On Anti-Waring Numbers

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Abstract

The results of Laughlin and Johnson [1] are generalized in this paper, and open problems left at the end of [1] are addressed. New values of Anti-Waring numbers are given, including $N(2,4)$, $N(2,5)$, $N(2,6)$, and $N(2,7)$.

1 Waring's Problem and "Anti-Waring" Numbers

The original conjecture of Waring [4] stated that for each positive integer k there is a number $g(k)$ such that every positive integer can be expressed as a sum of $g(k)$ or fewer k^{th} powers of positive integers. Waring's problem is to find the smallest such $g(k)$ for each k. The affirmation of Waring's Conjecture in 1909 added more foundation to Waring's problem, and subsequently, a second "Waring" type problem, namely to find the smallest integer $G(k)$ for each positive integer k such that every *sufficiently large* integer may be expressed as the sum of $G(k)$ or fewer k^{th} powers. Each of these problems have been thoroughly investigated; however several open problems still remain.

The "Anti-Waring" conjecture due to Johnson and Laughlin [1] is as follows: If k and r are positive integers, then every sufficiently large integer is the sum of r or more distinct k^{th} powers of positive integers. The smallest

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integer such that all integers from that one on are so expressible will be denoted $N(k,r)$. Looper and Saritzky in [2] proved that $N(k,r)$ exists for all k and r [2]. Johnson and Laughlin managed to find values for $N(2,1)$, $N(2,2)$, and $N(2,3)$, and in the following sections we will expand upon these results.

2 A General Result on Anti-Waring Numbers

We call an integer (k,r)-good if it can be expressed as the sum of r or more distinct k^{th} powers of positive integers. If an integer is not (k,r)-good, then it is (k,r)-bad. We begin this section by providing an extension of the methods of Laughlin and Johnson [1], followed by applications of our extension.

Theorem 1. *Suppose that k and r are positive nonzero integers. If there exist positive integers A, B, and C which satisfy the following:*

1. *C is a (k,r)-bad integer;*
2. *$C < A^k$ and for each integer $s \in [C+1,\dots,A^k]$, s is a (k,r)-good integer;*
3. *For each of the polynomials (1), (2), and (3) below, A is greater than its largest real root:*

$$x^k + \sum_{j=1}^{k} \binom{k}{j} x^{k-j}[2(-B)^j - 1] + \sum_{j=1}^{r-1} j^k \tag{1}$$

$$x^k + \sum_{j=1}^{k} \binom{k}{j} x^{k-j}[(-B)^j - 1] + 1 \tag{2}$$

$$C - 1 + \sum_{j=1}^{k} \binom{k}{j} x^{k-j}(-B)^j, \tag{3}$$

then $N(k,r) = C+1$.

Proof. To begin we note that $N(k,r) = C+1$ if and only if C is the largest (k,r)-bad integer.

Suppose that each integer in the interval $[C+1,\ldots,A^k]$ is (k,r)-good. We will show the following via induction on m: If $m \geq A$ and $n \leq m^k$ is (k,r)-bad, then $n \leq C$. This will finish the proof.

Since each integer $C+1,\ldots,A^k$ is (k,r)-good, at $m = A$ our statement holds. Now suppose that $m \geq A$ and that the statement is true for m; suppose that $n \leq (m+1)^k$ is a (k,r)-bad integer. We wish to show that $n \leq C$.

First, if $n \leq m^k$ then the conclusion follows from the induction hypothesis. Therefore suppose that $m^k + 1 \leq n \leq (m+1)^k$, or equivalently:

$$1 \leq z = n - m^k \leq (m+1)^k - m^k = \sum_{j=1}^{k} \binom{k}{j} m^{k-j}. \tag{4}$$

We now claim that $n - (m-B)^k$ is a (k,r)-bad integer. If

$$n - (m-B)^k = z + \sum_{j=1}^{k} \binom{k}{j} m^{k-j} (-1)^{j+1} B^j \tag{5}$$

were (k,r)-good then for some integer $t \geq r$ there exist positive integers $\alpha_1 < \alpha_2 < \cdots < \alpha_t$ such that $n-(m-B)^k = \sum_{j=1}^{t} \alpha_j^k$. Since n is (k,r)-bad then one of the α_j must be $m - B$. Therefore

$$n - (m-B)^k \geq (m-B)^k + \sum_{j=1}^{r-1} j^k$$

Consequently, using (4) and (5):

$$m^k + 2\sum_{j=1}^{k} \binom{k}{j} m^{k-j} (-B)^j + \sum_{j=1}^{r-1} j^k \leq z \leq \sum_{j=1}^{k} \binom{k}{j} m^{k-j}. \tag{6}$$

Inequality (6) implies that the value of

$$x^k + \sum_{j=1}^{k} \binom{k}{j} x^{k-j} [2(-B)^j - 1] + \sum_{j=1}^{r-1} j^k$$

at m, is non-positive. Therefore, m is no larger that the largest real root of this polynomial. However, this polynomial is the polynomial (1). Hence, the conclusion that m is no greater than its largest real root implies that $m < A$, a contradiction.

Now we do a similar manipulation to show that $n - (m - B)^k \le m^k$. If this were not the case then

$$\begin{aligned} m^k + 1 &\le z + \sum_{j=1}^{k} \binom{k}{j} m^{k-j}(-1)^{j+1}B^j \\ &\le \sum_{j=1}^{k} \binom{k}{j} m^{k-j} + \sum_{j=1}^{k} \binom{k}{j} m^{k-j}(-1)^{j+1}B^j. \end{aligned}$$

Thus m must be less than or equal to the largest real root of the following polynomial:

$$x^k + \sum_{j=1}^{k} \binom{k}{j} x^{k-j}[(-B)^j - 1] + 1.$$

By the hypothesis of the Theorem, we would then have that $m < A$, a contradiction.

Now we need only invoke the inductive hypothesis. We have

$$\begin{aligned} C \ge n - (m - B)^k &= z + \sum_{j=1}^{k} \binom{k}{j} m^{k-j}(-1)^{j+1}B^j \\ &\ge 1 + \sum_{j=1}^{k} \binom{k}{j} m^{k-j}(-1)^{j+1}B^j. \end{aligned}$$

Again we can conclude that m is no larger than the largest real root of the following polynomial:

$$C - 1 + \sum_{j=1}^{k} \binom{k}{j} x^{k-j}(-B)^j. \tag{7}$$

This implies $m < A$, contradicting $m \ge A$. Therefore $n \le m^k$ after all, so $n \le C$; the induction step has been taken.

□

Applying this theorem, we find A, B, and C to compute new values of $N(k, r)$. Below is a table giving values $[A, B, C]$ from which the value $N(k, r) = C + 1$ can be proven, for some pairs (k, r).

Table 1: Some values for $[A, B, C]$ giving $N(k, r)$, from Theorem 1:

$r\backslash k$	2	3	4
1	[18, 4, 128]	[33, 6, 12758]	?
2	[18, 4, 128]	[33, 6, 12758]	?
3	[18, 4, 128]	[33, 6, 12758]	?
4	[23, 6, 128]	[33, 6, 12758]	?
5	[23, 5, 197]	[33, 6, 12758]	?
6	[23, 6, 237]	?	?
7	[26, 8, 330]	?	?
8	[27, 9, 382]	?	?
9	[32, 10, 527]	?	?
10	[33, 12, 647]	?	?

In each case, in Table 1, $N(k, r) = C + 1$; for instance, $N(3, 5) = 12,759$. In each case the (k, r)-goodness of the integers $C + 1, \ldots A^k$ was checked directly; representations of those integers as sums of distinct k^{th} powers are available on request. Our results have extended the list of known Anti-Waring numbers from $N(2, r)$, $r = 1, 2, 3$ [1], to those indicated by Table 1. Progress on remaining Anti-Waring numbers gets more difficult; it has been verified by computer that $N(4, 1) > 550000$, and little is known about remaining Anti-Waring numbers; in particular we have no bounds on $N(k, r)$, although it appears that something could be extracted from [2].

3 On A Larger Problem

Johnson and Laughlin introduce a larger question in the final section of [1]. Let $(a_n) = (a_1, a_2, a_3 \ldots)$ be a strictly increasing sequence of positive integers, and let r be a positive integer. We say that (a_n) has property S_r if and only if each sufficiently large positive integer can be expressed as the sum of r or more distinct elements from the set $\{a_n | n = 1, 2, \ldots\}$. Furthermore, if (a_n) has property S_r for all r, we say that (a_n) has property S_∞. (In this respect, the existence of $N(k, r)$ for all k and r means that for all k the sequence (n^k) has property S_∞, as shown by Looper and Saritzky [2].)

Let $f(x)$ be a polynomial that maps integers to integers[1], and the sequence generated by $f(x)$ be the sequence $f(1), f(2), \ldots$. We define

[1] These polynomials are sometimes called *numerical polynomials* and their coefficients need not be integers.

$N(f(x), r)$ to be the smallest integer such that every integer from there on is representable as the sum of r or more distinct terms of the sequence generated by $f(x)$, if there is such an integer.

We are able to provide an additional infinite family of sequences which possess the property S_∞ by extending the methods of Looper and Saritzky [2]. To do this we use a Theorem of Roth and Szekeres [3]. We say a set of real numbers S is *complete* if all sufficiently large integers can be written as a sum of distinct elements of S.

Theorem 2 (Roth and Szekeres). *Let $f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0$, $\alpha_n > 0$ be a polynomial which maps integers into integers. Let $S(f)$ denote the set $\{f(j) | j = 1, 2, \dots\}$. Then $S(f)$ is complete if and only if for any prime p, there exists an integer m such that p does not divide $f(m)$.*

Using this result we may show the following.

Theorem 3. *If $f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \neq 0$, $\alpha_k \geq 0$, $\alpha_n > 0$, is a polynomial mapping $\mathbb{Z} \to \mathbb{Z}$ with the property that for any prime p there exists a integer m such that $p \nmid f(m)$, then $\{f(n) | n = 1, 2, \dots\}$ has the property S_∞.*

Proof. We proceed by induction on r. For $r = 1$, Theorem 2 implies that $N(f(x), 1)$ exists. Suppose $r \geq 1$, $N(f(x), r)$ exists, and $N(f(x), r) = B$ for some integer B. We note that

$$2f(x) - f(x+1) = \alpha_n \left[2x^n - (x+1)^n\right] + \cdots + \alpha_0[2-1] \to \infty$$

as $n \to \infty$.

Hence there exists a positive integer A such that for all $x > A$, $2f(x) > f(x+1) + B$. Now let m be an integer such that $m \geq f(A) + B$ so that $m - B \geq f(A)$. Let β be the greatest integer such that $f(\beta) \leq m - B$. Hence

$$f(\beta) \leq m - B < f(\beta + 1). \tag{8}$$

Combining inequalities yields

$$f(\beta) \leq m - B < f(\beta + 1) < 2f(\beta) - B. \tag{9}$$

From inequality (8), $B \leq m - f(\beta)$, so $m - f(\beta) = f(s_1) + \cdots + f(s_t)$, where s_j are distinct positive integers and $t \geq r$. Thus if any $s_j = \beta$, then $m \geq 2f(\beta)$, contradicting (9). Therefore $m = f(\beta) + \sum_{j=1}^{t} f(s_j)$ is the sum of $r+1$ or more distinct elements of the set $\{f(n)|n = 1, 2, \ldots\}$. Since m was an arbitrary integer greater than or equal to $f(A)+B$, by the induction hypothesis we are finished: $N(f(x), r+1)$ exists (and will be no greater than $f(A) + N(f(x), r)$).

□

Finally, we say that a sequence (a_n) has property $\hat{S}_r$ if and only if every tail of the sequence has property S_r; if (a_n) has this property for all r, then (a_n) has property $\hat{S}_\infty$. The following is a corollary pertaining to our family of sequences.

Corollary 4. *If $f(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \neq 0$, $\alpha_k \geq 0$, $\alpha_n > 0$, is a polynomial mapping $\mathbb{Z} \to \mathbb{Z}$ with the property that for any prime p there exists a positive integer $m \geq N$ such that $p \nmid f(m)$, then $\{f(n)|n = 1, 2, \ldots\}$ has the property $\hat{S}_\infty$.*

Proof. For each positive integer N, $g(x) = f(x + N - 1)$ satisfies the hypothesis of Theorem 3 and $(g(1), g(2), \ldots) = (f(N), f(N+1), \ldots)$. The conclusion of the corollary follows from Theorem 3.

□

4 Open Problems

In the pursuit of values for $N(k, r)$, Theorem 1 is very effective for verifying candidates found from searching. The algorithms for searching for candidate values all involve computing power sets for the set $\{1^k, 2^k, \ldots n^k\}$ where n^k is the smallest k^{th} power less than the number we are testing. Any upper bound on $N(k, r)$ or even $N(k, 1)$ would be extremely useful in searching for $N(k, r)$ by setting limits on the search for candidates. Also it is unknown whether $N(k, r) \leq N(k+1, r)$ and even this would be an interesting result.

For positive integers k, r, s, does there exist an integer $N(k, r, s)$ such that every $n \geq N(k, r, s)$ is expressible as a sum of r or more distinct k^{th} powers of positive integers in s or more different ways? Let $f_{k,r}(n)$ be the number of different ways of expressing n as a sum of r or more distinct k^{th} powers of positive integers. Then $N(k, r, s)$ exists for all k, r, and s if and only if $f_{k,r}(n) \to \infty$ as $n \to \infty$, for all k, r.

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