting coloring is also a Y_1-coloring with fewer red edges, a contradiction. ∎

Theorem 2.2 *If T is a tree of order at least 4 having no vertex of degree 2, then*

$$\chi'_{Y_1}(T) = \alpha''(T).$$

Proof. By Theorem 1.3, it remains to show that $\chi'_{Y_1}(T) \geq \alpha''(T)$. Assume, to the contrary, that $\chi'_{Y_1}(T) = k \leq \alpha''(T) - 1$. Let c be a minimum Y_1-coloring of T such that the edge independence number of the red subgraph $G_{c,r}$ is maximum. By Lemma 2.1, every non-end-vertex of T is incident with at least two blue edges. We claim that $E_{c,r}$ is an independent set of edges of T. For otherwise, suppose that uv and vw are adjacent edges in $E_{c,r}$. By Lemma 2.1, v is incident with two blue edges. If u is an end-vertex of T, then the coloring obtained from c by changing the color of uv to blue is an Y_1-coloring with fewer red edges than c. This is impossible since c is a minimum Y_1-coloring. Thus u is not an end-vertex. Similarly, w is not an end-vertex of T. Thus, we may assume that $N(u) = \{v, u_1, u_2, \ldots, u_\alpha\}$, $N(v) = \{u, w, v_1, v_2, \ldots, v_\beta\}$ and $N(w) = \{v, w_1, w_2, \ldots, w_\gamma\}$, where $\alpha, \beta, \gamma \geq 2$. By Lemma 2.1, we may assume that uu_1, uu_2, vv_1, vv_2, ww_1, ww_2 are blue. If there exists i with $1 \leq i \leq \alpha$ such that uu_i is red, then the coloring obtained from c by changing the color of uv to blue is an Y_1-coloring with fewer red edges than c, which is impossible. Hence all edges uu_i $(1 \leq i \leq \alpha)$ are blue. Similarly, all edges ww_i $(1 \leq i \leq \gamma)$ are blue. If there is i $(1 \leq i \leq \alpha)$ such that u_i is not incident with any red edge, then the coloring c' obtained from c by interchanging the colors uu_i and uv is a minimum Y_1-coloring with a larger number of independent edges in $E_{c',r}$, which contradicts the defining property of c. Thus each vertex u_i $(1 \leq i \leq \alpha)$ is incident with at least one red edge. Similarly, each vertex w_i $(1 \leq i \leq \beta)$ is incident with at least one red edge. By Lemma 2.1, each u_i $(1 \leq i \leq \alpha)$ is incident with at least two blue edges. Thus, the coloring obtained from c by changing the color of uv to blue is a Y_1-coloring with fewer red edges than c, which is impossible. Thus, as claimed, $E_{c,r}$ is an independent set of edges of T. Since $|E_{c,r}| = k \leq \alpha''(T) - 1$, it follows that $E_{c,r}$ is not a maximal independent set of edges of T. Thus there is a blue edge $e \notin E_{c,r}$ such that $E_{c,r} \cup \{e\}$ is an independent set of edges of T. However then, the blue edge e is not incident with any red edge and so c is not an Y_1-coloring, which is a contradiction. ∎

The condition in Theorem 2.2 that T has no vertex of degree 2 is necessary. For example, let k be an arbitrary positive integer and let T be the tree obtained from $P_{3k+1} = (v_1, v_2, \ldots, v_{3k+1})$ by adding the pendant edge vv_2 at the vertex v_2. Then $\chi'_{Y_1}(T) = 3k - 1$ and $\alpha''(T) = k$ and so $\chi'_{Y_1}(T) - \alpha''(T) = 2k - 1$, which can be arbitrarily large. In fact, this is also true for trees without non-claw edges. To see this, we first construct the tree T_0 from the subdivision graph $S(K_{1,3})$ of $K_{1,3}$ by adding two pendant edges at each end-vertex of $S(K_{1,3})$. Suppose that the central vertex of $K_{1,3}$ is t and t is adjacent to three vertices u, v, w of degree 2. Furthermore, suppose that u is adjacent to x, v is adjacent to y and w is adjacent to z. Thus each of x, y, z is adjacent to two end-vertices in T. Observe that $\alpha''(T_0) = 3$ and $\{ux, vy, wz\}$ is a maximal matching in T_0. Since T_0 has four edge-disjoint copies of $K_{1,3}$, any two of which have only an end-vertex in common, a Y_1-coloring of T_0 must assign red to at least one edge in each of these four copies of $K_{1,3}$ and so $\chi'_{Y_1}(T_0) \geq 4$. On the other hand, the red-blue coloring c with $E_{c,r} = \{tu, ux, vy, wz\}$ is a Y_1-coloring and so $\chi'_{Y_1}(T_0) = 4$. Let $T_1, T_2, \ldots, T_k$ be k copies of T_0. For each i with $1 \leq i \leq k$, let v_i be a vertex of degree 2 in T_i that corresponds to v in T_0. The tree T is then construed from $T_1, T_2, \ldots, T_k$ by adding the edges $v_i v_{i+1}$ for $1 \leq i \leq k-1$ (see Figure 4 for $k = 3$). Although T contains vertices of degree 2, it contains no non-claw edges. It can be shown that $\chi'_{Y_1}(T) = 4k$ and $\alpha''(T) = 3k$. Therefore, $\chi'_{Y_1}(T) - \alpha''(T) = k$, which can be arbitrarily large.

Theorems 1.1 and 2.2 provide us an upper bound for $\chi'_{Y_2}(T)$ of a tree T in terms of $\alpha''(T)$.

Corollary 2.3 *If T is a tree of order at least 4 having no vertex of degree 2, then*

$$\chi'_{Y_2}(T) \leq 3\alpha''(T).$$

The upper bound in Corollary 2.3 is sharp. In fact, for each positive odd integer k, there is a tree T_k having no vertex of degree 2 such that

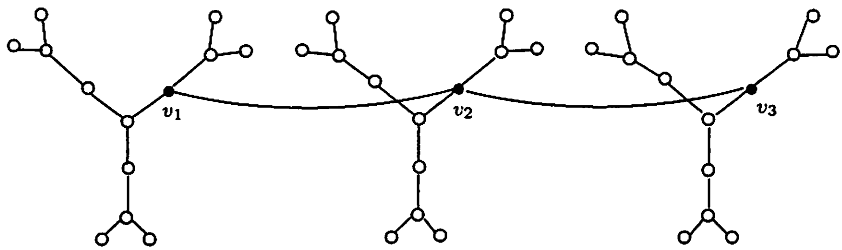

Figure 4: A tree T with $\chi'_{Y_1}(T) - \alpha''(T) = 3$

$\chi'_{Y_2}(T_k) = 3k$ and $\chi'_{Y_1}(T_k) = k$, as we show next. For each i with $1 \le i \le k$, let $S_i = S_{3,4}$ be the double star with central vertices u_i and v_i, where u_i is adjacent to the two end-vertices and v_i is adjacent to the three end-vertices one of which is w_i. If $k = 1$, let $T_1 = S_1$; while if $k \ge 3$, let $k = 2\ell + 1$ where $\ell \ge 1$ and we construct T_k in the following two steps: (1) For each i with $1 \le i \le \ell$, identifying w_i in S_i with w_{i+1} in S_{i+1} and labeling the identified vertex by x_i, resulting in a tree T' in which each x_i has degree 2 for $1 \le i \le \ell$; (2) For each i with $1 \le i \le \ell$, identifying $w_{\ell+1+i}$ in $S_{\ell+1+i}$ with the vertex x_i in T' constructed in (1), producing the tree T_k. The tree T_7 containing seven copies of $S_{3,4}$ is shown in Figure 5. Then T_k has the desired properties for each $k \ge 3$.

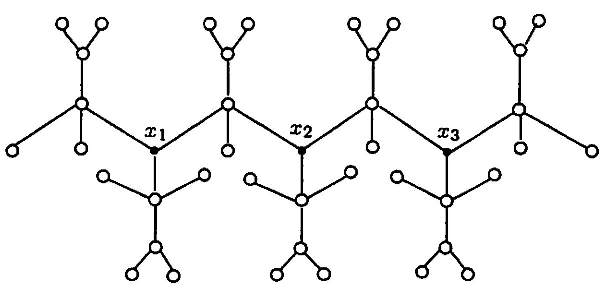

Figure 5: The tree T_7 with $\chi'_{Y_2}(T_7) = 21 = 3\chi'_{Y_1}(T_7)$

If G is a connected graph of order at least 4 having no vertex of degree 2 and c is a minimum Y_1-coloring of G, then the structure of the red subgraph induced by c can be determined. A graph H is a *galaxy* if each component of H is a star of order 2 or more.

Theorem 2.4 *Let G be a connected graph having no vertices of degree 2. If c is a minimum Y_1-coloring of G, then the red subgraph induced by c is a galaxy in G.*

Proof. Let c be a minimum Y_1-coloring of G and let $G_{c,r}$ be the red subgraph induced by c. Assume, to the contrary, that $G_{c,r}$ is not a galaxy. We then have the following two cases.

Case 1. *$G_{c,r}$ contains a path P of length* 3, *say* $P = (u, v, x, y)$. If vx is adjacent to a blue edge in G, then the red-blue coloring obtained from c by re-coloring vx blue is a Y_1-coloring with fewer red edges. This is a contradiction. Thus, we may assume that all edges adjacent to vx are red. Let $N(x) = \{v\} \cup \{x_1 = y, x_2, \ldots, x_p\}$, where then $p \geq 2$ and $xx_1, xx_2, \ldots, xx_p$ are all red edges in G. Hence x is incident with at least three red edges.

If there is some x_i $(1 \leq i \leq p)$ such that x_i is an end-vertex of G, say $x_i = x_1$, then the red-blue coloring obtained from c by re-coloring vx and xx_1 blue is a Y_1-coloring with fewer red edges, a contradiction. Thus $\deg_G x_i \geq 3$ for each i with $1 \leq i \leq p$. If there is some x_i $(1 \leq i \leq p)$ such that $\deg_{G_{c,r}} x_i \geq 2$, say $x_i = x_1$. Then the red-blue coloring obtained from c by re-coloring vx and xx_1 blue is a Y_1-coloring with fewer red edges. This is a contradiction. Thus $\deg_{G_{c,r}} x_i = 1$ for $1 \leq i \leq p$. For each integer i with $1 \leq i \leq p$, let $N(x_i) = \{x\} \cup \{x_{i,1}, x_{i,2}, \ldots, x_{i,q_i}\}$, where then $q_i \geq 2$ and $x_ix_{i,1}, x_ix_{i,2}, \ldots, x_ix_{i,q_i}$ are all blue for each i with $1 \leq i \leq p$.

Consider the vertex x_1. First, suppose that there is some j with $1 \leq j \leq q_1$ such that all edges incident with $x_{1,j}$ are blue, say $x_{1,j} = x_{1,1}$. Then the red-blue coloring c^* obtained from c by (1) interchanging the colors of xx_1 and $x_1x_{1,1}$ and (2) re-coloring vx to blue is a Y_1-coloring with fewer red edges, which is a contradiction. Next, suppose that every vertex $x_{1,j}$ is incident with at least one red edge for all j with $1 \leq j \leq q_1$. If there is j_0 with $1 \leq j_0 \leq q_1$ such that x_{1,j_0} is incident with exactly one blue edge (namely x_1x_{1,j_0}), then the red-blue coloring obtained by (1) interchanging the colors of xx_1 and x_1x_{1,j_0} and (2) re-coloring vx to blue is a Y_1-coloring with fewer red edges, which is a contradiction. Thus each vertex $x_{1,j}$ $(1 \leq j \leq q_1)$ is incident with at least two blue edges. Then the red-blue coloring obtained by re-coloring vx and xx_1 to blue is a Y_1-coloring with fewer red edges, which is a contradiction.

Case 2. *$G_{c,r}$ contains a 3-cycle C, say* $C = (u, v, w, u)$. Then each of u, v, w has degree at least 3. If one of u, v, w is incident with a blue edge, say, u is incident with a blue edge, then the red-blue coloring obtained from c by re-coloring uv blue is a Y_1-coloring with fewer red edges. This is a contradiction. Thus each of u, v, w is only incident with red edges and so each of u, v, w is incident with at least three red edges. However then, the red-blue coloring obtained from c by re-coloring uv, uw, vw to blue is a Y_1-coloring with fewer red edges, a contradiction.

By Cases 1 and 2, it follows that $G_{c,r}$ contains no path of length 3 and no 3-cycle, which implies that each component of $G_{c,r}$ is a star. Therefore,

$G_{c,r}$ is a galaxy. ∎

3 Minimal Y-Colorings

For a given color frame F, an F-coloring c of a graph G is a *minimal F-coloring* of G if no proper subset of $E_{c,r}$ is the set of red edges of an F-coloring of G. Thus a minimal F-coloring has the property that if any red edge of G is re-colored blue, then the resulting red-blue coloring of G is not an F-coloring of G. For example, a minimal Y_1-coloring of a tree with 7 red edges is shown in Figure 6, where each red edge is drawn in a bold line. The maximum number of red edges in a minimal F-coloring of G is the *upper F-chromatic index* $\chi''_F(G)$ of G. Since every minimum F-coloring of a graph G is minimal, $\chi'_F(G) \leq \chi''_F(G)$. For the tree T of Figure 6, $\chi''_{Y_1}(T) = 7$ and it follows by Theorem 2.2 that $\chi'_{Y_1}(T) = \alpha''(T) = 6$. The concepts of minimal F-colorings and upper F-chromatic indexes were introduced and studied in [16].

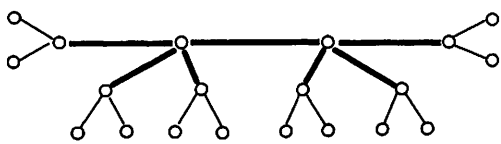

Figure 6: A minimal Y_1-coloring of a tree T

By Theorem 2.4, the red subgraph induced by a minimum Y_1-coloring in a connected graph having no vertices of degree 2 is a galaxy; while this may not be the case for a minimal Y_1-coloring. For example, the red subgraph induced by the minimal Y_1-coloring shown in Figure 6 is a double star.

By Theorem 1.3, if G is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(G) \leq \alpha''(G)$. We now show that $\chi''_{Y_1}(G) \geq \alpha'(G)$ for such a graph G.

Theorem 3.1 *If G is a connected graph having no vertices of degree 2, then*

$$\chi''_{Y_1}(G) \geq \alpha'(G).$$

Proof. Let M be a maximum matching. Then $|M| = \alpha'(G)$. Consider a red-blue coloring c of G that assigns the color red to every edge in M and the color blue to all other edges of G. Let $e = uv$ be a blue edge in G. Since M is a maximum matching, $M \cup \{e\}$ is not a matching and so either u or v is incident with at least one red edge in M, say v is incident with a red edge vw. Since G has no vertices of degree 2, it follows that $\deg v \geq 3$ and so v is incident two or more blue edges. Thus c is a Y_1-coloring. Next, we show that c is minimal. Let c' be a red-blue coloring obtained from c by changing the color of an edge $f \in M$ to blue. However then, the blue edge f in c' is not adjacent to any red edge in $M - \{f\}$ and so c' is not a Y_1-coloring. Therefore, c is minimal Y_1-coloring with $\alpha'(G)$ red edges and so $\chi''_{Y_1}(G) \geq |M| = \alpha'(G)$. ■

The lower bound in Theorem 3.1 is sharp. In order to show this, we determine the upper Y_1-chromatic index of the corona of n-cycle. It is shown in [3] that $\chi'_{Y_1}(G) = \lceil n/2 \rceil$ if G is the corona of an n-cycle where $n \geq 3$.

Proposition 3.2 *If G is the corona of an n-cycle where $n \geq 3$, then $\chi''_{Y_1}(G) = \alpha'(G)$.*

Proof. Let $G = cor(C_n)$ where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ for some integer $n \geq 3$. Suppose that u_iv_i is a pendant edge of G at v_i for $1 \leq i \leq n$. Since the order of G is $2n$ and $\{u_iv_i : 1 \leq i \leq n\}$ is a matching in G, it follows that $\alpha'(G) = n$. By Theorem 3.1, $\chi''_{Y_1}(G) \geq \alpha'(G) = n$. It remains to show that $\chi''_{Y_1}(G) \leq n$. Let c be a minimal Y_1-coloring of G with $|E_{c,r}| = \chi''_{Y_1}(G)$ and let G_r be the red subgraph induced by c. We claim that each vertex of C_n is incident with exactly one red edge in G_r. First, suppose that there is $v_i \in V(C_n)$ where $1 \leq i \leq n$ such that v_i is incident with no red edge of G_r. Then the blue edge v_iu_i does not belong to any copy of Y_1, which is impossible. Next, suppose that there is $v_j \in V(C_n)$ where $1 \leq j \leq n$ such that v_j is incident with at least two red edges of G_r, say $j = 1$. If the two red edges are v_nv_1 and v_1v_2, then the blue edge v_1u_1 does not belong to any copy of Y_1, a contradiction. If, on the other hand, one of these two red edges is v_1u_1, then the red-blue coloring obtained from c by changing the color of v_1u_1 to blue is a Y_1-coloring of G with fewer red edges, which is again a contradiction. Therefore, as claimed, every vertex of C_n is incident with exactly one red edge in G_r. This implies that $E_{c,r}$ is an independent set of edges in G and so $\chi''_{Y_1}(G) = |E_{c,r}| \leq \alpha'(G) = n$. Therefore, $\chi''_{Y_1}(G) = n$. ■

The value of $\chi''_{Y_1}(G) - \alpha'(G)$ can also be arbitrarily large for a connected graph G, as we show next. For each positive integer k, let $W_{6k} = C_{6k} + K_1$ be the wheel of order $6k + 1$, where the vertex v of W_{6k} is adjacent to every vertex of $C_{6k+1} = (v_1, v_2, \ldots, v_{6k}, v_{6k+1} = v_1)$ in W_{6k}. Let G_k be the graph obtained from W_{6k} by adding edges k edges v_iv_{i+3k} for each integer i with $i \equiv 2 \pmod 3$ and $2 \leq i \leq 3k - 1$. The order of G_k is $6k+1$. The graph G_2 of order 13 is shown in Figure 7. Since $M = \{v_iv_{i+1} : i \text{ is odd and } 1 \leq i \leq 6k - 1\}$ is a maximum matching in G_k, it follows that $\alpha'(G_k) = 3k$. The red-blue coloring c of G_k defined by

$$E_{c,r} = \{v_iv_{i+1}, v_{i+1}v_{i+2} : i \equiv 1 \pmod 3 \text{ and } 1 \leq i \leq 6k - 2\}$$

is a minimal Y_1-coloring of G_k. (The coloring c is shown in Figure 7 for $k = 2$.) Thus $\chi''_{Y_1}(G_k) \geq |E_{c,r}| = 4k$. Therefore, $\chi''_{Y_1}(G_k) - \alpha'(G_k) \geq k$, which can be arbitrarily large.

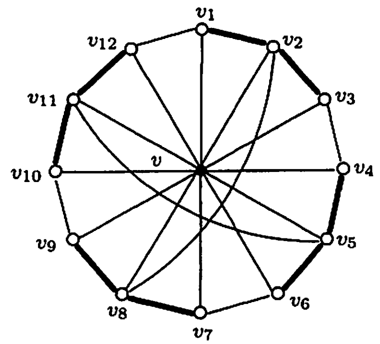

Figure 7: The graph G_2 and a minimal Y_1-coloring of G_2

The argument employed in the proof of Proposition 3.1 also shows that a *maximal* matching M in a graph G gives rise to a *minimal* Y_1-coloring c of G such that $E_{c,r} = M$.

Corollary 3.3 *If G is a connected graph order at least 4 having no vertices of degree 2 and M is maximal matching, then the Y_1-coloring c of G with $E_{c,r} = M$ is a minimal Y_1-coloring.*

The converse of Corollary 3.3 is not true in general; that is, there are minimal Y_1-colorings c of a connected graph order at least 4 having no vertices of degree 2 such that $E_{c,r}$ is not even a matching, as the graph of Figure 7 shows.

By Theorem 1.3, if G is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_2}(G) = \alpha''_2(G)$. By an argument similar to the proof of Theorem 3.1, we now show that $\chi''_{Y_2}(G) \geq \alpha'_2(G)$ for every such connected graph G.

Theorem 3.4 *If G is a connected graph having no vertices of degree 2, then*

$$\chi''_{Y_2}(G) \geq \alpha'_2(G).$$

Proof. Let X be Δ_2-set of maximum size where then $|X| = \alpha'_2(G)$. Consider a red-blue coloring c of G that assigns the color red to every edge in X and the color blue to all other edges of G. Let $e = uv$ be a blue edge in G. Since X is a maximum Δ_2-set, $X \cup \{e\}$ is not a Δ_2-set and so either u or v is incident with at least two red edge in X and e belongs to a copy of Y_2. Thus c is a Y_2-coloring. Next, we show that c is minimal. Let X' be a proper subset of X and let c' be the red-blue coloring of G such that $E_{c',r} = X'$. Now let $f \in X - X'$ be a blue edge in c'. Since $\Delta(G[X]) = 2$, it follows that the blue edge f in c' is not adjacent to two red edges in $X - \{f\}$ and so f does not belong to a copy of Y_2. Hence c' is not a Y_2-coloring. Therefore, c is minimal Y_2-coloring with $\alpha_2'(G)$ red edges and so $\chi''_{Y_2}(G) \geq |X| = \alpha_2'(G)$. ■

To show that the lower bound in Theorem 3.4 is sharp, we determine the upper Y_2-chromatic index of the corona of n-cycle. It is shown in [3] that $\chi'_{Y_2}(G) = n$ if G is the corona of an n-cycle where $n \geq 3$.

Proposition 3.5 *If G is the corona of an n-cycle where $n \geq 3$, then*

$$\chi''_{Y_2}(G) = \alpha_2'(G) = 2n - \lceil n/2 \rceil.$$

Proof. Let $G = cor(C_n)$ where $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ for some integer $n \geq 3$. Suppose that u_iv_i is the pendant edge of G at v_i for $1 \leq i \leq n$. We first show that $\chi''_{Y_2}(G) = 2n - \lceil n/2 \rceil$. First, we show that there is a minimal Y_2-coloring c of G having exactly $2n - \lceil n/2 \rceil$ red edges. For an even integer $n \geq 4$, let

$$E_{c,r} = \{v_iv_{i+1} : \ i \text{ is odd}, 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\} \qquad (1)$$

and so $E_{c,b} = \{v_iv_{i+1} : \ i \text{ is even}, 2 \leq i \leq n\}$. For an odd integer $n \geq 3$, let

$$E_{c,r} = \{v_iv_{i+1} : \ i \text{ is odd}, 1 \leq i \leq n\} \cup \{u_iv_i : 2 \leq i \leq n\} \qquad (2)$$

and so $E_{c,b} = \{v_iv_{i+1} : \ i \text{ is even}, 2 \leq i \leq n-1\} \cup \{u_1v_1\}$. (This coloring is shown in Figure 8 for $n = 8$ and $n = 9$.) Then $|E_{c,b}| = \lceil n/2 \rceil$ and so $\chi''_{Y_2}(G) \geq |E_{c,r}| = 2n - \lceil n/2 \rceil$.

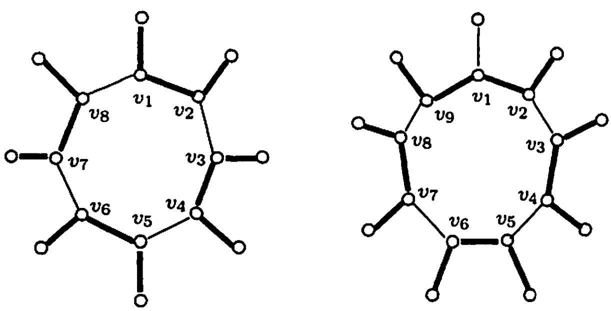

Figure 8: Illustrate the coloring c for $cor(C_8)$ and $cor(C_9)$

Next, we show that $\chi''_{Y_2}(G) \leq 2n - \lceil n/2 \rceil$. Assume, to the contrary, that

$$\chi''_{Y_2}(G) = t \geq 2n - \lceil n/2 \rceil + 1. \qquad (3)$$

Let c^* be a minimal Y_2-coloring of G having exactly t red edges and let G_r be the red subgraph induced by c^*. Thus the size of G_r is t. First, suppose that G_r contains a vertex v such that $\deg_{G_r} = 3$, say $v = v_2$ and v_1v_2, v_2u_2 and v_2v_3 are red. Then the red-blue coloring obtained from c^* by changing the color of v_2u_2 to blue is an Y_2-coloring, which is impossible. Thus $\deg_{G_r} v \leq 2$ for every vertex v of G_v. Since (i) the order of G_r is at most $2n$ and (ii) at most n vertices in G_r have degree 2 and the remaining vertices of G_r are end-vertices, the size t of G_r is at most $1/2(2n+n) = n+n/2$. By (3), $2n - \lceil n/2 \rceil + 1 \leq t \leq n + n/2$ or $n/2 + 1 \leq \lceil n/2 \rceil$, which is impossible. Therefore, $\chi''_{Y_2}(G) = 2n - \lceil n/2 \rceil$.

It remains to show that $\alpha'_2(G) = 2n - \lceil n/2 \rceil$. If n is even, then the subgraph induced by the set $E_{c,r}$ described in (1) is a Δ_2-set in G; while is n is odd, then the subgraph induced by the set $E_{c,r}$ described in (2) is a Δ_2-set in G. Thus $\alpha'_2(G) \geq |E_{c,r}| = 2n - \lceil n/2 \rceil$. Since $\alpha'_2(G) \leq \chi''_{Y_2}(G) = 2n - \lceil n/2 \rceil$ by Theorem 3.4, it follows that $\chi''_{Y_2}(G) = \alpha'_2(G)$. ∎

The value of $\chi''_{Y_2}(G) - \alpha'_2(G)$ can also be arbitrarily large for a connected graph G, as we show next. For each integer $k \geq 3$, let H and H' be two copy of $K_{2,k}$, where

$$V(H) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_k\} \text{ and } V(H') = \{u'_1, u'_2\} \cup \{v'_1, v'_2, \ldots, v'_k\}.$$

Let $U = \{u_1, u_2\}$, $V = \{v_1, v_2, \ldots, v_k\}$, $U' = \{u'_1, u'_2\}$ and $V' = \{v'_1, v'_2, \ldots, v'_k\}$. The graph H_k is obtained from H and H' by adding k new vertices in $W = \{w_1, w_2, \ldots, w_k\}$ and joining each w_i $(1 \leq i \leq k)$ to every vertex in $V \cup V'$. The order of H_k is $3k+4$. (The graph H_3 is shown in Figure 9.) Define a red-blue coloring c with $E_{c,r} = E(H) \cup E(H')$. The coloring c is shown in Figure 9 for $k = 3$. Since c is a minimal Y_2-coloring of H_k, it follows that $\chi''_{Y_2}(H_k) \geq |E_{c,r}| = 4k$. Next, let $X_k = \{v_iw_i, w_iv'_i : 1 \leq i \leq k\}$ and let

$$Y_k = \begin{cases} X_k \cup \{u_1v_1, u_1v_2, u_2v_3, u'_1v'_1, u'_1v'_2, u'_2v'_3\} & \text{if } k = 3 \\ X_k \cup \{u_1v_1, u_1v_2, u_2v_3, u_2v_4, u'_1v'_1, u'_1v'_2, u'_2v'_3, u'_2v'_4\} & \text{if } k \geq 4. \end{cases}$$

Since Y_k is a maximal Δ_2-set of H_k, it follows that $\alpha''_2(H_k) \leq |Y_k| = 2(k + \min\{k, 4\})$. Thus $\chi''_{Y_2}(H_k) - \alpha''_2(H_k) \geq 4k - 2(k + \min\{k, 4\}) = 2(k - \min\{k, 4\})$, which can be arbitrarily large.

4 Closing Statements

It was shown in [17] that if G is a graph and k is an integer with $\alpha''(G) \leq k \leq \alpha'(G)$, then G contains a maximal matching with k edges. It can be shown that if $Y \in \{Y_1, Y_2\}$ is a color frame of a claw and G is the corona of an n-cycle where $n \geq 3$, then for each integer k with $\chi'_Y(G) \leq k \leq \chi''_Y(G)$,

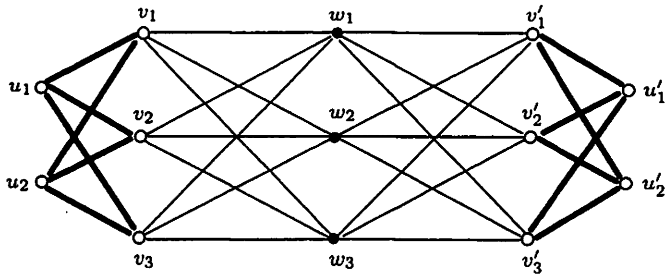

Figure 9: The graph H_3 and a minimal Y_2-coloring of H_3

there is a minimal Y-coloring of G using exactly k red edges. It gives rise to the following question.

Problem 4.1 *Let $Y \in \{Y_1, Y_2\}$ be a color frame of a claw. If G is a connected graph of order at least 4 and k is an integer with $\chi'_Y(G) \leq k \leq \chi''_Y(G)$, is there a minimal Y-coloring of G using exactly k red edges?*

For a connected graph G and a color frame F, if $\chi'_F(G) = a$ and $\chi''_F(G) = b$, then $a \leq b$ by the definitions of the F-chromatic index and upper F-chromatic index of G. Thus we conclude this paper with another question.

Problem 4.2 *Let $Y \in \{Y_1, Y_2\}$ be a color frame of a claw. For which pairs a, b of positive integers with $a \leq b$, does there exist a connected graph G such that $\chi'_Y(G) = a$ and $\chi''_Y(G) = b$?*

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