Roman domination critical graphs upon edge subdivision

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Abstract

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A graph G is Roman domination critical upon edge subdivision if the Roman domination number increases whenever an edge is subdivided. In this paper we study the Roman domination critical graphs upon edge subdivision. We present several properties, bounds and general results for these graphs.

Keywords: Roman domination; Subdivision, Critical.

2000 Mathematical subject classification: 05C69.

1 Introduction

For notation and graph theory terminology, we in general follow [8]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and size $|E(G)| = m$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N(v)$. The degree of v is $\deg_G(v) = |N_G(v)|$. If the graph G is clear from the context, then we simply write $N(v)$ and $\deg(v)$ rather than $N_G(v)$ and $d_G(v)$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A set of vertices S in G is a *dominating set*, (or just DS), if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a DS of G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A set of vertices S in G is an *independent dominating set*, if S is a DS and the induced subgraph $G[S]$ has no edge. The *independent domination number*, $i(G)$, of G is the minimum cardinality of an independent dominating set of G. A set $S \subseteq V(G)$ is a 2-packing if for every two different vertices $x, y \in S$, $N[x] \cap N[y] = \emptyset$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A Roman dominating function $f : V(G) \to \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of $V(G)$, (or (V_0^f, V_1^f, V_2^f) to refer to f), where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$. A function $f = (V_0, V_1, V_2)$ is called a γ_R-function (or $\gamma_R(G)$-function to refer to G), if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$. Roman domination has been studied, for example, in [1, 2, 4, 5, 6, 9].

Independent Roman domination in graphs was studied by Adabi et al. in [1]. An RDF $f = (V_0, V_1, V_2)$ in a graph G is an independent RDF, or just IRDF, if $V_1 \cup V_2$ is independent. The *independent Roman domination number*, $i_R(G)$, is the minimum weight of an IRDF of G. An IRDF with minimum weight is called an i_R-function.

Unique response Roman domination in graphs was studied by Ebrahimi et al. in [5]. A function $f : V(G) \longrightarrow \{0, 1, 2\}$ with ordered partition (V_0, V_1, V_2) is a *unique response Roman function* if $x \in V_0$ implies that $|N(x) \cap V_2| \leq 1$ and $x \in V_1 \cup V_2$ implies that $|N(x) \cap V_2| = 0$. A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a *unique response Roman dominating function* if it is a unique response Roman function and a Roman dominating function. The *unique response Roman domination number* of G, denoted by $u_R(G)$, is the minimum weight of a unique response Roman dominating function.

A *cycle* on n vertices is denoted by C_n, while a *path* on n vertices is denoted by P_n. We denote by K_n the *complete graph* on n vertices. An *r-partite graph* G is a graph whose vertex set $V(G)$ can be partitioned into r sets of pair-wise non-adjacent vertices. For positive integers $p_1, p_2, \ldots, p_r$, the *complete r-partite graph* $K_{p_1,p_2,\ldots,p_r}$ is the r-partite graph with partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ such that $|V_i| = p_i$ for $1 \leq i \leq r$ and such that every two vertices belonging to different partition sets are adjacent to each other. A *star* is a complete bipartite graph of the form $K_{1,n}$, and a *double star* is a graph obtained from two stars by joining their centers. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. A *strong* support vertex is a vertex which is adjacent to at least two leaves.

The *subdivision* of an edge uv is the operation of replacing uv with a path uwv through a new vertex w. Given a graph G and an edge $e \in E(G)$, we denote by G^e the graph obtained from G by subdividing the edge e.

Roman domination vertex critical graphs were studied by Hansberg et al. [6]. A graph G is *Roman domination vertex critical*, or just γ_R-vertex critical, if for any vertex v of $V(G)$, $\gamma_R(G-v) < \gamma_R(G)$.

Theorem 1 *[6] For a vertex v in a graph G, $\gamma_R(G-v) < \gamma_R(G)$ if and only if there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ such that $v \in V_1$.*

The concept of Roman domination critical graphs upon edge addition, edge removal and edge contraction has been studied, [3, 6, 7, 9, 10]. In this paper we will study Roman domination critical graphs upon subdivision of an edge. A graph G is *Roman domination critical upon edge subdivision* if the Roman domination number increases whenever an edge is subdivided. We present several properties, bounds and general results for these graphs.

The *corona* $cor(G)$ of a graph G is a graph obtained from G by attaching a leaf to each vertex. The *2-corona* of a graph H, denoted by $H \circ P_2$, is the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex-disjoint.

2 General results and bounds

We begin by investigating which effects the subdivision of an edge has on the on the Roman domination number.

Proposition 2 *For any edge e in a graph G, $\gamma_R(G) \leq \gamma_R(G^e) \leq \gamma_R(G) + 1$.*

Proof. Let $e = xy \in E(G)$, and xwy be the subdivision of e. If f is any $\gamma_R(G)$-function, then let g be the function defined by

$g(w) = 1$ and $g(u) = f(u)$ for $u \neq w$. Clearly, g is an RDF for G^e, implying that $\gamma_R(G^e) \leq \gamma_R(G) + 1$. Now let g be a $\gamma_R(G^e)$-function. If $g(w) \neq 2$, then $g|_{V(G)}$ is an RDF for G implying that $\gamma_R(G) \leq \gamma_R(G^e)$. Thus assume that $g(w) = 2$. Then h defined on $V(G)$ by $h(u) = \max\{g(u), 1\}$ if $u \in \{x, y\}$ and $h(u) = g(u)$ if $u \notin \{x, y\}$ is an RDF for G, implying that $\gamma_R(G) \leq \gamma_R(G^e)$. ■

We call a graph G *Roman domination critical upon edge subdivision*, or just γ_{Rsd}-critical, if $\gamma_R(G^e) > \gamma_R(G)$ for any edge $e \in E(G)$. Thus if G is γ_{Rsd}-critical, then for any edge e, $\gamma_R(G^e) = \gamma_R(G)+1$. If G is a γ_{Rsd}-critical graph and $\gamma_R(G) = k$, then we call G, $k - \gamma_{Rsd}$-critical. In the case that a graph G has no edge, we define it to be γ_{Rsd}-critical.

Observation 3 *A disconnected graph G is γ_{Rsd}-critical if and only if every component of G is γ_{Rsd}-critical.*

Proposition 4 *Any connected graph G of order $n \geq 3$ with $\gamma_R(G) \in \{2, 3\}$ is γ_{Rsd}-critical.*

Proof. Let G be a connected graph of order $n \geq 3$, and $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function. Assume that $\gamma_R(G) = 2$. Then $V_1^f = \emptyset$ and $|V_2^f| = 1$. Let $V_2^f = \{x\}$. Then $\deg(x) = n - 1$. Let $e \in E(G)$. If $e = xy$, where $y \in N(x)$, and xwy is the subdivision of e in G^e, then for any $\gamma_R(G^e)$-function g, $g(x) + g(y) + g(w) \geq 2$, and since $n \geq 3$, we find that $w(g) \geq 3$. Similarly if $e = yz$, where $y, z \in N(x)$, then $\gamma(G^e) \geq 3$. By Proposition 2, $\gamma_R(G^e) = 3$ and thus G is γ_{Rsd}-critical.
Next assume that $\gamma_R(G) = 3$. Then $|V_1^f| = |V_2^f| = 1$, and clearly $\Delta(G) < n - 1$. Let $V_2^f = \{x\}$, $V_1^f = \{y\}$, and $e \in E(G)$. If $e = xz$, where $z \in N(x)$, and xwz is the subdivision of e in G^e, then for any $\gamma_R(G^e)$-function g, $g(x) + g(z) + g(w) \geq 2$, and since $\deg(x) < n-1$ and $\deg(z) < n-1$, we find that $w(g) \geq 4$. Similarly, if $e = zy$ or $e = zt$, where $z \in N(x) \cap N(y)$ and $t \in$

$N(x)$, then for any $\gamma_R(G^e)$-function g, $w(g) \geq 4$. Consequently, $\gamma(G^e) \geq 4$. By Proposition 2, $\gamma_R(G^e) = 4$ and thus G is γ_{Rsd}-critical. ■

Corollary 5 *For any $n \geq 3$, K_n is γ_{Rsd}-critical.*

However a graph G of order n with $\gamma_R(G) = 4$ is not necessarily γ_{Rsd}-critical. Let G be a double star with x, y as its two central vertices such that $\deg(x) \geq 3$ and $\deg(y) \geq 3$. Then $\gamma_R(G) = \gamma_R(G^{xy}) = 4$.

Proposition 6 *For any $k \geq 4$, there is a γ_{Rsd}-critical graph G with $\gamma_R(G) = k$.*

Proof. Let $m = \lfloor \frac{k}{2} \rfloor$. Let G be a graph obtained from K_m by adding at least three leaves to each vertex of K_m, and then subdividing a pendant edge if k is odd. Then it can be easily seen that $\gamma_R(G) = k$, and G is not γ_{Rsd}-critical. ■

Lemma 7 *[4] For paths P_n and cycles C_n, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$.*

In the next lemma we present some classes of γ_{sd}-critical graphs.

Lemma 8 *(1) A path P_n is γ_{Rsd}-critical if and only if $n \not\equiv 2$ (mod 3).*
(2) A cycle C_n is γ_{Rsd}-critical if and only if $n \not\equiv 2$ (mod 3).
(3) If $n_1 \leq n_2 \leq ... \leq n_k$, then $K_{n_1,n_2,...,n_k}$ is γ_{Rsd}-critical if and only if $n_1 \leq 2$.

Proof. (1) and (2) follow from Lemma 7. Note that $\lceil \frac{2n}{3} \rceil = \lceil \frac{2(n+1)}{3} \rceil$ if and only if $n \not\equiv 2$ (mod 3).

(3) Assume that $G = K_{n_1,n_2,\ldots,n_k}$ is γ_{Rsd}-critical. Let $X_1, X_2, \ldots, X_k$ be the partite sets of $G = K_{n_1,n_2,\ldots,n_k}$. Suppose that $n_1 \geq 3$. Thus $\gamma_R(G) = 4$. Let $e = xy$ be an arbitrary edge of G. Then $(V(G) - \{x,y\}, \emptyset, \{x,y\})$ is an RDF for G^e implying that $\gamma_R(G) = \gamma_R(G^e)$, a contradiction. Thus $n_1 \leq 2$. Conversely, assume that $n_1 \leq 2$. If $n_1 = 1$ then $\gamma_R(G) = 2$, and if $n_1 = 2$ then $\gamma_R(G) = 3$. By Proposition 4, G is γ_{Rsd}-critical. ∎

In the following we give a characterization for γ_{Rsd}-critical graphs.

Theorem 9 *A graph G is γ_{Rsd}-critical if and only if for every $\gamma_R(G)$-function (V_0, V_1, V_2), V_1 is independent and V_2 is a 2-packing.*

Proof. ($\Longrightarrow$) Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function. If V_1^f is not independent then we let x, y be two adjacent vertices in $G[V_1^f]$, and xwy be the subdivision of xy in G^{xy}. Then g defined on G^{xy} by $g(w) = 2$, $g(x) = g(y) = 0$, and $g(u) = f(u)$ if $u \notin \{x, y, w\}$, is an RDF for G^{xy} with weight $\gamma_R(G)$, a contradiction. Thus V_1^f is independent. Next assume that V_2^f is not a 2-packing. Then there are two vertices $x, y \in V_2^f$ such that $N[x] \cap N[y] \neq \emptyset$. If $x \in N(y)$, then g defined on G^{xy} by $g(w) = 0$ and $g(u) = f(u)$ if $u \neq w$, is an RDF for G^{xy}, a contradiction. Thus $x \notin N(y)$. Let $z \in N(x) \cap N(y)$. Then g defined on G^{xz} by $g(w) = 0$ and $g(u) = f(u)$ if $u \neq w$, is an RDF for G^{xz}, a contradiction. Thus V_2^f is a 2-packing.

($\Longleftarrow$) Suppose that G is not γ_{Rsd}-critical. Then there is an edge $e = xy$ such that $\gamma_R(G^e) = \gamma_R(G)$. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G^e)$-function, and let xwy be the subdivision of xy in G^e. If $f(w) = 1$, then $f|_{V(G)}$ is an RDF for G with weight less than $\gamma_R(G)$, a contradiction. If $f(w) = 2$, then g defined on G by $g(x) = \max\{1, f(x)\}$, $g(y) = \max\{1, f(y)\}$ and $g(u) = f(u)$ if $u \notin \{x, y\}$, is an RDF for G with weight less than $\gamma_R(G^e)$, a contradiction. Thus $f(w) = 0$. Without loss of generality we may assume that $f(x) = 2$. If $f(y) = 1$, then $f|_{V(G)}$ is a $\gamma_R(G)$-

function such that $f(x) = 2$ and $f(y) = 1$, a contradiction. If $f(y) = 2$, then $g = f|_{V(G)}$ is a $\gamma_R(G)$-function such that V_2^g is not a 2-packing. Thus $f(y) = 0$. Since there is a vertex $v \in N(y) - \{x\}$ with $f(v) = 2$, we find that $h = f|_{V(G)}$ is a $\gamma_R(G)$-function such that V_2^h is not a 2-packing, a contradiction. ■

If $f = (V_0^f, V_1^f, V_2^f)$ is a $\gamma_R(G)$-function, then a vertex $x \in V_0^f$ is a *private neighbor* of a vertex $y \in V_2^f$ if $N(x) \cap V_2^f = \{y\}$. As a consequence we have the following.

Corollary 10 *If $f = (V_0^f, V_1^f, V_2^f)$ is a $\gamma_R(G)$-function in a γ_{Rsd}-critical graph G, then:*
(1) Any vertex of V_2^f has at least two private neighbors in V_0^f.
(2) Any vertex of V_0^f is adjacent to at most one vertex of V_1^f.

Proof. (1) If a vertex x of V_2^f has no private neighbor in V_0^f, then replacing $f(x)$ by 1 produces an RDF with weight less than $\gamma_R(G)$, a contradiction. Assume that a vertex x of V_2^f has precisely one private neighbor y in V_0^f. Then replacing $f(x)$ and $f(y)$ by 1 produces a $\gamma_R(G)$-function g such that V_1^g is not independent, a contradiction to Theorem 9.
(2) If a vertex x of V_0^f is adjacent to at least two vertices of V_1^f, then replacing $f(x)$ by 2 and $f(y)$ by 0 for each vertex $y \in N(x) \cap V_1^f$, produces an RDF g such that either $w(g) < \gamma_R(G)$ or V_2^g is not a 2-packing, a contradiction to Theorem 9. ■

It is shown in [5] that for any graph G, $\gamma_R(G) \leq i_R(G) \leq u_R(G)$. Let G be a γ_{Rsd}-critical graph. By Theorem 9 any $\gamma_R(G)$-function is also a $u_R(G)$-function, and thus $\gamma_R(G) = i_R(G) = u_R(G)$.

It is well-known that for an isolate-free graph G of order n, $\gamma_R(G) = n$ if and only if n is even and $G = \frac{n}{2}K_2$.

Observation 11 *A graph G of order n is $n - \gamma_{Rsd}$-critical if and only if $G = \overline{K_n}$.*

Proof. Let G be a $n - \gamma_{Rsd}$-critical graph of order n. Let A be the set of isolated vertices of G, and $A_1 = V(G) - A$. Suppose that $A_1 \neq \emptyset$. Let G_1 be the subgraph induced by A_1. Then G_1 is isolate-free. Now $\gamma_R(G_1) = |V(G_1)|$ and so any component of G_1 is a K_2. Since K_2 is not γ_{Rsd}-critical, by Observation 3, G is not γ_{Rsd}-critical, a contradiction. Thus $A_1 = \emptyset$. Consequently, $G = \overline{K_n}$. The converse is obvious. ∎

Chambers et al. [2] proved the following.

Theorem 12 *[2] If G is a connected graph of order $n \geq 3$, then $\gamma_R(G) \leq \frac{4n}{5}$.*

Proposition 13 *A connected graph G of order n is $(n-1) - \gamma_{Rsd}$-critical if and only if $G \in \{P_3, P_4, C_3, C_4\}$.*

Proof. Let G be a graph of order n with $\gamma_R(G) = n - 1$. From Theorem 12 we find that $n \leq 5$. If $\Delta(G) \geq 3$, then $(N(x), V(G) - N[x], \{x\})$ is an RDF for G, where x is a vertex of maximum degree, and thus $\gamma_R(G) \leq n - \deg(x) + 1 \leq n - 2$, a contradiction. Thus $\Delta(G) = 2$ and so G is a path or a cycle. Now by Lemma 8 we obtain that $G \in \{P_3, P_4, C_3, C_4\}$. The converse is obvious. ∎

Let $S(K_{1,3})$ be the subdivided graph obtained from $K_{1,3}$ by subdivision of all its edges. Let $\mathcal{E}$ be the set

$$\mathcal{E} = \{cor(K_3), P_6, P_7, C_6, C_7, S(K_{1,3}), S(K_{1,2}), S\},$$

where S is the graph obtained from $S(K_{1,3})$ by identifying two leaves.

Proposition 14 *A connected graph G of order n with $\gamma_R(G) > 3$ is $(n-2) - \gamma_{Rsd}$-critical if and only if $G \in \mathcal{E}$.*

Proof. It is a routine matter to see that every graph in $\mathcal{E}$ is $(n-2)-\gamma_{Rsd}$-critical. Let G be a $(n-2)-\gamma_{Rsd}$-critical graph of order n with $\gamma_R(G) > 3$. If $\Delta(G) \geq 4$, then, for any vertex x of maximum degree, $(N(x), V(G)-N[x], \{x\})$ is an RDF for G of weight $n-\Delta(G)+1 \leq n-3$, a contradiction. Thus $\Delta(G) \leq 3$. Since G is connected and $\gamma_R(G) > 3$, we find that $\Delta(G) \geq 2$. If $\Delta(G) = 2$, then G is a path or a cycle, and by Lemma 8, $G \in \{P_6, P_7, C_6, C_7\}$. Thus assume that $\Delta(G) = 3$. Let x be a vertex of maximum degree, $N(x) = \{y, z, w\}$, and $H = G - N[x]$. Clearly $|V(H)| \geq 2$, since $\gamma_R(G) > 3$. Since $f(N(x), V(G)-N[x], \{x\})$ is a $\gamma_R(G)$-function, Theorem 9 implies that $V(G)-N[x]$ is independent. If there are two vertices $a, b \in V(H)$ with a common neighbor in $N(x)$, say y, then $f = (N(x)-\{y\}, V(G)-(N[x] \cup \{a,b\}), \{x,y\})$ is a $\gamma_R(G)$-function, a contradiction to Theorem 9, since $\{x,y\}$ is not a 2-packing. Thus no pair of vertices in $V(H)$ have a common neighbor. In particular, $2 \leq |V(H)| \leq 3$. We consider the following cases.

Case 1. $N(x)$ is not independent. Without loss of generality assume that y is adjacent to z. Since $\gamma_R(G) > 3$, there is a vertex of $V(H)$ which is adjacent to y or z. Let $y_1 \in V(H) \cap N(y)$. If w is adjacent to some vertex in $V(H)-\{y_1\}$, then $f = (N(y), V(G)-N[y], \{y\})$ is a $\gamma_R(G)$-function, where $V(G)-N[y]$ is not independent, a contradiction to Theorem 9. So w is not adjacent to any vertex of $V(H)-\{y_1\}$. Since $|V(H)| \geq 2$, we find that $|V(H)| = 2$ and there is a vertex $z_1 \in V(H)$ such that $z_1 \in N(z)$. If $w \in N(y_1)$, then $g = (N(z), V(G)-N[z], \{z\})$ is a $\gamma_R(G)$-function, where $V(G)-N[z]$ is not independent, a contradiction. Thus $\deg(w) = 1$. Consequently $G = cor(K_3) \in \mathcal{E}$.

Case 2. $N(x)$ is independent. Since $\gamma_R(G) > 3$, we observe that $G \in \{S(K_{1,3}), cor(K_{1,2}), S\} \subset \mathcal{E}$. ∎

We say that $\gamma_R(G)$ and $u_R(G)$ are *strongly equal* for G, denoted by $\gamma_R(G) \equiv u_R(G)$, if every $\gamma_R(G)$-function is a $u_R(G)$-function.

Similarly we say that $\gamma_R(G)$ and $i_R(G)$ are *strongly equal* for G, denoted by $\gamma_R(G) \equiv i_R(G)$, if every $\gamma_R(G)$-function is a $i_R(G)$-function. We observe that if G is γ_{Rsd}-critical, then $\gamma_R(G) \equiv i_R(G)$ and $\gamma_R(G) \equiv u_R(G)$.

Let $\alpha_0(G)$ denote the independence number of G, that is, the maximum cardinality of an independent set of G.

Proposition 15 *If G is a γ_{Rsd}-critical graph of order $n \geq 2$, then $\gamma_R(G) \leq 2\alpha_0(G)$, and this bound is sharp.*

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function. Then $V_1^f \cup V_2^f$ is an independent set in G. Now $\gamma_R(G) = |V_1^f| + 2|V_2^f| \leq \alpha_0(G) + |V_2^f| \leq \alpha_0(G) + \frac{\gamma_R(G)}{2}$, and so the result follows. To see the sharpness consider a complete graph. ■

Proposition 16 *If a graph G with $\gamma_R(G) \geq 4$ is γ_{Rsd}-critical, then* $\operatorname{diam}(G) \geq 3$.

Proof. Let G be a γ_{Rsd}-critical graph with $\gamma_R(G) \geq 4$. Let f be a $\gamma_R(G)$-function, and let x be a vertex with $f(x) = 2$. Assume that $\operatorname{diam}(G) \leq 2$. Since $\gamma_R(G) \geq 4$, we find that $\operatorname{diam}(G) = 2$. Let $A = N(x)$ and $B = V(G) - N[x]$. By Theorem 9, for any vertex $b \in B$, $f(b) = 1$ and B is an independent set. If there are two vertices $b_1, b_2 \in B$ such that $N(b_1) \cap N(b_2) \neq \emptyset$, then we replace $f(b_1)$ and $f(b_2)$ by 0, and $f(z)$ by 2, where $z \in N(b_1) \cap N(b_2)$, to obtain a $\gamma_R(G)$-function g in which V_2^g is not a 2-packing, a contradiction. Thus no two vertices of B have a common neighbor. Now if $b_1, b_2 \in B$, then $d(b_1, b_2) \geq 4$, a contradiction. ■

Proposition 17 *For any $m \geq 4$, there is a $m - \gamma_{Rsd}$-critical graph of diameter* 3.

Proof. Let $m \geq 4$, and $G = cor(K_{m-1})$. Then it is straightforward to see that G is $m - \gamma_{Rsd}$-critical of diameter 3. ■

Theorem 18 *There is no induced subgraph characterization of γ_{Rsd}-critical graphs.*

Proof. Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, and $H = G \circ P_2$. Let $V(H) = \{v_1, v_2, ..., v_n\} \cup \{x_{ij} | i = 1, 2, ..., n, j = 1, 2\}$, where x_{i1} is adjacent to v_i, and x_{i2} is adjacent to x_{i1} for $i = 1, 2, ..., n$. It is easy to see that $\gamma_R(H) = 2n$ and $(V(G) \cup \{x_{i2} : i = 1, 2, ..., n\}, \emptyset, \{x_{i1} : i = 1, 2, ..., n\})$ is the only $\gamma_R(H)$-function, and then H is γ_{Rsd}-critical. ■

A graph G is called *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. Next we study Roman γ_{Rsd}-critical graphs.

Theorem 19 *[4] A graph G is Roman if and only if there is a $\gamma_R(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that $V_1^f = \emptyset$.*

Let $\delta^*(G) = \min\{\deg(v) : v \in V(G) - L(G)\}$, where $L(G)$ is the set of all leaves of G.

Theorem 20 *If G is a γ_{Rsd}-critical Roman graph, then $\gamma_R(G) \leq \frac{2n}{1+\delta^*(G)}$. Equality holds if and only if there is a $\gamma_R(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that $V_1^f = \emptyset$ and every vertex of V_2^f is of degree $\delta^*(G)$.*

Proof. Let G be a γ_{Rsd}-critical Roman graph. By Theorem 19 there is a $\gamma_R(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that $V_1^f = \emptyset$. By Corollary 10, any vertex of V_2^f has at least two private neighbors in V_0^f. So $V_2^f \cap L(G) = \emptyset$. Since V_2^f is a 2-packing by Theorem 9, every vertex of V_2^f dominates at least $\delta^*(G) + 1$ vertices of G, and therefore V_2^f dominates at least

$|V_2^f|(1+\delta^*(G)) = \frac{\gamma_R(G)}{2}(1+\delta^*(G))$ vertices of G. This implies that $\gamma_R(G) \le \frac{2n}{1+\delta^*(G)}$.

Next assume that equality holds. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function such that $V_1^f = \emptyset$. Thus $|V_2^f| = \frac{\gamma_R(G)}{2} = \frac{n}{1+\delta^*(G)}$. Since V_2^f is a 2-packing by Theorem 9 and $n = |V_2^f|(1+\delta^*(G)$, it follows that every vertex of V_2^f is of degree $\delta^*(G)$. The converse is similarly verified. ∎

Since $\delta^*(G) \ge 2$, we obtain the following.

Corollary 21 *If G is a γ_{Rsd}-critical Roman graph of order n, then $\gamma_R(G) \le \frac{2n}{3}$. Equality holds if and only if there is a $\gamma_R(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that $V_1^f = \emptyset$ and every vertex of V_2^f is of degree 2.*

Similarly the following is verified.

Theorem 22 *If G is a γ_{Rsd}-critical Roman graph, then $\gamma_R(G) \ge \frac{2n}{1+\Delta(G)}$, and equality holds if and only if there is a $\gamma_R(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that $V_1^f = \emptyset$ and every vertex of V_2^f is of degree $\Delta(G)$.*

Proof. Let G be a γ_{Rsd}-critical Roman graph. By Theorem 19 there is a $\gamma_R(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that $V_1^f = \emptyset$. Every vertex of V_2^f dominates at most $\Delta(G)+1$ vertices of G, and therefore V_2^f dominates at most $|V_2^f|(1+\Delta(G)) = \frac{\gamma_R(G)}{2}(1+\Delta(G))$ vertices of G. This implies that $\gamma_R(G) \ge \frac{2n}{1+\Delta(G)}$.

Next assume that equality holds. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function such that $V_1^f = \emptyset$. Thus $|V_2^f| = \frac{\gamma_R(G)}{2} = \frac{n}{1+\Delta(G)}$. Since V_2^f is a 2-packing by Theorem 9 and $n = |V_2^f|(1+\Delta(G)$, it follows that every vertex of V_2^f is of degree $\Delta(G)$. The converse is similarly verified. ∎

For realizability of equality in Theorems 20 and 22, note that C_n is a γ_{Rsd}-critical Roman graph for $n \equiv 0 \pmod 3$, and $\gamma_R(C_n) = \frac{2n}{3} = \frac{2n}{1+\delta^\bullet(C_n)} = \frac{2n}{1+\Delta(C_n)}$. A consequence of Theorems 20 and 22 leads to the following.

Theorem 23 *A regular Roman graph G is γ_{Rsd}-critical if and only if $\gamma_R(G) = \frac{2n}{1+\Delta(G)}$.*

3 Trees

In this section we give necessary conditions for a tree T to be γ_{Rsd}-critical. We call a vertex v in a graph G a *non-γ_R-critical* vertex if $\gamma_R(G - v) \geq \gamma_R(G)$. Let $\mathcal{T}$ be the family of unlabelled trees T that can be obtained from a sequence $T_1, ..., T_j$ $(j \geq 1)$ of trees such that T_1 is a star $K_{1,r}$ for $r \geq 2$, and, if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations.

- **Operation $\mathcal{O}_1$.** Let $T_i \in \mathcal{T}$ and v be a non-γ_R-critical vertex of T_i. Then the tree T_{i+1} is obtained from T_i by attaching a leaf to v.

- **Operation $\mathcal{O}_2$.** Let $T_i \in \mathcal{T}$ and v be a vertex of T_i. Then the tree T_{i+1} is obtained from T_i by joining v to a leaf of a star $K_{1,m}$ for some $m \geq 2$.

Theorem 24 *If a tree T of order $n \geq 3$ is γ_{Rsd}-critical, then $T \in \mathcal{T}$.*

Proof. We proceed by induction on the Roman domination number $\gamma_R(T)$ of a γ_{Rsd}-critical tree T to show that $T \in \mathcal{T}$. Since $n \geq 3$, we find that $\gamma_R(G) \geq 2$. If $\gamma_R(T) = 2$, then clearly

T is a star, and thus $T \in \mathcal{T}$. Suppose the result is true for all γ_{Rsd}-critical trees with Roman domination number at most $q \geq 2$. Let T be a γ_{Rsd}-critical tree with $\gamma_R(T) = q + 1 > 2$. Clearly $\text{diam}(T) \geq 3$, since $\gamma_R(T) > 2$.

If $\text{diam}(T) = 3$, then T is a double-star. Let x, y be the central vertices of T. If $\deg(x) \geq 3$, and $\deg(y) \geq 3$, then $\gamma_R(T) = 4$, and clearly $\gamma_R(T^{xy}) = 4 = \gamma_R(T)$, a contradiction. Thus we may assume without loss of generality that $\deg(x) = 2$. Then $\gamma_R(T) = 3$. Let x_1 be the leaf adjacent to x, and $T_1 = T - x_1$. Then $\gamma_R(T_1) = 2 = \gamma_R(T_1 - x)$, and so x is a non-γ_R-critical vertex in T_1. Then $T_1 \in \mathcal{T}$. Hence, T is obtained from $T_1 \in \mathcal{T}$ by operation $\mathcal{O}_1$.

Assume now that $\text{diam}(T) \geq 4$. Let $x_0 - x_1 - x_2 - ... - x_k$ be a diametrical path in T between two leaves x_0 and x_k, where $k = \text{diam}(T)$.

Assume that $\text{diam}(T) = 4$. If $\deg(x_1) \geq 3$, then there is a $\gamma_R(G)$-function f such that $f(x_1) = 2$. By Theorem 9, $f(x_3) \neq 2$. If $f(x_3) = 1$, then $f(x_4) = 1$, and thus replacing $f(x_3)$ by 2 and $f(x_4)$ by 0 produces a contradiction. Thus $f(x_3) = 0$, and so $f(x_2) = 2$, a contradiction. We deduce that $\deg(x_1) = 2$, and by symmetry $\deg(x_3) = 2$. Moreover any support vertex adjacent to x_2 is of degree two. By Lemma 8, $T \neq P_5$, and thus $\deg(x_2) \geq 3$. Let l be the number of support vertices adjacent to x_2. Then $\gamma_R(T) = 2 + l$. Let $T_1 = T - x_0$. Then $\gamma_R(T_1) = 1 + l$, and $f_1 = f|_{V(T_1)}$ is the unique $\gamma_R(T_1)$-function of T_1 and $V_1^{f_1}$ is independent and $V_2^{f_1}$ is a 2-packing. Thus T_1 is a γ_{Rsd}-critical tree, and by the inductive hypothesis $T_1 \in \mathcal{T}$. But $\gamma_R(T_1 - x_1) = \gamma_R(T_1)$, and thus x_1 is a non-γ_R-critical vertex of T_1. Now T is obtained from $T_1 \in \mathcal{T}$ by operation $\mathcal{O}_1$.

We thus assume that $\text{diam}(T) \geq 5$.

We root T at x_0. There are the following cases.

Case 1. $\deg(x_{k-2}) = 2$. We consider the following subcases.

Subcase 1.1. There is a $\gamma_R(T)$-function f that $x_{k-1} \in V_2^f$. Let $T_1 = T - N[x_{k-1}]$. By Theorem 9 $f(x_{k-2}) \neq 2$ and thus $f|_{V(T_1)}$ is an RDF for T_1 and so $\gamma_R(T_1) \leq \gamma_R(T) - 2$. On the other hand any $\gamma_R(T_1)$-function can be extended to an RDF for T by assigning 2 to x_{k-1} and 0 to any vertex in $N(x_{k-1})$. This implies that $\gamma_R(T) \leq \gamma_R(T_1) + 2$. Thus $\gamma_R(T) = \gamma_R(T_1) + 2$. If T_1 is not γ_{Rsd}-critical, then there is a $\gamma_R(T_1)$-function g such that either V_1^g is not independent or V_2^g is not a 2-packing. But then we extend g to a $\gamma_R(T)$-function h by assigning 2 to x_{k-1} and 0 to any vertex in $N(x_{k-1})$, such that either V_1^h is not independent or V_2^h is not a 2-packing, a contradiction. We conclude that T_1 is γ_{Rsd}-critical. By the inductive hypothesis, $T_1 \in \mathcal{T}$. Now T is obtained from T_1 by using operation $\mathcal{O}_2$.

Subcase 1.2. There is no $\gamma_R(T)$-function f that $x_{k-1} \in V_2^f$. Let f be any $\gamma_R(T)$-function. Then $f(x_{k-1}) = 0$, and thus $f(x_{k-2}) = 2$ and $f(x_k) = 1$. As a consequence we have $\deg(x_{k-1}) = 2$. Let $T_1 = T - x_k$. It is obvious that $f|_{V(T_1)}$ is an RDF for T_1, and so $\gamma_R(T) \geq \gamma_R(T_1) + 1$. On the other hand it is clear that any $\gamma_R(T_1)$-function can be extended to an RDF for T by assigning 1 to x_k, implying that $\gamma_R(T) \leq \gamma_R(T_1) + 1$. Thus $\gamma_R(T) = \gamma_R(T_1) + 1$. If T_1 is not γ_{Rsd}-critical then there is a $\gamma_R(T_1)$-function g such that either V_1^g is not independent or V_2^g is not a 2-packing. But then we extend g to a $\gamma_R(T)$-function g_1 by assigning 1 to x_k such that either $V_1^{g_1}$ is not independent or $V_2^{g_1}$ is not a 2-packing, a contradiction. Thus T_1 is γ_{Rsd}-critical. Suppose now that $\gamma_T(T_1 - x_{k-1}) < \gamma_R(T_1)$. By Theorem 1, there is a $\gamma_R(T_1)$-function h such that $h(x_{k-1}) = 1$. Then we extend h to a $\gamma_R(T)$-function h_1 by assigning 1 to x_k such that $V_1^{h_1}$ is not independent, a contradiction. Thus $\gamma_R(T_1 - x_{k-1}) \geq \gamma_R(T_1)$, and so x_{k-1} is a non-γ_R-critical vertex of T_1. Now T is obtained from T_1 by operation $\mathcal{O}_1$.

Case 2. $\deg(x_{k-2}) \geq 3$.

Subcase 2.1. x_{k-1} is a strong support vertex. If there is a child $y \neq x_{k-1}$ of x_{k-2} such that y is a support vertex, then there is a $\gamma_R(T)$-function f such that $y, x_{k-1} \in V_2^f$ and so V_2^f is not a 2-packing, a contradiction to Theorem 9. Thus any child $y \neq x_{k-1}$ of x_{k-2} is a leaf. If $\deg(x_{k-2}) \geq 4$, then there is a $\gamma_R(T)$-function f such that $x_{k-2}, x_{k-1} \in V_2^f$, and so V_2^f is not a 2-packing, a contradiction to Theorem 9. Thus $\deg(x_{k-2}) = 3$. Let z be the leaf adjacent to x_{k-2} as its child. Let f be a $\gamma_R(T)$-function. Then $f(x_{k-1}) = 2$ and $f(z) = 1$. Let $T_1 = T - z$. It is obvious that $f|_{V(T_1)}$ is an RDF for T_1 implying that $\gamma_R(T_1) \leq \gamma_R(T) - 1$. On the other hand any $\gamma_R(T_1)$-function can be extended to an RDF of T by assigning 1 to z which implies that $\gamma_R(T) \leq \gamma_R(T_1) + 1$. Thus $\gamma_R(T) = 1 + \gamma_R(T_1)$. If T_1 is not γ_{Rsd}-critical, then then there is a $\gamma_R(T_1)$-function g such that either V_1^g is not independent or V_2^g is not a 2-packing. But then we extend g to a $\gamma_R(T)$-function h by assigning 1 to z such that either V_1^h is not independent or V_2^h is not a 2-packing, a contradiction. We conclude that T_1 is γ_{Rsd}-critical. By the inductive hypothesis, $T_1 \in \mathcal{T}$. If $\gamma_R(T_1 - x_{k-2}) < \gamma_R(T_1)$, then by Theorem 1 there is a $\gamma_R(T_1)$-function h_1 such that $h_1(x_{k-2}) = 1$. Then h_1 can be extended to a $\gamma_R(T)$-function h_2 by assigning 1 to z, and thus $V_1^{h_2}$ is not independent, a contradiction. Thus $\gamma_R(T_1 - x_{k-2}) \geq \gamma_R(T_1)$. Now T is obtained from $T_1 \in \mathcal{T}$ by using operation $\mathcal{O}_1$.

Subcase 2.2. x_{k-1} is not a strong support vertex. So $\deg(x_{k-1}) = 2$ and we may assume that no child of x_{k-2} is a strong vertex. Furthermore, any child of x_{k-2} is either a leaf or a support vertex of degree two. Let k_1 be the number of children of x_{k-2} that are support vertices.

If $k_1 \geq 2$, then there is a $\gamma_R(T)$-function f, such that $f(x_{k-2}) = 2$. Then $f(x_k) = 1$. Let $T_1 = T - x_k$. Then $f|_{V(T_1)}$ is an RDF for T_1 implying that $\gamma_R(T_1) \leq \gamma_R(T) - 1$, and we can easily see that $\gamma_R(T) = \gamma_R(T_1) + 1$. If T_1 is not γ_{Rsd}-critical, then there is a $\gamma_R(T_1)$-function g such that either V_1^g is not independent or V_2^g is not a 2-packing. Then we extend g to a $\gamma_R(T)$-function h by assigning 1 to x_k such that either V_1^h is not independent or V_2^h is not a 2-packing, a contradiction. Thus T_1 is γ_{Rsd}-critical. If $\gamma_R(T_1 - x_{k-1}) < \gamma_R(T_1)$ then by Theorem 1 there is a $\gamma_R(T_1)$-function g_1 such that $x_{k-1} \in V_1^{g_1}$. Then we extend g_1 to a $\gamma_R(T)$-function h_1 by assigning 1 to x_k, and thus $V_1^{h_1}$ is not independent, a contradiction. Thus $\gamma_R(T_1 - x_{k-1}) \geq \gamma_R(T_1)$. Now T is obtained from T_1 by using operation $\mathcal{O}_1$.

Thus we assume that $k_1 = 1$. Since $\deg(x_{k-2}) \geq 3$, there is a $\gamma_R(T)$-function f, such that $f(x_{k-2}) = 2$. Then $f(x_k) = 1$. Let $T_1 = T - x_k$. Then $f|_{V(T_1)}$ is an RDF for T_1, and we can easily see that $\gamma_R(T) = \gamma_R(T_1) + 1$. Furthermore, as in the previous cases, we can see that T_1 is γ_{Rsd}-critical, and $\gamma_R(T_1 - x_{k-1}) \geq \gamma_R(T_1)$. Thus $T_1 \in \mathcal{T}$, and T is obtained from T by using operation $\mathcal{O}_1$.
■

We close with the following problem.

Problem 25 *Find a constructive characterization for γ_{Rsd}-critical trees.*

Acknowledgements
I would like to thank the referee for his/her careful review of the manuscript and a lot of helpful suggestions which improved the presentation of the manuscript.

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