OPTIMAL STOPPING TIME ON A MINORITY COLOR IN A 2-COLOR URN SCHEME

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ABSTRACT. An urn contains $2n + 1$ balls in two colors. The number of balls of a particular color is a random variable having binomial distribution with $p = \frac{1}{2}$. We sample the urn removing balls one by one without replacement. Our aim is to stop the process maximizing the probability that the color of the last selected ball is the minority color. We give an algorithm for an optimal stopping time, evaluate the probability of success and its asymptotic behavior.

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1. INTRODUCTION

The aim of the paper is to find an optimal stopping time in a process involving objects of two types. Like in the most famous model, namely the secretary problem, the objects are coming one by one in a random permutation. A decision about stopping at time t is based solely on the partial sequence observed up to time t, but the outcome (win, or loss, or some payment) is determined after the whole sequence is revealed.

In the classic secretary problem, solved originally in 1961 [9], the objects are linearly ordered and we want to maximize the probability of stopping on the maximal element. The book by Berezovsky and Gnedin [1] and Ferguson's paper [3] provide lovely history of the original problem and its generalizations.

Later the secretary problem was studied for partial orders: complete binary tree [10], two parallel linear orders [4], general partial order [11], [5], and threshold stopping times [6]. In [8] a broader graph-theoretic approach to optimal stopping time was introduced. For a given graph (or a digraph), we want to choose in the online decision process a vertex from a predefined subset of vertices. This generalization does not require a partial order structure, and effective algorithms for such a choice may be applicable, for instance, in searches for appropriate servers which are a part of a known computer network. In [8] the case of searching for an end-vertex of a directed path of order n was solved.

In our model, the objects are of two types (balls of different colors) and the objective is to stop on the ball of the minority color. Therefore, this process might be thought as a game played on a bipartite graph whose order $2n+1$ is known but the sizes of partite sets are random. A simpler version of the game with known sizes of partite sets was solved recently in [7].

In this paper we will not use graph-theoretical terminology and the process will be described using random binary sequences.

2. Description of the Game

Consider an urn containing an odd number of balls, say $2n+1$, where $n \geq 1$. The balls are in two colors and the number of balls of a specific color is a random variable having binomial distribution with $p = \frac{1}{2}$; the probability that the number of balls in a specific color is k equals $\frac{\binom{2n+1}{k}}{2^{2n+1}}$. Because we have an odd number of balls, the color classes have different sizes. We refer to the color of the smaller class as the minority color. We pick the balls randomly one by one without replacement revealing their colors. We want to stop the process maximizing the probability that the color of the ball selected in the last step is the minority color. Of course, during the process we do not know which color is the minority color. We label the two colors by 1 and -1. Let x_i denote the color of the ith ball drawn. The assumption about binomial distribution of a size of a particular color class implies that $x_1, x_2, \ldots, x_{2n+1}$ are independent random variables having Bernoulli distribution with $p = \frac{1}{2}$. Our goal is to determine the stopping policy for which the probability of selecting a ball in the minority color is a maximum. Notice that x_t is the minority color if and only if $x_t \sum_{i=1}^{2n+1} x_i < 0$. Let $\mathcal{F}_0$ denote the trivial σ-algebra and let $\mathcal{F}_i = \sigma(x_1, x_2, \ldots, x_i)$ be the σ-algebra generated by random variables $x_1, x_2, \ldots, x_i$, $1 \leq i \leq 2n+1$. Define

$$W_t = W_t(x_1, x_2, \ldots, x_{2n+1}) = \begin{cases} 1, & \text{if } x_t \sum_{i=1}^{2n+1} x_i < 0 \\ 0, & \text{if } x_t \sum_{i=1}^{2n+1} x_i > 0, \end{cases}$$

and

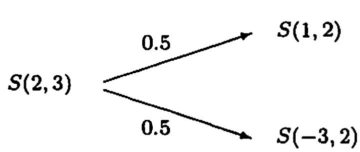
$$Z_t = E(W_t | \mathcal{F}_t).$$

The first line in the definition of W_t represents the event that x_t is the minority color, the second line - the majority color. A **stopping time** for the stochastic sequence $\{Z_t, \mathcal{F}_t\}_{t=1}^{2n+1}$ is an integer valued random variable T, $1 \leq T \leq 2n+1$, for which $\{T \leq t\} \in \mathcal{F}_t$, $1 \leq t \leq 2n+1$. The last condition says that our decision to stop at t is based only on the values of x_i for $i \leq t$ and does not depend on the future events $x_{t+1}, \ldots, x_{2n+1}$. We want to find a stopping time τ, referred to as an **optimal stopping time**, for which $E(Z_\tau) = \max E(Z_T)$ where the maximum is taken over the set of all stopping times T.

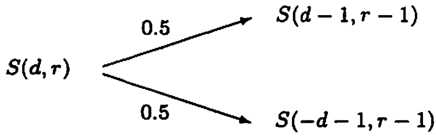
Let us notice that in our notation the probability of stopping on a ball of the minority color is equal to

$$P[x_\tau \sum_{i=1}^{2n+1} x_i < 0] = P[W_\tau = 1].$$

One can think about maximizing $P[W_\tau = 1]$ as a game, called $G(0, 2n+1)$, played against nature, where $2n+1$ denotes the length of the game and 0 denotes the initial difference between sizes of color classes of balls observed so far. Because our game is finite, the optimal stopping time will be determined by the method of backward induction (see [CRS]). For our process the value of Z_t depends not really on the whole history $x_1, x_2, \ldots, x_t$ of the process till time t, but on x_t, $\sum_{i=1}^{t} x_i$, and $(2n+1) - t$, the number of unrevealed balls. In this backward induction, which in this case is really a normal induction on the number of unreaveled balls, we need to compare the value of Z_t when we stop the process at time t with the expected value of the process when we stop later. In the later case, we consider a shorter game determined by a new smaller length and an updated initial difference between sizes of color classes of balls observed so far. Instead of introducing new games, we prefer to call those games the **states** of the original game $G(0, 2n+1)$. They are defined as follows. A state $S(d, r)$ consists of all sequences of length $2n+1$ whose initial partial segments of length $t = 2n+1-r$ have the property that the number of balls of color $-x_t$ minus the number of balls of the color x_t is equal to d. Of course, r represents the number of balls remaining in the urn, balls whose colors have not been revealed yet. For example, each of the following sequences of length 9 belongs to the state $S(2,3)$: $(1,1,1,-1,1,-1,x_7,x_8,x_9), (-1,-1,-1,1,-1,1,x_7,x_8,x_9),$ $(1,-1,-1,-1,-1,1,x_7,x_8,x_9)$, for $x_7, x_8, x_9 \in \{-1,1\}$. Notice that d can be evaluated as $d = -x_t \sum_{i=1}^{t} x_i$, where $t = (2n+1) - r$. If $d < 0$, then x_t is a majority color till time t. If $d > 0$, then x_t is a minority color so far. When the t initial terms of a sequence $\bar{x}$ determine that $\bar{x}$ is in the state $S(2,3)$ and the next term x_{t+1} of $\bar{x}$ is revealed, then either $\bar{x} \in S(1,2)$ or $\bar{x} \in S(-3,2)$. For example, $(1,1,1,-1,1,-1,x_7,x_8,x_9) \in S(2,3)$ but $(1,1,1,-1,1,-1,-1,x_8,x_9) \in S(1,2)$ and $(1,1,1,-1,1,-1,1,x_8,x_9) \in S(-3,2)$. Therefore, from the state $S(2,3)$ we can go either to $S(1,2)$ or $S(-3,2)$, to each with probability 0.5. The transition diagram in this case is:



In general, the transition diagram for $S(d, r)$ is:



Let's notice that in any state $S(d, r)$, the integers d and r are of different parity. Also, from a state $S(0, r)$ we always go to $S(-1, r-1)$. Being at the state $S(d, r)$, we should decide whether to stop on the color $x_{(2n+1)-r}$ or wait (and stop later). Let $p(d, r)$ denote the value of the game that starts from the state $S(d, r)$. This value is the probability of success when an optimal stopping time is used in the game of length r and the difference d between color classes. At time t, the knowledge about $x_1, x_2, \ldots, x_t$ allows us to evaluate d and r. An optimal stopping time τ is given by

$$\tau = \min\{t : 1 \leq t \leq 2n+1 \text{ and } Z_t \geq p(d, r)\}$$

3. Recurrence Formulas

First, we will determine the values of $p(d, r)$ for all r and d such that $1 \leq r \leq 2n+1$ and $-(2n+1) \leq d \leq 2n+1$. Let's start with a couple of simple observations.

Fact 1. If $d > r$, then $p(d, r) = 1$.

This is true because the last ball is guaranteed to be of the minority color and we can stop on it.

Fact 2. If $d < 0$ and $-d > r$, then $p(d, r) = 1 - \frac{1}{2^r}$.

The assumption $-d > r$ guarantees that the last ball is of the majority color, so we have to wait for a ball of the other(minority) color. There is a chance of $(\frac{1}{2})^r$ that such a ball will never come.

Fact 3. For every d and $r \geq 1$, $p(d, r) \geq \frac{1}{2}$.

Suppose that $r = 1$. Then d is even. If $d \geq 2$, stopping guarantees the success. If $d = 0$, stopping gives the probability of success $\frac{1}{2}$. If $d \leq -2$, then waiting and stopping on the last ball gives the probability of success $\frac{1}{2}$. Because the optimal strategy must give the probability of success at least as high as the strategy described above, the fact follows for $r = 1$. For $r \geq 2$, waiting till $r = 1$ and then using the strategy described above gives the probability of success at least $\frac{1}{2}$. The optimal stopping strategy cannot be worse.

It is not difficult to establish recurrence formulas for $p(d,r)$.

Theorem 1. *(a) If $d < 0$, then*

$$p(d,r) = \frac{1}{2}p(d-1,r-1) + \frac{1}{2}p(-d-1,r-1) \tag{1}$$

(b) If $d \geq 0$ and $d < r$, then

$$p(d,r) = max \begin{cases} \frac{1}{2}p(d-1,r-1) + \frac{1}{2}p(-d-1,r-1); \\ 1 - \frac{\binom{r}{0}+\binom{r}{1}+\dots+\binom{r}{\frac{r-d-1}{2}}}{2^r}. \end{cases} \tag{2}$$

Proof. (a) Stopping at the state with $d < 0$ gives the probability of success less than $\frac{1}{2}$. By Fact 3, $p(d,r) \geq \frac{1}{2}$. So we have to wait and get the average of the probabilities in two possible states $S(d-1,r-1)$ or $S(-d-1,r-1)$. (b) We are comparing the probabilities of success when waiting (the first line) and when stopping (the second line in the formula involving maximum). When we stop on a current ball, we will loose if, among the remaining r balls, the number c of balls of the other color is $0, 1, 2, \dots$, or $\frac{1}{2}(r-d-1)$. Then the color of the current ball will become the majority color, since $d+(c-(r-c)) = d+2c-r \geq d+2\cdot\frac{1}{2}(r-d-1)-r = -1$. The probability that this happens is equal to $2^{-r}(\binom{r}{0}+\binom{r}{1}+\dots+\binom{r}{\frac{r-d-1}{2}})$. Otherwise, the current color will remain the minority color and we win. □

We will establish initial conditions, i.e. probabilities $p(d,1)$ for all possible values of d when d is even. For $d=0$, if we stop on a current ball we win with the probability $\frac{1}{2}$. Waiting and stopping on the last ball guarantees failure, so $p(0,1) = \frac{1}{2}$. From Fact 1 and Fact 2 we have that $p(d,1) = 1$ for $d \geq 2$ and $p(d,1) = \frac{1}{2}$ for $d \leq -2$.

These initial conditions together with recurrence formulas from Theorem 1 allow us to evaluate probabilities $p(d,r)$. We would like to construct a table of their values, but since all of $p(d,r)$ are fractions with denominators being powers of 2, we give a table of values of $P(d,r) = 2^r p(d,r)$. Of course, the initial conditions for $P(d,r)$ are $P(d,1) = 2$ for $d \geq 2$ and $P(d,1) = 1$ for $d \leq 0$. Then the recurrence formulas from Theorem 1 become

$$P(d,r) = P(d-1,r-1) + P(-d-1,r-1), \text{ for } d < 0, \tag{3}$$

and

$$P(d,r) = max \begin{cases} P(d-1,r-1) + P(-d-1,r-1); \\ 2^r - \binom{r}{0} - \binom{r}{1} - \dots - \binom{r}{\frac{r-d-1}{2}}. \end{cases} \quad \text{for } 0 \leq d < r. \tag{4}$$

We also have, $P(d,r) = 2^r$ for $d > r$.

The values of $P(d,r)$ for $1 \leq r \leq 9$ and $-12 \leq d \leq 12$ are given in Table 1.

d \r	1	2	3	4	5	6	7	8	9
0	1		4		20		90		382
1		3		11		45		191	
-1		2		10		45		191	
2	2		7		26		101		411
-2	1		6		25		101		411
3		4		15		57		220	
-3		3		14		56		220	
4	2		8		31		120		466
-4	1		7		30		119		466
5		4		16		63		247	
-5		3		15		62		246	
6	2		8		32		127		502
-6	1		7		31		126		501
7		4		16		64		255	
-7		3		15		63		254	
8	2		8		32		128		511
-8	1		7		31		127		510
9		4		16		64		256	
-9		3		15		63		255	
10	2		8		32		128		512
-10	1		7		31		127		511
11		4		16		64		256	
-11		3		15		63		255	
12	2		8		32		128		512
-12	1		7		31		127		511

TABLE 1. Table of values of $P(d,r)$.

Looking at the entries $P(d,r)$ in Table 1 one can notice that there is a triangular region (a subset of the region above the main diagonal $d = r$) for which $P(d,r) = P(-d,r)$, whereas below this region we have $P(d,r) = P(-d,r) + 1$ for $d > 0$. In the region where $P(d,r) = P(-d,r)$ stopping is not better than waiting. For the states $S(d,r)$ in the other region, stopping gives larger probability of success than waiting. In the remaining part of the paper we find compact formulas for $P(d,r)$, describe the shape of the boundary between these two regions, and examine the asymptotic behavior of the probability of winning as the number of balls in the urn increases without bound ($2n+1 \to \infty$). The value of the game $G(0, 2n+1)$ is $E(Z_\tau) = \frac{P(0,2n+1)}{2^{2n+1}}$. Any random sequence $(x_1, x_2, \ldots, x_{2n+1})$ determines the random walk in the Table 1 that starts at $S(0, 2n+1)$ and proceeds northwest or southwest through different states $S(d,r)$ from which one can read the probability of success when optimal stopping time is used.

Table 1 has a significant shortcoming. In the region where $P(d,r) = P(-d,r)$, the probability of success when stopping is not larger than the probability of success when waiting. However, the entries of the table do not show the difference between corresponding probabilities. To overcome these shortcomings we introduce new two-dimentional arrays $a(d,r)$ and $A(d,r)$ defined for all integer values of d and r such that $d \geq 0$ and $r \geq 1$.

$$(5) \qquad a(d,r) = p(-d,r) - \left(1 - \frac{\binom{r}{0} + \binom{r}{1} + \ldots + \binom{r}{\frac{r-d-1}{2}}}{2^r}\right).$$

$$(6) \qquad A(d,r) = 2^r a(d,r).$$

In other words, the number $a(d,r)$ represents the difference in probability of success using the optimal strategy at the state $S(-d,r)$ (when $-d \leq 0$ and we have to wait) and the probability of success when we stop at state $S(d,r)$. Informally, one might think about $a(d,r)$ as "the advantage of waiting over stopping". The numbers $A(d,r)$, introduced in order to avoid fractions, satisfy nice recurence formulas that are the subject of Theorem 2. Before formulating this theorem, let us notice that

$$(7) \quad A(d,r) = 2^r a(d,r) = P(-d,r) + \binom{r}{0} + \binom{r}{1} + \ldots + \binom{r}{\frac{r-d-1}{2}} - 2^r.$$

Theorem 2. *(a) If $d > 0$ and $A(d-1, r-1) \geq 0$, then*

$$(8) \qquad A(d,r) = A(d-1, r-1) + A(d+1, r-1)$$

(b) If $d = 0$ and $r \geq 3$, then

$$(9) \qquad A(d,r) = A(0,r) = 2A(1, r-1) + \binom{r-1}{\frac{r-1}{2}}$$

Proof. (a) $A(d,r) = 2^r a(d,r) = P(-d,r) + \binom{r}{0} + \binom{r}{1} + \ldots + \binom{r}{\frac{r-d-1}{2}} - 2^r$
$= P(-d-1, r-1) + P(d-1, r-1) + \binom{r-1}{0} + \binom{r-1}{1} + \ldots + \binom{r-1}{\frac{r-d-3}{2}} +$
$+ \binom{r-1}{0} + \binom{r-1}{1} + \ldots + \binom{r-1}{\frac{r-d-3}{2}} + \binom{r-1}{\frac{r-d-1}{2}} - 2^{r-1} - 2^{r-1}$
because of Formula (3) and properties of binomial coefficients.

Since $A(d-1, r-1) \geq 0$, we have that $P(d-1, r-1) = P(-d+1, r-1)$ and the last sum can be written as
$[P(-d-1, r-1) + \binom{r-1}{0} + \binom{r-1}{1} + \ldots + \binom{r-1}{\frac{r-d-3}{2}} - 2^{r-1}] +$
$+ [P(-d+1, r-1) + \binom{r-1}{0} + \binom{r-1}{1} + \ldots + \binom{r-1}{\frac{r-d-1}{2}} - 2^{r-1}]$
$= A(d+1, r-1) + A(d-1, r-1).$

(b) For $r \geq 3$, we have

$$\begin{aligned}
&2A(1, r-1) + \binom{r-1}{\frac{r-1}{2}} \\
&= 2[P(-1, r-1) + \binom{r-1}{0} + \binom{r-1}{1} + \ldots + \binom{r-1}{\frac{r-3}{2}} - 2^{r-1}] + \binom{r-1}{\frac{r-1}{2}} \\
&= 2P(-1, r-1) + \binom{r}{0} + \binom{r}{1} + \ldots + \binom{r}{\frac{r-1}{2}} - 2^r \\
&= A(0, r).
\end{aligned}$$

□

Theorem 2 gives no formula for $A(d,r)$ in those cases when $A(d-1,r-1)$ is negative. The next result will establish a pattern of negative entries of the array $A(d,r)$ and justify that the only possible negative value of $A(d,r)$ is -1.

Theorem 3. *(a) If $A(d,r) < 0$, then $A(d,r) = -1$.*
(b) If $A(d,r) = -1$, then $A(d',r) = -1$ for $d' > d$.
(c) If $A(d,r) = -1$, then $A(d+1,r+1) = -1$.

Proof. We will prove (a) and (b) by induction on r. Part (c) will follow from the proof.

For $r = 1$, using (7), we have $A(0,1) = P(0,1) + \binom{1}{0} - 2^1 = 1+1-2 = 0$. For $d \geq 2$, $A(d,1) = P(-d,1) - 2^1 = 1-2 = -1$, so (a) and (b) are true for $r = 1$.

Assume that (a) and (b) are satisfied for some $r \geq 1$. Consider the entry $A(d,r+1)$ in the next column. If $d = 0$, then $r+1 \geq 3$ and $A(0,r+1) \geq 3$. Therefore, if $A(d,r+1) < 0$, then $d \geq 1$. Consider the element $A(d-1,r)$ from the previous column. If $A(d-1,r) \geq 0$, then, from (8), $A(d,r+1) = A(d-1,r) + A(d+1,r) \geq 0 + (-1) = -1$.
If $A(d-1,r) < 0$, then, using the inductive assumption, $A(d-1,r) = -1$ and also $A(d+1,r) = -1$. This implies that

$$\begin{aligned}
A(d, r+1) &= P(-d, r+1) + \binom{r+1}{0} + \binom{r+1}{1} + \ldots + \binom{r+1}{\frac{r+1-d-1}{2}} - 2^{r+1} \\
&= P(-d-1, r) + P(d-1, r) + \binom{r}{0} + \binom{r}{1} + \ldots + \binom{r}{\frac{r-d}{2}} + \\
&+ \binom{r}{0} + \binom{r}{1} + \ldots + \binom{r}{\frac{r-d-2}{2}} - 2^r - 2^r.
\end{aligned}$$

Since $A(d-1,r) = -1$, we have $P(d-1,r) = P(-d+1,r)+1$, and the last expression equals to

$$\begin{aligned}
&[P(-d-1,r) + \binom{r}{0} + \binom{r}{1} + \ldots + \binom{r}{\frac{r-d}{2}} - 2^r] + 1 + \\
&+ [P(-d+1,r) + \binom{r}{0} + \binom{r}{1} + \ldots + \binom{r}{\frac{r-d-2}{2}} - 2^r] \\
&= A(d+1,r) + 1 + A(d-1,r) = -1 + 1 - 1 = -1.
\end{aligned}$$

Therefore, $A(d,r+1) = -1$. If $d' > d$, then $A(d',r+1) = -1$ by analogous calculations in which we use entries $A(d'-1,r)$ and $A(d'+1,r)$. □

Theorem 2 and Theorem 3 allow us to evaluate the entries of $A(d,r)$ recursively. The values of $A(d,r)$ for $0 \leq d \leq 12$ and $1 \leq r \leq 15$ are presented in Table 2.

d \r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0		0		4		26		126		562		2416		10180
1		-1		-1		3		28		155		746		3374	
2	-1		-1		-1		2		29		184		968		4571
3		-1		-1		-1		1		29		212		1197	
4	-1		-1		-1		-1		0		28		239		1463
5		-1		-1		-1		-1		-1		27		265	
6	-1		-1		-1		-1		-1		-1		26		290
7		-1		-1		-1		-1		-1		-1		25	
8	-1		-1		-1		-1		-1		-1		-1		24
9		-1		-1		-1		-1		-1		-1		-1	
10	-1		-1		-1		-1		-1		-1		-1		-1
11		-1		-1		-1		-1		-1		-1		-1	
12	-1		-1		-1		-1		-1		-1		-1		-1

TABLE 2. Table of values of $A(d,r)$.

Looking at this table one can observe that the below diagonal pattern of negative entries of $A(d,r)$ has small disturbances. Notice that $A(4,9) = 0$

and, by using (8) from Theorem 2, $A(5,10) = -1$. The entry $A(5,10)$ is the beginning of a new infinite diagonal line of negative entries in $A(d,r)$. Those -1's cause the entries above them to form a decreasing sequence reaching 0 in entry $A(32,39)$. Starting from $A(33,40)$ and going diagonally down, we will have a new line of -1's. Without these disturbances, it would be easy to find closed formulas for $A(d,r)$ and for $P(d,r)$ as well. The disturbances produce some adjustments to those formulas. Fortunately, the disturbances are rare, adjustments are relatively small, and they do not effect the asymptotic behavior of probabilities.

4. Finding Optimal Stopping Time

To find a formula for $A(d,r)$ it is more convenient to work with modified coefficients we call $A'(d,r)$. $A'(d,r)$ are the same as $A(d,r)$ if $A(d,r) \geq 0$ or $A(d,r) = -1$, but this -1 is the first negative entry in column r. Otherwise $A'(d,r) = 0$. In other words, the modified table $A'(d,r)$ is obtained from $A(d,r)$ by replacing all -1 entries (except the first -1 in each column) by 0's. The values of modified $A'(d,r)$ are presented in Table 3 in which all -1 entries are boxed.

d \ r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0		0		4		26		126		562		2416		10180
1		-1		-1		3		28		155		746		3374	
2	-1		-1		-1		2		29		184		968		4571
3		0		0		-1		1		29		212		1197	
4	0		0		0		-1		0		28		239		1463
5		0		0		0		-1		-1		27		265	
6	0		0		0		0		-1		-1		26		290
7		0		0		0		0		0		-1		25	
8	0		0		0		0		0		0		-1		24
9		0		0		0		0		0		0		-1	
10	0		0		0		0		0		0		0		-1
11		0		0		0		0		0		0		0	
12	0		0		0		0		0		0		0		0

TABLE 3. Table of values of modified $A'(d,r)$.

We will need the following properties of binomial coefficient.

Property 1. $\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 2^{2n}$.

The next property uses binomial coefficients and applies to the Pascal triangle, but can be easily generalized to any rectangular array $S(k,n)$ of numbers satisfying recurrence formula $S(k,n) = S(k-1,n-1) + S(k+1,n-1)$.

Property 2. Let $S(k,n), n \geq 1, k \in Z$, be an array of numbers satisfying the recurrence relation

$$S(k,n) = S(k-1,n-1) + S(k+1,n-1)$$

for every $n \geq 2$ and $k \in Z$. Then for all integers n,k,u such that $n > u \geq 1$ we have

$$S(k,n) = \sum_{i=0}^{u} \binom{u}{i} S(k-u+2i, n-u).$$

Therefore, the entry $S(k,n)$ can be expressed as a linear combination of $u+1$ entries from the line $n-u$ (u units to the left from the line n) with coefficients $\binom{u}{i}$ counting the number of southwest-nortwest paths of length u from (k,n) to $(k-u+2i, n-u)$. The number i represents how many northwest steps, out of the total of u steps, we take. For example, the entry $A'(5,14)$ of the Table 3 can be expressed as:
for $u=1$: $A'(5,14)$= 265 = 239 + 26;
for $u=2$: 265 = 212 + 2(27) -1;
for $u=3$: 265 = 184+ 3(28) + 3(-1) + 0;
for $u=4$: 265 = 155 + 4(29 + 6(-1) +4(0) + 0.

The array of modified entries $A'(d,r)$ was defined only for $d \geq 0$ but , for computational purposes only, we can consider the extended array of $A'(d,r)$ for any $d \in Z$ by defining $A'(-d,r) = A'(d,r)$ (the mirror image of the entries in Table 3 with respect to the row $r=0$).

We would like to express the entries of the array $A'(d,r)$ in terms of entries of the 4-th column because this column contains only two nonzero entries, namely $A'(1,4) = A'(-1,4) = -1$. We would like to use Property 2 for the array $A'(d,r)$. However, the recurrence formula $A'(d,r) = A'(d-1,r-1) + A'(d+1,r-1)$ is not valid for $d=0$. The extra term $\binom{r-1}{\frac{r-1}{2}}$ on the right-hand-side of Formula (9) contributes to $A'(0,r)$. There are also contributions from other terms of the form $\binom{r-1-2k}{\frac{r-1-2k}{2}}$ to $A'(0,r)$. The weights of these contributions are equal to the number of diagonal paths from $(0,r)$ to $(0,r-2k)$, that is $\binom{2k}{k}$. Therefore, using Property 2, we can express

$A'(0,r)$ for $r = 2n+1$, where $5 \le 2n+1 \le 15$, as follows:

$$A'(0,r) = A'(0,2n+1)$$
$$= A'(1,4)\binom{2n+1-4}{n-2} + A'(-1,4)\binom{2n+1-4}{n-1} + \sum_{k=2}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}$$
$$= -\binom{2n-3}{n-2} - \binom{2n-3}{n-1} + \sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} - \binom{2n}{n} - \binom{2}{1}\binom{2n-2}{n-1}.$$

Since $\binom{2n-3}{n-2}+\binom{2n-3}{n-1} = \binom{2n-2}{n-1}$ and, by Property 1, $\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} = 2^{2n}$, we have

$$A'(0,2n+1) = 2^{2n} - \binom{2n}{n} - 3\binom{2n-2}{n-1}.$$

For example, if $2n+1=5$, then $n=2$ and $A'(0,5) = 2^4 - \binom{4}{2} - 3\binom{2}{1} = 4$. On the other hand, if $2n+1 = 15$, then $n=7$ and $A'(0,15) = 2^{14} - \binom{14}{7} - 3\binom{12}{6} = 10180$.

For $2n+1 \ge 17$, the formula for $A'(0,2n+1)$ will contain more terms. The entry $A'(7,10)$, which appears on some diagonal paths from $(0,2n+1)$, does not satisfy the recurrence relation $A'(7,10) = A'(8,9) + A'(6,9)$, since $A'(7,10) = 0, A'(8,9) = 0$ and $A'(6,9) = -1$. By the way, neither $A'(3,2)$ nor $A'(3,)$ satisfies the recurrence relation but we do not go back beyond the fourth column in our calculations. The first extra term in $A'(0,2n+1)$ for $2n+1 \ge 17$ is $2\binom{2n+1-10}{\frac{2n+1-10-7}{2}} = 2\binom{2n-9}{n-8}$. The factor 2 is present because the identical adjustments are produced by entries (7, 10) and (-7, 10). The binomial coefficient counts the number of diagonal paths from $(0,2n+1)$ to (7, 10).

For larger values of $2n+1$ similar adjustments are produced by zero entries of $A'(d,r)$ ending the diagonally down sequences of -1's. Following $A'(7,10)$, the two next entries of this type are $A'(35,40)$ and $A'(624,631)$ ending sequences of -1's, boxed in Table 3, of lengths 30 and 591, respectively. Let us denote the coordinates of s^{th} such entry by (w_s, k_s), where $(w_1,k_1) = (7,10)$. After including all those extra terms, the general formula for $A'(0,2n+1)$ is

$$A'(0,2n+1) = 2^{2n} - \binom{2n}{n} - 3\binom{2n-2}{n-1} + 2\sum_{s=1}^{z}\binom{2n+1-k_s}{n+s-1-k_s},$$

where z is the largest integer such that $n+z-1-k_z \ge 0$.

We will find recurrence formulas for (w_s,k_s). We will prove that k_s (as well as w_s) grow very quickly (super-exponentially) and the number of those entries that effect $A'(0,2n+1)$ is very small (sub-logarithmic). In addition, the positions of those entries will allow us to determine the shape of the boundary (boxed -1's in Table 3) separating the region where the optimal strategy tells us to wait from the region where the optimal strategy tells us to stop. The boundary we are interested in consists of line segments going down diagonally, that are translated two units up just before reaching the entry $A'(w_s, k_s)$. The recurrence formula (8) from Theorem 2 explains this phenomenon. Let us look at the entries just above this boundary. The entry $A'(0,5) = 4$ starts the decreasing sequence of entries going diagonally down and ending with 0. The entry above this zero, $A'(2,9) = 29$, is repeated (because of Formula 8) as $A'(3,10) = 29$, which stars a new decreasing sequence of entries going diagonally down and ending with 0. This pattern continues producing the shape of the boundary depicted in Figure 1.

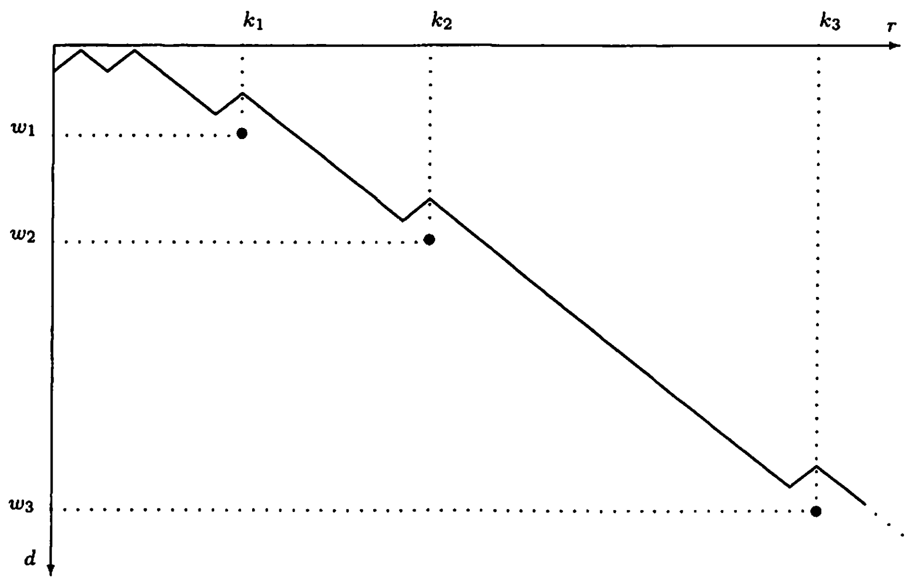


FIGURE 1. The shape of the boundary in $A'(d,r)$.

The consecutive entries ending the sequences of (-1)'s have coordinates $(w_1, k_1) = (7,10)$ and $(w_2, k_2) = (35,40)$. The entries above $(w_s, k_s), s \geq 1$, are equal to $A(w_s - 2, k_s) = -1$. Let's denote the entries above them by $A(w_s - 4, k_s) = l_s$. Notice that l_s determines the length of the next diagonal segment of (-1)'s. More precisely, we have $k_{s+1} = k_s + l_s + 1$ for every $s \geq 1$. Therefore, a formula for l_s will allow us to find recursively k_s for $s \geq 2$. Notice that w_s can be eliminated because the boundary equation is given by

$$
(10) \qquad d = \begin{cases} r-3, & \text{for } 3 < r \le 10 = k_1; \\ r-5, & \text{for } 10 < r \le 40 = k_2; \\ \vdots & \\ r-2s-1, & \text{for } k_{s-1} < r \le k_s, \\ \vdots & \end{cases}
$$

where d is the row number and r is the column number.

Therefore, $(w_s, k_s) = (k_s - 2s - 1, k_s)$ and $l_s = A'(w_s - 4, k_s) = A'(k_s - 2s - 5, k_s)$. Let's describe how to find k_s and l_s for $s \ge 1$. We have $k_1 = 10$, $l_1 = 29$, and $k_2 = 40$. Knowing the values of $k_1, k_2, \dots, k_s$ allows us to find l_s and k_{s+1} according to the following recurrence formulas:

$$
l_s = \sum_{i=0}^{s} \binom{k_s-2s-5+2i}{i}\binom{2s+4-2i}{s+2-i} - \binom{k_s-3}{s+1} + \sum_{j=1}^{s-1} \binom{k_s-k_j}{s+2-j} + \sum_{j=1}^{s-1} \binom{k_s-k_j}{s+2j+2-k_j},
$$

$$
(11) \qquad k_{s+1} = k_s + l_s + 1.
$$

The recurrence formula for l_s looks very unpleasant. Fortunately, for a fixed value of n, the number of terms in the summation present in l_s is very small. This observation will follow from several technical lemmas about the rate of growth of the sequence $\{k_s\}_{s \ge 1}$.

Lemma 1. *For $n \ge 4, A(0, 2n+1) < 6A(1, 2n)$.*

Proof.

$$
\begin{aligned}
A(0, 2n+1) &= 2A(1,2n) + \binom{2n}{n} \le 2A(1,2n) + 4\binom{2n-2}{n-1} \\
&< 2A(1,2n) + 4A(0,2n-1) < 2A(1,2n) + 4A(1,2n) = 6A(1,2n).
\end{aligned}
$$

□

Before proving the next lemma, let us observe that l_s is the largest entry on the diagonal line containing it, which means that for every $i \ge 0$,

$$
A(i, k_s - w_s + 4 + i) \le l_s.
$$

Also in the diagonal line below it $A(i, k_s - w_s + 2 + i) \le l_{s-1}$ for all $i \ge 0$.

Lemma 2. *For every $s \ge 2, l_s < k_s(k_s - 1)$.*

Proof. For every $s \geq 2$, using the recurrence relation $(w_s - 4)$ times, we get

$$l_s = A(w_s - 4, k_s) = A(w_s - 5, k_s - 1) + A(w_s - 3, k_s - 1)$$
$$= A(w_s - 6, k_s - 2) + A(w_s - 4, k_s - 2) + A(w_s - 3, k_s - 1)$$
$$= A(w_s - 7, k_s - 3) + A(w_s - 5, k_s - 3) + A(w_s - 4, k_s - 2) + A(w_s - 3, k_s - 1)$$
$$= \ldots$$
$$= A(0, k_s - w_s + 4) + \sum_{i=2}^{w_s-3} A(i, k_s - w_s + 2 + i)$$
$$< A(0, k_s - w_s + 4) + (w_s - 4)l_{s-1} < 6A(1, k_s - w_s + 3) + (w_s - 4)l_{s-1}$$
$$\leq 6l_{s-1} + (w_s - 4)l_{s-1} \leq (w_s + 2)l_{s-1}$$
$$\leq (k_s - 1)(k_s - k_{s-1} - 1) < k_s(k_s - 1)$$

which concludes the proof. □

Lemma 3. *For every* $s \geq 1, k_{s+1} \leq {k_s}^2$.

Proof. Using Lemma 2, we can write

$k_{s+1} = k_s + l_s + 1 < k_s + k_s(k_s - 1) + 1 = {k_s}^2 + 1$, so $k_{s+1} \leq {k_s}^2$.

□

Before proving the next lemma, let's notice that for the number $l_s = A(w_s - 4, k_s)$, the entry below it is equal to $A(w_s - 2, k_s) = -1$ and the diagonal including this -1 and going northwest in Table 3 has entries $0, 1, 2, \ldots, l_{s-1}$. From the recurrence relation, all these entries contribute to l_s and, therefore,

$$l_s = A(w_s - 4, k_s) \geq \sum_{i=1}^{l_{s-1}} i = \frac{1}{2} l_{s-1}(l_{s-1} + 1).$$

Lemma 4. *For every* $s \geq 1, k_{s+1} > \frac{{k_s}^2}{4}$.

Proof. We use induction on s.

The inequality is satisfied for $s = 1$, since $k_2 = 40 > \frac{10^2}{4} = \frac{{k_1}^2}{4}$.

For $s \geq 2$, assuming that $k_s > \frac{(k_{s-1})^2}{4}$, we have

$$k_{s+1} = k_s + l_s + 1 \geq k_s + \tfrac{1}{2} l_{s-1}(l_{s-1} + 1) + 1 = k_s + \tfrac{1}{2}(k_s - k_{s-1} - 1)(k_s - k_{s-1}) + 1$$

$$= k_s + \frac{1}{2}(k_s - k_{s-1})^2 - \frac{1}{2}(k_s - k_{s-1}) + 1 \geq \frac{1}{2}[(k_s - k_{s-1})^2 + k_s + 2]$$
$$> \frac{1}{2}[(k_s - 2\sqrt{k_s})^2 + k_s + 2] \geq \frac{(k_s)^2}{4}.$$

The last inequality is true since $k_s \geq 40$ for $s \geq 2$, and the previous inequality follows from the inductive assumption. □

Lemma 5. *For every* $s \geq 1, k_s \geq \frac{10^{2^{s-1}}}{4^{(2^{s-1}-1)}}$.

Proof. We use induction on s.

The inequality is satisfied for $s = 1$, since $k_1 = 10 \geq \frac{10^1}{4^0}$.

Assume that the inequality is true for some $s \geq 1$. Then, by Lemma 4,

$$k_{s+1} > \frac{1}{4}(k_s)^2 \geq \frac{1}{4}\left(\frac{10^{2^{s-1}}}{4^{(2^{s-1}-1)}}\right)^2 = \frac{10^{2^s}}{4^{(2^s-1)}}.$$

□

From Lemma 5, we have that $k_{s+1} > (2.5)^{2^s}$ for every $s \geq 1$. Therefore, if $(2.5)^{2^s} > 2n+1$ or, equivalently, $s > \frac{\ln\left(\frac{\ln(2n+1)}{\ln 2.5}\right)}{\ln 2}$, then $k_{s+1} > 2n+1$. In other words, for every integer n at most $\left\lceil \frac{\ln\left(\frac{\ln(2n+1)}{\ln 2.5}\right)}{\ln 2} \right\rceil$ values of s produce $k_s \leq 2n+1$.

5. Optimal Strategy, the Value of the Game, and Asymptotics

We are ready to describe an algorithm giving the optimal stopping time and the probability of success. Informally, the knowledge of the value of $2n+1$ allows us to find an equation of the boundary. When we sample the urn, we construct a random $\{-1, 1\}$ sequence which determines the states $S(d, r)$ with r decreasing from the starting value of $2n+1$. This sequence of states determines a random walk in any of the table 1, 2, or 3. At some time, that is also a random variable, we hit the boundary. When this happens the last move had to be in the southwest direction which means that the last selected ball is in the current majority color and, therefore, d is negative. Since we cannot stop then, the algorithm tells us to wait for the first ball in the other color and stops on it. The value $p(0, 2n+1) = \frac{P(0,2n+1)}{2^{2n+1}}$ from Table 1 gives the expected value of the game, the probability of stopping on the minority color when the optimal stopping time τ is used. Of course, for a particular realization of a random sequence these probabilities change with time and also can be found in Table 1. If we are more lucky and the sequence of balls revealed so far is more unbalanced ($|d|$ is large), then the boundary will be hit earlier and the probability of success will be larger.

Algorithm:
Input: $2n+1$
Output: The optimal stopping time τ and $E(Z_\tau)$.

(1) Using recurrence formula (11) evaluate $k_1, \ldots, k_z$ until $k_z \leq 2n+1$ and $k_{z+1} > 2n+1$.

(2) Values $k_1, k_2, \ldots, k_z$ give the equation of the boundary, the first z diagonal line segments of (10). The optimal stopping time τ is defined as follows:
   (a) Let t be the first time such that (d, r), where $r = 2n+1-t$, satisfies the equation of the boundary.
   (b) At that time t the value of d is negative.
   (c) $\tau = min\{T : T > t \text{ and } x_T = -x_t\}$ or equivalently, τ is the first moment in which we see the ball of the color different that x_t; with the assumption that $min\ \emptyset = 2n+1$.
(3) The probability of succes when the optimal strategy τ is used is

$$E(Z_\tau) = p(0, 2n+1) = \frac{1}{2} + a(0, 2n+1) = \frac{1}{2} + \frac{A'(0, 2n+1)}{2^{2n+1}}$$

$$= 1 - 2^{-(2n+1)} \left\{ \binom{2n}{n} + 3\binom{2n-2}{n-1} - 2\sum_{s=1}^{z} \binom{2n+1-k_s}{n+s-1-k_s} \right\}.$$

We will establish the asymptotic behavior of $E(Z_\tau)$ for the game $G(0, 2n+1)$. Let's denote the value of this game by $E_{2n+1}(Z_\tau)$. It turns out that

$$E_{2n+1}(Z_\tau) = 1 - O(\frac{1}{\sqrt{n}}).$$

More precisely,

$$\lim_{n\to\infty} [1 - E_{2n+1}(Z_\tau)]\sqrt{n} = c,$$

where the constant c satisfies the inequalities $0.4925 < c < 0.4926$ and can be easily estimated with higher accuracy. One can justify the last statement by using Stirling's formula for binomial coefficients present in the formula for $E(Z_\tau)$. Then we obtain

$$\lim_{n\to\infty} [1 - E_{2n+1}(Z_\tau)]\sqrt{n} = c = \frac{2}{\sqrt{\pi}} \left\{ \frac{7}{16} - \sum_{i=1}^{\infty} 2^{-k_i} \right\}.$$

The exact value of c can be easily approximated because the terms of the series $\sum_{i=1}^{\infty} 2^{-k_i}$ approach zero extremely fast; recall that $k_3 = 631$, so the third term of the sum is 2^{-631}.

As an illustration, if $2n+1 = 1001$, or $n = 500$, then the exact value of the game $G(0, 1001)$ is

$$E_{1001}(Z_\tau) = 1 - \frac{\binom{1000}{500} + 3\binom{998}{499} - 2\binom{991}{490} - 2\binom{961}{461}}{2^{1001}},$$

which is approximately 0.9779652. The asymptotic formula with $n = 500$ and $c = 0.49255$ gives $E_{1001}(Z_\tau) \simeq 1 - 0.49255\frac{1}{\sqrt{500}} \simeq 0.97797$.

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