CALCULATING THE FREQUENCY OF TOURNAMENT SCORE SEQUENCES

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ABSTRACT. We indicate how to calculate the number of round-robin tournaments realizing a given score sequence. This is obtained by inductively calculating the number of tournaments realizing a score function. Tables up to 18 participants are obtained.

1. TOURNAMENTS AND SCORE SEQUENCES

A (round-robin) *tournament* on a set P of n vertices (participants, teams, ...) is a directed graph obtained by orienting the complete graph K_n on P. In other words, a tournament is a directed graph on the vertex set P having exactly one arc connecting each pair in P.

Clearly there are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs in P to be connected by one of two possible arcs, and thus the total number of tournaments of size n is $2^{\binom{n}{2}}$.

The *score function* f_t of a tournament t on P gives for each $p \in P$ the outdegree $f_t(p)$ of p, i.e. $f_t(p)$ is the number of arcs of t leaving p. When the values of a score function f_t are ordered (nondecreasingly, by convention) we obtain a *score sequence*. We say that this score sequence is realized by the tournament t.

Three questions immediately arise concerning score sequences:

(1) Which sequences are the score sequence of some tournament ?
(2) How many different score sequences exist ?
(3) How many tournaments realize a given score sequence ?

The first question was solved by Landau [7] when investigating dominance relations within animal societies by the following result.

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Thanks to Dennis Van Den Broeck for bringing this question to my attention.

TABLE 1. Number of score sequences. Narayana and Bent (1964)

n	T_n	n	T_n
1	0	19	73996100
2	1	20	259451116
3	2	21	951695102
4	4	22	3251073303
5	9	23	11605141649
6	22	24	41631194766
7	59	25	150021775417
8	167	26	542875459724
9	490	27	1972050156181
10	1486	28	7189259574618
11	4639	29	26295934251565
12	14805	30	96478910768821
13	48107	31	354998461378719
14	158808	32	1309755903513481
15	531469	33	4844523965710167
16	1799659	34	17961489379744400
17	6157068	35	66742666423989519
18	21258104	36	248530319605591021

Theorem 1.1 (Landau). *Let $s = (s_1, s_2, \ldots, s_n)$ be a nondecreasing sequence of nonnegative integers. Then s is the score sequence of some tournament if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} = \frac{k(k-1)}{2} \tag{1.1}$$

for all $k = 1, 2, \ldots, n$, with an equality for $k = n$.

Using this characterization it is not hard to develop algorithms to generate all possible score sequences of given length. A relatively recent one may found in Hemasinha [6], who gives further references. One of the problems is, however, that the number T_n of possible score sequences rises sharply with n. Determining T_n is the second question mentioned above. No explicit formula seems to be known, but some asymptotic bounds were obtained, see e.g. Winston and Kleitman [9]. Narayana and Bent [5] gave the first recursive formulas to calculate this number for any n, and obtained table 1 for $n = 1, \ldots, 36$. Note the correction for T_{11}, incorrectly stated as 4649.

The third question is particularly of interest for studying statistical hypothesis testing in sports concerning equivalence of teams or players, or in experiments involving the method of paired comparisons, as used in psychology (see David [4]). To the best of our knowledge the only results concerning this question have been published by Bradley and Terry [3], Bradley [2] and by David [4]. The first two papers form a series and give frequency tables up to $n = 5$, while the latter gives tables up to $n = 8$, obtained using a generating function approach, that calls for symbolic algebra tools. This note is intended to continue in the direction of solving the frequency question in a more direct combinatorial way using induction.

In view of the sizes indicated in table 1 it is not possible to print much larger tables than those published previously, but complete tables up to $n = 18$ have been generated and are available online.

2. Frequency of a score function

A function $f : P \to [n]$, where $[n] \stackrel{\text{def}}{=} \{ 0, \dots, n-1 \}$, is a *score function* on P if $f = f_t$, the score function of some tournament t on P; in this case we say that t *realizes* f. We start by deriving an induction formula for the number of tournaments realizing f, that we call the frequency of f and denote by $F(f)$.

2.1. First induction formula. Let t be any tournament on P that realizes f. For any fixed $q \in P$ the set P partitions into $\{ q \}$, $t^+(q) = \{ r \in P \mid (q,r) \in t \}$ and $t^-(q) = \{ p \in P \mid (p,q) \in t \}$, and $f(q) = f_t(q) = |t^+(q)|$, while $|t^-(q)| = n - 1 - f(q)$.

Deletion of q from P, together with all arcs in t that contain q, defines a tournament t' on the set $P' = P \setminus \{ q \}$.

Inversely the tournament t on $P = P' \cup \{ q \}$ is uniquely determined by the tournament t' on P' and the subset Q of $n - 1 - f(q)$ elements of P' that is to constitute $t^-(q)$, by the operation

$$t = t' \cup \{ (p,q) \mid p \in Q \} \cup \{ (q,r) \mid r \in P' \setminus Q \} \tag{2.1}$$

Evidently, when either the choice of t' changes or another subset $Q \subset P'$ is selected, the resulting tournament t will be different.

The relation between the score functions of t and t' is as follows

$$f_{t'}(p) = \begin{cases} f_t(p) - 1 & \text{when } p \in t^-(q) \\ f_t(p) & \text{when } p \in t^+(q) \end{cases} \tag{2.2}$$

Thus $f_{t'}$ is obtained from f through following two-step modification:

(1) drop the element $f(q)$
(2) choose $n - 1 - f(q)$ other elements of f and decrease them by 1.

Let $\Phi_q(f)$ denote the set of all such q-predecessors of f, i.e. functions $v : P' \to [n-1]$ obtainable from f by this kind of two-step modification.

We may then write a first induction formula

$$F(f) = \sum_{v \in \Phi_q(f)} F(v) \tag{2.3}$$

The number of q-predecessors of f depends on the choice of q, more precisely on the value $f(q)$, and is given by $\binom{n-1}{n-1-f(q)}$. Therefore, one possible strategy for choosing q may consist in minimizing this number, which is obtained by selecting among the minimum and maximum value of $f(q)$ the one farthest from $(n-1)/2$.

The simplest situation arises when either $f(q) = 0$ or $f(q) = n-1$ and the number of q-predecessors of f equals 1. One easily sees that since f satisfies the Landau condition (1.1) at most one 0 and at most one $n-1$ may appear in f. One then simply obtains $F(f) = F(v)$ for the only q-predecessor of f that is found as follows: in either case one drops $f(q)$ from f, and in case $f(q) = 0$ one also decreases each remaining $f(p)$ by 1. In this case this particular choice of q is evidently to be preferred.

It should also be noted that it is not guaranteed that q-predecessors of f always satisfy the Landau condition (1.1). Since in such a case it is the score function of no tournament we have $F(v) = 0$. Alternatively we may test each potential q-predecessor and retain as valid only those satisfying condition (1.1).

As a simple example of this phenomenon one may consider any score function f on P with $f(p) = n-1$ for some $p \in P$, then any choice of $q \neq p$ and $Q \not\ni p$ will lead to a q-predecessor v of f with $v(p) = n-1$, which cannot be a score function on P' because the maximal allowed value appearing in it is $n-2$. Hence such choices of Q should be avoided. In fact we saw before that for such f the choice $q = p$ is a much better one that additionally totally avoids the difficulty. However, the phenomenon arises also quite often in less simple situations, as exemplified next.

Example 2.1. Consider as an example the set $P = \{ a_1, a_2, a_3, a_4, a_5 \}$, and the score function $f = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 3 & 3 & 2 & 1 & 1 \end{pmatrix}$. Taking $q = a_3$ and applying the two step modification above choosing for $Q = \{ a_1, a_4 \}$ with $n - 1 - f(q) = 5 - 1 - 2 = 2$ elements, we obtain the q-predecessor of f : $\begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ 2 & 3 & 0 & 1 \end{pmatrix}$.

The set Q may be chosen in $\binom{4}{2}$ ways, obtaining the following 6 elements of $\Phi_q(f)$

$$v_1 = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ 2 & 2 & 1 & 1 \end{pmatrix}, v_2 = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ 2 & 3 & 0 & 1 \end{pmatrix}, v_3 = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ 2 & 3 & 1 & 0 \end{pmatrix},$$
$$v_4 = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ 3 & 2 & 0 & 1 \end{pmatrix}, v_5 = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ 3 & 2 & 1 & 0 \end{pmatrix}, v_6 = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ 3 & 3 & 0 & 0 \end{pmatrix}.$$

Note that v_6 does not satisfy the Landau condition (1.1), so is not the score function of a tournament, thus $F(v_6) = 0$. □

Observe that for calculation purposes the formula (2.3) is far from efficient, since many of the terms $F(v)$ are equal; e.g. in the example above we will have $F(v_2) = F(v_3) = F(v_4) = F(v_5)$ for reasons of symmetry.

2.2. **Improved induction formula.** For any permutation π of P and any score function g on P, $g \circ \pi$ is also a score function. Indeed for any tournament t on P realizing g the tournament $\pi(t) = \{ (\pi(p), \pi(q)) \mid (p, q) \in t \}$ realizes $g \circ \pi$. We also have $F(g) = F(g \circ \pi)$.

For a score function g on P we define its repetition function $\rho_g : [n] \to [n]$ that gives for each $k = 0, \ldots, n-1$ the number of times k appears as score in g:

$$\rho_g(k) \stackrel{\text{def}}{=} |\{ p \in P \mid g(p) = k \}|$$

The repetition function is the basic invariant of score functions: for two score functions g, h there exists a permutation π on P such that $h = g \circ \pi$ if and only if $\rho_g = \rho_h$. It follows that if $\rho_g = \rho_h$ we have $F(g) = F(h)$, and hence the frequency $F(f_\rho)$ of any score function f_ρ with repetition function ρ depends on ρ only.

Repetition functions of (the score function of) tournaments on P satisfy following properties:

$$\sum_{k=0}^{n-1} \rho(k) = n \tag{2.4}$$

$$\sum_{k=0}^{m} k\rho(k) \geq \binom{m+1}{2} \quad \text{when } 0 < m < n-1 \tag{2.5}$$

$$\sum_{k=0}^{n-1} k\rho(k) = \binom{n}{2} \tag{2.6}$$

Example 2.2. (continuation of example 2.1) The score function f has as repetition function $\rho_f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 2 & 0 \end{pmatrix}$. □

The q-predecessors set of f, $\Phi_q(f)$ may be split into classes, each class containing all score functions on P' with repetition function equal to some fixed $\rho : [n-1] \to [n-1]$.

As repetition function of some q-predecessor of f such a ρ is found as follows.

(1) choose $j = f(q)$

(2) Choose numbers t_k ($k \in [n]$) such that

$$\begin{array}{rcll} 0 \le t_k & \le & \rho_f(k) & \text{when } k \neq j \\ 0 \le t_j & \le & \rho_f(j) - 1 & \\ \sum_{k\in[n]} t_k & = & n-1-j & \end{array}$$

(3) Define ρ by

$$\begin{array}{rcll} \rho(k) & = & \rho_f(k) - t_k + t_{k+1} & \text{when } j \neq k \in [n-1] \\ \rho(j) & = & \rho_f(j) - 1 - t_j + t_{j+1} & \text{except when } j = n-1 \end{array}$$

We denote by $\Psi_q(f)$ the set of such repetition functions.

The exceptional case $j = n-1$ corresponds to $f(q) = n-1$. In this case we must have $\sum_{k\in[n]} t_k = 0$ and hence all $t_k = 0$, so $\Psi_q(f)$ is a singleton. Its only element ρ is given by $\rho(k) = \rho_f(k)$ for all $k \in [n-1]$ (while $\rho(k)$ remains undefined for $k = j = n-1$, as it should). This is the repetition function of the only q-predecessor of f, obtained by simply deleting from f the element at position q.

In case $f(q) = 0$ we have $j = 0$. Note that this choice means $\rho_f(0) = 1$ (higher values being excluded for tournament score functions), so we must choose $t_0 = 0$. So $\sum_{k\in[n]} t_k = n - 1 = \sum_{k\in[n]} \rho_f(k) - 1 = \sum_{k=1}^{n-1} \rho_f(k)$ and it follows that we have as only possible choice $t(k) = \rho_f(k)$ for all $k = 1, \ldots, n-1$. So again $\Psi_q(f)$ is a singleton with only element ρ given by $\rho(k) = \rho_f(k+1)$ for all $k \in [n-1]$. This 'shifted down' repetition function corresponds with the only q-predecessor of f, obtained by deleting from f the element at position q and decreasing all other elements by 1.

Example 2.3. (continuation of example 2.2) Taking for $q = a_3$ as in example 2.1, we choose $j = f(q) = 2$. Then the numbers t_k must satisfy $t_0 = t_2 = t_4 = 0$, while $0 \le t_1, t_3 \le 2$ and $t_1 + t_3 = n - 1 - j = 2$. Hence there are only three choices possible for (t_1, t_3): $(2, 0)$, $(1, 1)$, $(0, 2)$. and these yield as elements of $\Psi_q(f)$ respectively

$$\rho_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 0 & 2 \end{pmatrix}, \rho_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \rho_3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 2 & 0 \end{pmatrix}.$$

ρ_1 is not a tournament's repetition function, since $\sum_{k=0}^{1} k\rho_1(k) = 0 < \binom{2}{2} = 1$ contradicting property (2.5). In fact it is the repetition function of the invalid score function v_6. ρ_2 is the repetition function of the permutation-equivalent score functions v_2, v_3, v_4 and v_5. ρ_3 is the repetition function of v_1. □

Each element of $\Psi_q(f)$ is uniquely determined by the numbers $t = (t_k)_{k\in[n]}$ and we denote it by $\rho_f^{q,t}$. The q-predecessors of f with repetition function $\rho = \rho_f^{q,t}$ correspond exactly to those obtained by way of a set

Q such that

$$|Q \cap \{ p \in P \setminus \{ q \} \mid f(p) = k \}| = t_k.$$

This means that Q selects for every $k \neq j$ exactly t_k elements out of the $\rho_f(k)$ elements $p \in P$ for which $f(p) = k$, and t_j elements out of the $\rho_f(j) - 1$ elements of $p \in P \setminus \{ q \}$ for which $f(p) = f(q) = j$. Clearly the number of such choices for Q equals

$$N_\rho \stackrel{\text{def}}{=} \binom{\rho_f(j) - 1}{t_j} \prod_{\substack{k=0 \\ k \neq j}}^{n-1} \binom{\rho_f(k)}{t_k}$$

and so this is the number of q-predecessors of f with repetition function $\rho = \rho_f^{q,t}$.

We therefore obtain the second induction formula

$$F(f) = \sum_{\rho \in \Psi_q(f)} N_\rho F(f_\rho) \tag{2.7}$$

where f_ρ denotes any score function with repetition function ρ. As representative score function for ρ we have opted in what follows when needed to select the only score function with nonincreasing values. This allows to distinguish them from the nondecreasing score sequences, except in the case of constant score functions, where we let the context guide the reader.

Example 2.4. (continuation of example 2.3) Using $\rho_f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 2 & 0 \end{pmatrix}$, $j = f(q) = 2$ and encoding t as $(t_0, t_1, \ldots, t_4)$ we obtain the following elements of $\Psi_q(f)$:

- $t = (0,2,0,0,0)$ leads to ρ_1, so $N_{\rho_1} = \binom{1-1}{0}\binom{0}{0}\binom{2}{2}\binom{2}{0}\binom{0}{0} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$, which is the cardinality of $\{ v_6 \}$.
- $t = (0,1,0,1,0)$ leads to ρ_2, so $N_{\rho_2} = \binom{1-1}{0}\binom{0}{0}\binom{2}{1}\binom{2}{1}\binom{0}{0} = 1 \cdot 1 \cdot 2 \cdot 2 \cdot 1 = 4$, that equals the cardinality of $\{ v_2, v_3, v_4, v_5 \}$.
- $t = (0,0,0,2,0)$ leads to ρ_3, so $N_{\rho_3} = \binom{1-1}{0}\binom{0}{0}\binom{2}{0}\binom{2}{2}\binom{0}{0} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$ which is the cardinality of $\{ v_1 \}$.

The induction formula (2.7) therefore yields

$$F(f) = F(f_{\rho_1}) + 4 \cdot F(f_{\rho_2}) + F(f_{\rho_3}) = 4 \cdot F(f_{\rho_2}) + F(f_{\rho_3})$$

because $F(f_{\rho_1}) = 0$. □

2.3. Some simple rules.

2.3.1. *Maximum score.* The situation where the maximum possible score $n-1$ appears in a score function of length n has been evoked several times before. We can now finalise these remarks in the following useful lemma.

Let g be a score function of length $n-1$, then we denote by $[n-1, g]$ the score function of length n that is obtained by augmenting g by the score $n-1$.

Lemma 2.5. $F([n-1, g]) = F(g)$

Proof
Since g is of length $n-1$ it contains only values at most $n-2$, so does not contain the value $n-1$. There is only one q-predecessor of $f = [n-1, g]$ when q is chosen such that $f(q) = n-1$, which is obtained from f by dropping the element at position q (which is the score $n-1$). No other modifications are necessary since $|Q| = n-1-f(q) = 0$. In terms of repetition functions we have $\rho_f(k) = \rho_g(k)$ for $k < n-1$, and $\rho_f(n-1) = 1$.

Thus the only q-predecessor of f is g (with repetition function ρ_g). To calculate N_{ρ_g} we observe that all its factors are $\binom{\rho_f(k)}{0} = 1$ for $k < n-1$ and $\binom{1}{1} = 1$, so $N_{\rho_g} = 1$. Induction formula (2.7) then yields $F([n-1, g]) = F(f) = N_{\rho_g} F(g) = F(g)$. $\square$

2.3.2. *Minimum score.* Similarly we have a simple induction rule for score functions containing the minimal value 0. Let g be a score function of length $n-1$, then we denote by $[g+1, 0]$ the score function obtained by adding 1 to each score in g and augmenting it in length with the value 0.

Lemma 2.6. $F([g+1, 0]) = F(g)$

Proof
Clearly $f = [g+1, 0]$ contains only one score of 0. Choosing q such that $f(q) = 0$, we obtain only one q-predecessor of f by dropping the element at position q and subtracting 1 from each of the $n-1-0$ other scores (those of $g+1$), in other words, we obtain g as only q-predecessor of f. In terms of repetition functions we have $\rho_f(k) = \rho_g(k-1)$ for $k > 0$, and $\rho_f(0) = 1$.

Thus the only q-predecessor of f is g (with repetition function ρ_g). To calculate N_{ρ_g} we observe that all its factors are $\binom{\rho_f(k)}{\rho_f(k)} = 1$ for $k > 0$ and $\binom{0}{0} = 1$, so $N_{\rho_g} = 1$. Induction formula (2.7) then yields $F([g+1, 0]) = F(f) = N_{\rho_g} F(g) = F(g)$. $\square$

2.3.3. *Complementation.* The complement t^{-1} of a tournament t is obtained by inversion of all arcs of t, which also yields a tournament. The score function of t^{-1} counts the indegrees of t, so $f_{t^{-1}}(k) = n-1-f_t(k)$. This is the complement $f^c \stackrel{\text{def}}{=} n-1-f$ of the score function f.

Complementation works on repetition functions as follows $\rho_{f_c}(k) = \rho_f(n-1-k)$.

Lemma 2.7. $F(f^c) = F(f)$

Proof

Complementation is an involutory bijection on all tournaments, i.e. $(t^{-1})^{-1} = t$. It follows that complementary score functions have the same frequency. □

Application of this simple rule almost halves the work of calculating frequencies of score functions (and score sequences). In fact Lemma 2.6 may be seen as the combination of lemma 2.5 and lemma 2.7.

2.4. **Induction initialization and first steps.** In what follows we use shorthand notation for score functions, their frequencies and repetition functions, exemplified by:

- $[5,2,4,2,1,2]$ is the score function $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 5 & 2 & 4 & 2 & 1 & 2 \end{pmatrix}$ on the set $P = \{ a_1, a_2, a_3, a_4, a_5, a_6 \}$.
- Its frequency $F([5,2,4,2,1,2])$ is simplified to $F[5,2,4,2,1,2]$.
- $\langle 0,1,3,0,1,1 \rangle$ is its repetition function $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 0 & 1 & 1 \end{pmatrix}$.
- As in example 2.4 we denote the vectors t needed for the construction of $\Psi_q(f)$ as $(t_0, t_1, \dots, t_{n-1})$.

As explained in next section there is a one-one correspondence between repetition functions and score sequences. It follows that for each n the number of different repetition functions of tournaments to be considered equals T_n as given in table 1. These are listed in lexicographical order of their representative score function.

When possible we will apply lemmas 2.6, 2.5 and 2.7 without further reference.

2.4.1. **n = 2**. For $n = 2$ only one possible repetition function exists.

(1) $[1,0]$ represents $\langle 1,1 \rangle$. It may evidently be realized by just 1 tournament.
 Hence $F[1,0] = 1$.

2.4.2. **n = 3**. For $n = 3$ we have two possible repetition functions.

(1) $[2,1,0]$ represents $\langle 1,1,1 \rangle$.
 $F[2,1,0] = F[1,0] = 1$.
(2) $[1,1,1]$ represents $\langle 0,3,0 \rangle$.
 We choose $j = 1$ (there is no other choice anyway), and there is only one possible choice for t:

- $t = (0,1,0)$, yielding the reduced $\rho_1 = \langle 0,1\rangle$ with $f_{\rho_1} = [1,0]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{2}{1}\binom{0}{0} = 1\cdot 2\cdot 1 = 2$.

Hence $F[1,1,1] = N_{\rho_1}\cdot F[1,0] = 2\cdot 1 = 2$.

2.4.3. **n = 4.** For $n = 4$ we have 4 possible repetition functions.

(1) $[3,2,1,0]$ represents $\langle 1,1,1,1\rangle$.
  $F[3,2,1,0] = F[2,1,0] = 1$.

(2) $[3,1,1,1]$ represents $\langle 0,3,0,1\rangle$.
  $F[3,1,1,1] = F[1,1,1] = 2$.

(3) $[2,2,2,0]$ represents $\langle 1,0,3,0\rangle$. This is the complement of the previous one.
  $F[2,2,2,0] = F[3,1,1,1] = 2$.

(4) $[2,2,1,1]$ represents $\langle 0,2,2,0\rangle$.
  We choose $j = 2$, and there are two choices for t:
  - $t = (0,1,0,0)$, yielding the reduced $\rho_1 = \langle 1,1,1\rangle$ with $f_{\rho_1} = [2,1,0]$.
    We have $N_{\rho_1} = \binom{0}{0}\binom{2}{1}\binom{1}{0}\binom{0}{0} = 1\cdot 2\cdot 1\cdot 1 = 2$.
  - $t = (0,0,1,0)$, yielding the reduced $\rho_2 = \langle 0,3,0\rangle$ with $f_{\rho_1} = [1,1,1]$.
    We have $N_{\rho_2} = \binom{0}{0}\binom{2}{0}\binom{1}{1}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1 = 1$.

  Hence $F[2,2,1,1] = N_{\rho_1}\cdot F[2,1,0] + N_{\rho_2}\cdot F[1,1,1] = 2\cdot 1 + 1\cdot 2 = 4$.

2.4.4. **n = 5.** For $n = 5$ there are 9 cases to consider.

(1) $[4,3,2,1,0]$ represents $\langle 1,1,1,1,1\rangle$.
  $F[4,3,2,1,0] = F[3,2,1,0] = 1$.

(2) $[4,3,1,1,1]$ represents $\langle 0,3,0,1,1\rangle$.
  $F[4,3,1,1,1] = F[3,1,1,1] = 2$.

(3) $[4,2,2,2,0]$ represents $\langle 0,0,3,0,1\rangle$.
  $F[4,2,2,2,0] = F[2,2,2,0] = 2$.

(4) $[4,2,2,1,1]$ represents $\langle 0,2,2,0,1\rangle$.
  $F[4,2,2,1,1] = F[2,2,1,1] = 4$.

(5) $[3,3,3,1,0]$ represents $\langle 1,1,0,3,0\rangle$.
  $F[3,3,3,1,0] = F[2,2,2,0] = 2$.

(6) $[3,3,2,2,0]$ represents $\langle 1,0,2,2,0\rangle$.
  $F[3,3,2,2,0] = F[2,2,1,1] = 4$.

(7) $[3,3,2,1,1]$ represents $\langle 0,2,1,2,0\rangle$.
  We choose $j = 3$ and there are three choices for t:
  - $t = (0,1,0,0,0)$, yielding the reduced $\rho_1 = \langle 1,1,1,1\rangle$ with $f_{\rho_1} = [3,2,1,0]$.
    We have $N_{\rho_1} = \binom{0}{0}\binom{2}{1}\binom{1}{0}\binom{1}{0}\binom{0}{0} = 1\cdot 2\cdot 1\cdot 1\cdot 1 = 2$.
  - $t = (0,0,1,0,0)$, yielding the reduced $\rho_2 = \langle 0,3,0,1\rangle$ with $f_{\rho_2} = [3,1,1,1]$.
    We have $N_{\rho_2} = \binom{0}{0}\binom{2}{0}\binom{1}{1}\binom{1}{0}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.

- $t = (0,0,0,1,0)$, yielding the reduced $\rho_3 = \langle 0,2,2,0\rangle$ with $f_{\rho_3} = [2,2,1,1]$.
  We have $N_{\rho_3} = \binom{0}{0}\binom{2}{0}\binom{1}{0}\binom{1}{1}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.

Hence $F[3,3,2,1,1] = N_{\rho_1}\cdot F[3,2,1,0] + N_{\rho_2}\cdot F[3,1,1,1] + N_{\rho_3}\cdot F[2,2,1,1] = 2\cdot 1 + 1\cdot 2 + 1\cdot 4 = 8$.

In fact this was the case developed in examples 2.1, 2.2, 2.3 and 2.4 through the different choice of $j = 2$, but with the same result $F[3,3,2,1,1] = 4\cdot F[3,2,1,0] + F[2,2,1,1] = 4\cdot 1 + 4 = 8$.

This illustrates that the outcome is independent of the choice of j, as it should.

(8) $[3,2,2,2,1]$ represents $\langle 0,1,3,1,0\rangle$.
We choose $j = 3$ and there are two choices for t:
- $t = (0,1,0,0,0)$, yielding the reduced $\rho_1 = \langle 1,0,3,0\rangle$ with $f_{\rho_1} = [2,2,2,0]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{1}{1}\binom{3}{0}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.
- $t = (0,0,1,0,0)$, yielding the reduced $\rho_2 = \langle 0,2,2,0\rangle$ with $f_{\rho_1} = [2,2,1,1]$.
  We have $N_{\rho_2} = \binom{0}{0}\binom{1}{0}\binom{3}{1}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 3\cdot 1\cdot 1 = 3$.

Hence $F[3,2,2,2,1] = N_{\rho_1}\cdot F[2,2,2,0] + N_{\rho_2}\cdot F[2,2,1,1] = 1\cdot 2 + 3\cdot 4 = 14$.

(9) $[2,2,2,2,2]$ represents $\langle 0,0,5,0,0\rangle$.
The only choice is $j = 2$ and a single t:
- $t = (0,0,2,0,0)$, yielding the reduced $\rho_1 = \langle 0,2,2,0\rangle$ with $f_{\rho_1} = [2,2,1,1]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{0}{0}\binom{4}{2}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 6\cdot 1\cdot 1 = 6$.

Hence $F[2,2,2,2,2] = N_{\rho_1}\cdot F[2,2,1,1] = 6\cdot 4 = 24$.

2.4.5. **n = 6**. For $n = 6$ there are 22 cases to consider.

(1-9) All 9 cases of type $[5,g]$ with g of length 5, for which $F[5,g] = F[g]$.

(10) $F[4,4,4,2,1,0] = F[3,3,3,1,0] = 2$.

(11) $[4,4,4,1,1,1]$ represents $\langle 0,3,0,0,3,0\rangle$.
Choosing $j = 4$ there are two possible choices for t:
- $t = (0,1,0,0,0,0)$, yielding $\rho = \langle 1,2,0,0,2\rangle$ with $f_\rho = [4,4,1,1,0]$ that does not satisfy Landau's condition.
- $t = (0,0,0,0,1,0)$, yielding $\rho_1 = \langle 0,3,0,1,1\rangle$ with $f_{\rho_1} = [4,3,1,1,1]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{3}{0}\binom{0}{0}\binom{0}{0}\binom{2}{1}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 1 = 2$.

Hence $F[4,4,4,1,1,1] = N_{\rho_1}\cdot F[4,3,1,1,1] = 2\cdot 2 = 4$.

(12) $F[4,4,3,3,1,0] = F[3,3,2,2,0] = 4$.

(13) $F[4,4,3,2,2,0 = F[3,3,2,1,1] = 8$.

(14) $[4,4,3,2,1,1]$ represents $\langle 0,2,1,1,2,0\rangle$.
Choosing $j = 4$ there are four possible choices for t:

- $t = (0,1,0,0,0,0)$, yielding $\rho_1 = \langle 1,1,1,1,1\rangle$ with $f_{\rho_1} = [4,3,2,1,0]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{2}{1}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{0} = 1\cdot 2\cdot 1\cdot 1\cdot 1\cdot 1 = 2$.
- $t = (0,0,1,0,0,0)$, yielding $\rho_2 = \langle 0,3,0,1,1\rangle$ with $f_{\rho_2} = [4,3,1,1,1]$.
  We have $N_{\rho_2} = \binom{0}{0}\binom{2}{0}\binom{1}{1}\binom{1}{0}\binom{1}{0}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.
- $t = (0,0,0,1,0,0)$, yielding $\rho_3 = \langle 0,2,2,0,1\rangle$ with $f_{\rho_3} = [4,2,2,1,1]$.
  We have $N_{\rho_3} = \binom{0}{0}\binom{2}{0}\binom{1}{0}\binom{1}{1}\binom{1}{0}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.
- $t = (0,0,0,0,1,0)$, yielding $\rho_4 = \langle 0,2,1,2,0\rangle$ with $f_{\rho_4} = [3,3,2,1,1]$.
  We have $N_{\rho_4} = \binom{0}{0}\binom{2}{0}\binom{1}{0}\binom{1}{0}\binom{1}{1}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.

Hence $F[4,4,3,2,1,1] = N_{\rho_1}\cdot F[4,3,2,1,0] + N_{\rho_2}\cdot F[4,3,1,1,1] + N_{\rho_3}\cdot F[4,2,2,1,1] + N_{\rho_4}\cdot F[3,3,2,1,1] = 2\cdot 1 + 1\cdot 2 + 1\cdot 4 + 1\cdot 8 = 16$.

(15) $[4,4,2,2,2,1]$ represents $\langle 0,1,3,0,2,0\rangle$.
Choosing $j = 1$ there are two possible choices for t:
- $t = (0,0,3,0,1,0)$, yielding $\rho_1 = \langle 0,3,0,1,1\rangle$ with $f_{\rho_1} = [4,3,1,1,1]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{0}{0}\binom{3}{3}\binom{0}{0}\binom{2}{1}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 1 = 2$.
- $t = (0,0,2,0,2,0)$, yielding $\rho_2 = \langle 0,2,1,2,0\rangle$ with $f_{\rho_2} = [3,3,2,1,1]$.
  We have $N_{\rho_2} = \binom{0}{0}\binom{0}{0}\binom{3}{2}\binom{0}{0}\binom{2}{2}\binom{0}{0} = 1\cdot 1\cdot 3\cdot 1\cdot 1\cdot 1 = 3$.

Hence $F[4,4,3,2,1,1] = N_{\rho_1}\cdot F[4,3,1,1,1] + N_{\rho_2}\cdot F[3,3,2,1,1] = 2\cdot 2 + 3\cdot 8 = 28$.

(16) $F[4,3,3,3,2,0] = F[3,2,2,2,1] = 14$.

(17) $F[4,3,3,3,1,1] = F[4,4,2,2,2,1] = 28$.

(18) $[4,3,3,2,2,1]$ represents $\langle 0,1,2,2,1,0\rangle$.
Choosing $j = 4$ there are three possible choices for t:
- $t = (0,1,0,0,0,0)$, yielding $\rho_1 = \langle 1,0,2,2,0\rangle$ with $f_{\rho_1} = [3,3,2,2,0]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{1}{1}\binom{2}{0}\binom{2}{0}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.
- $t = (0,0,1,0,0,0)$, yielding $\rho_2 = \langle 0,2,1,2,0\rangle$ with $f_{\rho_2} = [3,3,2,1,1]$.
  We have $N_{\rho_2} = \binom{0}{0}\binom{1}{0}\binom{2}{1}\binom{2}{0}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 2\cdot 1\cdot 1\cdot 1 = 2$.
- $t = (0,0,0,1,0,0)$, yielding $\rho_3 = \langle 0,1,3,1,0\rangle$ with $f_{\rho_2} = [3,2,2,2,1]$.
  We have $N_{\rho_3} = \binom{0}{0}\binom{1}{0}\binom{2}{0}\binom{2}{1}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 2\cdot 1\cdot 1 = 2$.

Hence $F[4,3,3,2,2,1] = N_{\rho_1}\cdot F[3,3,2,2,0] + N_{\rho_2}\cdot F[3,3,2,1,1] + N_{\rho_3}\cdot F[3,2,2,2,1] = 1\cdot 4 + 2\cdot 8 + 2\cdot 14 = 48$.

(19) $[4,3,2,2,2,2]$ represents $\langle 0,0,4,1,1,0\rangle$.
Choosing $j = 4$ there are two possible choices for t:

- $t = (0,0,1,0,0,0)$, yielding $\rho_1 = \langle 0,1,3,1,0\rangle$ with $f_{\rho_1} = [3,2,2,2,1]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{0}{0}\binom{4}{1}\binom{1}{0}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 4\cdot 1\cdot 1\cdot 1 = 4$.
- $t = (0,0,0,1,0,0)$, yielding $\rho_2 = \langle 0,0,5,0,0\rangle$ with $f_{\rho_2} = [2,2,2,2,2]$.
  We have $N_{\rho_2} = \binom{0}{0}\binom{0}{0}\binom{4}{0}\binom{1}{1}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.

Hence $F[4,3,2,2,2,2] = N_{\rho_1}\cdot F[3,2,2,2,1] + N_{\rho_2}\cdot F[2,2,2,2,2] = 4\cdot 14 + 1\cdot 24 = 80$.

(20) $F[3,3,3,3,3,0] = F[2,2,2,2,2] = 24$.

(21) $F[3,3,3,3,2,1] = F[4,3,2,2,2,2] = 80$.

(22) $[3,3,3,2,2,2]$ represents $\langle 0,0,3,3,0,0\rangle$.
Choosing $j = 3$ there are three possible choices for t:
- $t = (0,0,2,0,0,0)$, yielding $\rho_1 = \langle 0,2,1,2,0\rangle$ with $f_{\rho_1} = [3,3,2,1,1]$.
  We have $N_{\rho_1} = \binom{0}{0}\binom{0}{0}\binom{3}{2}\binom{2}{0}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 3\cdot 1\cdot 1\cdot 1 = 3$.
- $t = (0,0,1,1,0,0)$, yielding $\rho_2 = \langle 0,1,3,1,0\rangle$ with $f_{\rho_2} = [3,2,2,2,1]$.
  We have $N_{\rho_2} = \binom{0}{0}\binom{0}{0}\binom{3}{1}\binom{2}{1}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 3\cdot 2\cdot 1\cdot 1 = 6$.
- $t = (0,0,0,2,0,0)$, yielding $\rho_3 = \langle 0,0,5,0,0\rangle$ with $f_{\rho_2} = [2,2,2,2,2]$.
  We have $N_{\rho_3} = \binom{0}{0}\binom{0}{0}\binom{3}{0}\binom{2}{2}\binom{0}{0}\binom{0}{0} = 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$.

Hence $F[3,3,3,2,2,2] = N_{\rho_1}\cdot F[3,3,2,1,1] + N_{\rho_2}\cdot F[3,2,2,2,1] + N_{\rho_3}\cdot F[2,2,2,2,2] = 3\cdot 8 + 6\cdot 14 + 1\cdot 24 = 132$.

2.4.6. $\mathbf{n > 6}$. As the lists for higher n quickly explode in length T_n (see table 1) we do not continue this listing here. We first programmed the calculations in Delphi-Pascal using the highest precision available integer type Int64 , and were able to obtain complete lists of score function frequencies up to $n = 12$. There are 'only' $T_{12} = 14805$ of these, so this looks as a relatively easy task. However, the size of the frequencies increases quite dramatically and lead this quickly to Int64 overflow. As an example table 2 lists the almost uniform score functions of increasing length and their frequency. Therefore for higher n we were obliged to abandon exact integer precision and to move to floating point calculations. This allowed to push up the calculations to $n = 18$, while at the next induction step we ran out of memory.

Complete tables of results for $n = 7,\ldots,18$ may be downloaded at `http://homepages.vub.ac.be/~faplastr/Tournaments.html`.

3. Frequency and probability of a score sequence

A (nondecreasing) sort of f is any bijection $\sigma : P \to [n]$ such that for all $k = 1,\ldots,n-1$

$$f(\sigma(k-1)) \leq f(\sigma(k))$$

TABLE 2. Frequencies for almost uniform score functions

n	f	$F(f)$
2	10	1
3	111	2
4	2211	4
5	22222	24
6	333222	132
7	3333333	2640
8	44443333	46144
9	444444444	3230080
10	5555544444	191474240
11	55555555555	48251508480
12	666666555555	10073269059840
13	6666666666666	9307700611292160
14	77777776666666	7.01106745286988006E18
15	777777777777777	2.40619834982494284E22
16	8888888877777777	6.64993943699674823E25
17	88888888888888888	8.55847205541481497E29
18	999999999888888888	8.78450602892399261E33

Applying any sort to f results in the score sequence $s(f)$ of f. The score sequence is another invariant of score functions: it simply lists $\rho_f(k)$ copies of k for each k from 0 to $n-1$ in sequence.

It follows that all score functions having a same score sequence s also have a same repetition function ρ_s, so all have equal frequency. And the number of such score functions is the number of ways one may split the set P into n parts P_k ($k \in [n]$) having respectively $\rho_s(k)$ elements. This number is well known as the multinomial number (see e.g. [1])

$$\binom{n}{\rho_s} = \binom{n}{\rho_s(0), \rho_s(1), \ldots, \rho_s(n-1)} = \frac{n!}{\rho_s(0)!\rho_s(1)! \ldots, \rho_s(n-1)!}$$

It follows that the frequency $F(s)$ for a given score sequence s is obtained as follows from any score function f with $s(f) = s$.

$$F(s) = \binom{n}{\rho_s} F(f) \tag{3.1}$$

Observe that the sum of all these frequencies over all possible score sequences of length n must equal the total number of tournaments $2^{\binom{n}{2}}$.

For uniformly distributed tournaments we then obtain the probability for a score sequence as

$$P(s) = \frac{F(s)}{2^{\binom{n}{2}}} \tag{3.2}$$

TABLE 3. Frequency of score sequences for $n = 4$

f	$F(f)$	ρ_f	$\binom{n}{\rho_s}$	s_f	$F(s_f)$	$P(s_f)$
3210	1	1111	24	0123	24	0.375
3111	2	0301	4	1113	8	0.125
2220	2	1030	4	0222	8	0.125
2211	4	0220	6	1122	24	0.375
				Total	64	1.000

Example 3.1. (continued) Let $s = (1, 1, 2, 3, 3)$ be a score sequence of length 5.

The score function f from example 2.1 has score sequence $s(f) = s$. Also the repetition function of s is $\rho_s = \rho_f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 2 & 0 \end{pmatrix}$.

Then we find for the frequency of s

$$F(s) = \binom{n}{\rho_s} F(f) = \binom{5}{0,2,1,2,0} F(f) = \frac{5!}{0!2!1!2!0!} F(f) = 30 \cdot 8 = 240$$

and for its probability

$$P(s) = \frac{F(s)}{2^{\binom{n}{2}}} = \frac{240}{2^{10}} = 0.234375$$

□

For **n = 2** we have

$s_1 = (01)$: $F(s_1) = \binom{2}{1,1} F(f_{s_1}) = 2 \cdot 1 = 2$, $P(s_1) = F(s_1)/2^{\binom{2}{2}} = 2/2 = 1$

For **n = 3** we have

$s_1 = (012)$: $F(s_1) = \binom{3}{1,1,1} F(f_{s_1}) = 6 \cdot 1 = 6$, $P(s_1) = F(s_1)/2^{\binom{3}{2}} = 0 \cdot 75$

$s_2 = (111)$: $F(s_2) = \binom{3}{0,3,0} F(f_{s_2}) = 1 \cdot 2 = 2$, $P(s_2) = F(s_2)/2^{\binom{3}{2}} = 0 \cdot 25$

Tables 3, 4 and 5 list the complete calculations for $n = 4, 5, 6$. Further tables may be downloaded from `http://homepages.vub.ac.be/~faplastr/Tournaments.html` for $n = 7, \ldots, 18$ Since the case $n = 12$ gave rise to overflow in Int64 mode, we had to move to floating point calculations resulting in a slight loss of precision.

4. CONCLUDING REMARKS

We have calculated the frequency of tournament score sequences through the frequencies of score functions. These latter are obtained by induction on the length of the score function and a listing of all possible predecessor

TABLE 4. Frequency of score sequences for $n = 5$

f	$F(f)$	ρ_f	$\binom{n}{\rho_s}$	s_f	$F(s_f)$	$P(s_f)$
43210	1	11111	120	01234	120	0.1171875
43111	2	03011	20	11134	40	0.0390625
42220	2	10301	20	02224	40	0.0390625
42211	4	02201	30	11224	120	0.1171875
33310	2	11030	20	01333	40	0.0390625
33220	4	10220	30	02233	120	0.1171875
33211	8	02120	30	11233	240	0.2343750
32221	14	01310	20	12223	280	0.2734375
22222	24	00500	1	22222	24	0.0234375
				Total	1024	1.0000000

TABLE 5. Frequency of score sequences for $n = 6$

f	$F(f)$	ρ_f	$\binom{n}{\rho_s}$	s_f	$F(s_f)$	$P(s_f)$
543210	1	111111	720	012345	720	0.02197265625
543111	2	030111	120	111345	240	0.00732421875
542220	2	103011	120	022245	240	0.00732421875
542211	4	022011	180	112245	720	0.02197265625
533310	2	110301	120	013335	240	0.00732421875
533220	4	102201	180	022335	720	0.02197265625
533211	8	021201	180	112335	1440	0.04394531250
532221	14	013101	120	122235	1680	0.05126953125
522222	24	005001	6	222225	144	0.00439453125
444210	2	111030	120	012444	240	0.00732421875
444111	4	030030	20	111444	80	0.00244140625
443310	4	110220	180	013344	720	0.02197265625
443220	8	102120	180	022344	1440	0.04394531250
443211	16	021120	180	112344	2880	0.08789062500
442221	28	013020	60	122244	1680	0.05126953125
433320	14	101310	120	023334	1680	0.05126953125
433311	28	020310	60	113334	1680	0.05126953125
433221	48	012210	180	122334	8640	0.26367187500
432222	80	004110	30	222234	2400	0.07324218750
333330	24	100500	6	033333	144	0.00439453125
333321	80	011400	30	123333	2400	0.07324218750
333222	132	003300	20	222333	2640	0.08056640625
				Total	32768	1.00000000000

score functions of length one less. In other words, we have proposed here a simple induction of order one.

This is adequate as long as full lists of frequencies of score functions of length one less are available. As the size of these lists grows extremely fast, such lists may not be available. Therefore higher order induction, where a given score function of length n is analyzed by a split into two parts of any size p and q ($p + q = n$) may be of interest. This turns out not to be that easy, and leads to new enumeration problems that seem presently unpublished. Such a technique is currently being investigated.

The simplest case of such a splitting may be obtained as follows for some particular score functions, generalizing lemma 2.5 (and 2.6).

We call the score function f on P *splittable* on $A \subsetneq P$ if $\sum_{a \in A} f(a) = \binom{|A|}{2}$. In this case any tournament t realizing f will have all $|A|.(|P| - |A|)$ arcs between A and $B = P \setminus A$ oriented from B to A. Such tournaments have also been called reducible, see [8]. We denote the two subtournaments of t induced on A and B as t_A and t_B. If the tournament t realizes f then t_A has score function $f_A = f|_A$ and t_B has score function $f_B = f|_B - |A|$, where $f|_C$ denotes the trace of f on $C \subset P$. Furthermore when f is splittable on A any tournament t realizing f is fully determined by t_A and t_B by addition of all possible arcs from B to A:

$$t = t_A \cup t_B \cup (B \times A)$$

It follows therefore

Lemma 4.1. *If the score function f on P is splittable on $A \subset P$ we have*

$$F(f) = F(f_A) \cdot F(f_B)$$

The case $A = \{ q \}$ with $f(q) = n - 1$ corresponds to lemma 2.5, since in this case $f_A = [0]$ and $F(f_A) = 1$.

To illustrate this property further we consider $f = [444210]$ that is splittable on subset $A = \{ a_4, a_5, a_6 \}$. Then $f_A = [210]$ and $f_B = [111]$, so

$$F[444210] = F[210] \cdot F[111] = 1 \cdot 2 = 2$$

Similarly from $F[432222] = 80$ and $F[32221] = 14$, and taking $f_A = [432222]$ and $f_B = [32221]$, we derive that

$$F[98887432222] = 80 \cdot 14 = 1120,$$

while taking $f_A = [32221]$ and $f_B = [432222]$ we obtain

$$F[98777732221] = 14 \cdot 80 = 1120.$$

However, most score functions are not splittable. In fact, asymptotically there are almost no tournaments with splittable score function, as shown by Moon and Moser [8] (see also Wright [10]), and so for most score functions the calculations will be (much) more tedious in general.

Similarly to the remarks in David [4] one may expect that the methodology presented here may be extended to multiple-round tournaments. In case of non-uniformly distributed tournaments the calculation of probabilities of score functions and score sequences seems to be much less evident, however.

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