On the Erdős-Sós Conjecture and double-brooms

Gary Tiner
Faulkner University
gtiner@faulkner.edu

Abstract

Let G be a graph with average degree greater than $k-2$. Erdős and Sós conjectured that G contains every tree on k vertices. A *star* is a tree consisting of one center vertex adjacent to all the other vertices, and a *double-broom* is a tree made up of two stars and a path connecting the center of one star with the center of the other. If the path connecting the two stars has length 2 or 3, then G contains the double-broom (unpublished). In this paper, we prove that G contains every double-broom on k vertices.

1 Introduction

The average degree of graph G is denoted $\bar{d}(G)$ and is equal to $2e(G)/|V(G)|$. Erdős and Gallai [3] proved that if $\bar{d}(G) > k-2$, then G contains a path on k vertices. Subsequently, Erdős and Sós conjectured the following.

Erdős-Sós Conjecture. *If G is a graph with $\bar{d}(G) > k-2$, then G contains every tree on k vertices.*

Various special cases of the conjecture have been proven. Many place restrictions on the graph G. The cases where G has number of vertices $k, k+1$, $k+2$, or $k+3$ were proved by Zhou [11], Slater, Teo, and Yap [6], Woźniak [10], and Tiner [8], respectively. The number of edges on a path is its *length*. The *diameter* of a graph G, *diam*(G), is the length of a longest path in G. Eaton and Tiner [2] proved the conjecture holds if diam(G) is less than $k+3$.

Other cases that have been proven place restrictions on the class of trees. Sidorenko proved the conjecture holds for every tree with a vertex having at least $\lceil \frac{k}{2} \rceil - 1$ leaf-neighbors. Eaton and Tiner [1] improved this to include every tree with a vertex having at least $\lceil \frac{k}{2} \rceil - 2$ leaf-neighbors.

A *spider* is a tree with one vertex of degree at least 3, called the *center*, and all others with degree at most 2. Woźniak [10] proved the conjecture for spiders of diameter at most 4. McLennan [5] improved this to include all trees of diameter at most 4. A *leg* of the spider is a path from the center to a vertex of degree one. Fan and Sun [4] proved the conjecture for spiders with no leg of length greater than 4. Sun [7] improved this to include all spiders of diameter at most 9.

A *double-broom* is a tree made up of two stars and a path connecting the center of one star with the center of the other. It is mentioned in [10] that if $\bar{d}(G) > k - 2$ and the path between the two stars of the double-broom has either 2 or 3 edges, then G contains the double-broom (unpublished). We prove the following.

Theorem 1. *If G is a graph with $\bar{d}(G) > k - 2$, then G contains every double-broom on k vertices.*

For standard notation and terminology in graph theory, see [9]. Let G be a graph. For two subgraphs $C, D \subseteq G$, the set of edges with one endpoint in $V(C)$ and one in $V(D)$ is $E(C, D)$; the number of edges in $E(C, D)$ is $e(C, D)$. The subgraph induced by $V(C)$ is $G[C]$, its edge-set is $E(C, C)$ or simply $E(C)$, and $e(C)$ is the number of edges in $E(C)$. The subgraph $G - V(C)$, or simply $G - C$, is obtained from G by deleting $V(C)$ and the set of edges with an endpoint in $V(C)$. Choose $A, B \subseteq V(G)$ and let $a \in A$ and $b \in B$. If $ab \in E(G)$, then the vertex a *hits* B and the subset B *hits* A.

The number of edges with at least one endpoint in A is $e^*_G(A)$ or simply $e^*(A)$. Notice that $e^*(A) = \sum_{v \in A} d(v) - e(A) = e(A) + e(A, G - A)$, and

$$\sum_{v \in A} d(v) = 2e^*(A) - e(A, G - A). \tag{1}$$

A proof of the following lemma is in [1].

Lemma 2. *Let G be a graph with $\bar{d}(G) > k - 2$. Let $W \subsetneq V(G)$ and $G' = G - W$. If $e^*(W) \le \frac{1}{2}(k - 2)|W|$, then $\bar{d}(G') > k - 2$.*

The minimum degree among all vertices in G is $\delta(G)$. For a natural number m, a graph G is *minimal with $\bar{d}(G) > m$* if $\bar{d}(G') \le m$ whenever G' is a proper subgraph of G.

The following corollary follows from Lemma 2 and Identity (1) above.

Corollary 3. *If a graph G is minimal with $\bar{d}(G) > k-2$ and $W \subsetneq V(G)$, then $e^*_G(W) > \frac{1}{2}|W|(k-2)$. In particular,*

$$(i)\ \delta(G) \geq \lfloor \frac{k}{2} \rfloor, \text{ and } (ii) \sum_{v \in W} d(v) > |W|(k-2) - e(W, G-W).$$

2 Proof of Theorem 1

In this section we prove that if G is a graph with $\bar{d}(G) > k-2$, then G contains every double-broom on k vertices.

Lemma 4. *Let G be a graph that is minimal with $\bar{d}(G) > k-2$, and let W be a subset of $V(G)$, where $\lfloor \frac{k}{2} \rfloor \leq |W| \leq k-2$. If each vertex in W has degree at most $|W|$, then a vertex in W hits $(k-1)-|W|$ vertices outside of W.*

Proof. Let $W' = V(G-W)$ and to the contrary, suppose that each vertex in W hits at most $(k-2)-|W|$ vertices in W'. It follows that $e(W, W') \leq |W|(k-2-|W|)$. By Corollary 3$(ii)$, we have

$$\sum_{v \in W} d(v) > |W|(k-2) - |W|(k-2-|W|) = |W|^2$$

which implies a vertex in W has degree greater than $|W|$, a contradiction. Therefore, a vertex in W hits at least $(k-1)-|W|$ vertices in $V(G-W)$. □

Let P be an r-path in a graph G, where $P = v_1, \ldots, v_r$. A *path on the vertex set V(P)*, or simply a *path on V(P)*, is an r-path in G whose vertex set is $V(P)$. For distinct vertices v_i and v_j on the path P, if there is a path on $V(P)$ whose end-vertices are v_i and v_j, then it is a *v_i,v_j-path on* $V(P)$. For a vertex v_t on the path P,

$$\alpha(P, v_t) = \{v_s \in V(P) : \text{there is a } v_s, v_t\text{-path on } V(P)\}.$$

For each $v_i \in N_P(v_1)$, the path $v_{i-1}, \ldots, v_1, v_i, \ldots, v_r$ is a v_{i-1}, v_r-path on $V(P)$. It follows that $v_{i-1} \in \alpha(P, v_r)$, and $e(v_1, P) \leq |\alpha(P, v_r)|$. We state this more generally in the following lemma.

Lemma 5. *If P is a path in a graph G, where $P = v_1, \ldots, v_r$, then $e(v_i, P) \leq |\alpha(P, v_r)|$ for all $v_i \in \alpha(P, v_r)$.*

Lemma 6. *Let G be a graph that is minimal with $\bar{d}(G) > k-2$. Let Q be a path in G, where $Q = v_1, \ldots, v_r$, and let $W = \alpha(Q, v_r)$. If $N(W) \subseteq V(Q)$, then W hits a vertex in $\{v_{k-1}, \ldots, v_r\}$ and $r \geq k-1$.*

Proof. We will prove that W hits a vertex in $\{v_{k-1}, \ldots, v_r\}$ which implies that $r \geq k-1$. If $|W| \geq k-1$, then either $W = \{v_1, \ldots, v_{k-1}\}$ or the maximal i such that $v_i \in W$ is greater than $k-1$, and in either case, the claim is obvious. Otherwise $|W| \leq k-2$. Since $N(v_1) \subseteq V(Q)$ and $d(v_1) \geq \delta(G) \geq \lfloor \frac{k}{2} \rfloor$ (by Corollary 3(i)), we see that $|W| \geq \lfloor \frac{k}{2} \rfloor$ (by Lemma 5). By Lemma 4, a vertex v_i in W hits at least $(k-1)-|W|$ vertices outside of W. Therefore v_i hits a vertex in $\{v_{k-1}, \ldots, v_r\}$. □

Let G be a graph with $\bar{d}(G) > k-2$. Erdős and Gallai [3] proved that G contains a path on k vertices. Now suppose G is minimal with $\bar{d}(G) > k-2$. For an arbitrary vertex $v \in V(G)$, let P be a longest path in G having v as one end-vertex. By our choice of P, we see that $N(\alpha(P,v)) \subseteq V(P)$, and therefore the path P has at least $k-1$ vertices (by Lemma 6). We state this as a corollary.

Corollary 7. *Let G be a graph that is minimal with $\bar{d}(G) > k-2$. If $v \in V(G)$, then there is a $(k-1)$-path in G having v as one end-vertex.*

Lemma 8. *Let G be a graph that is minimal with $\bar{d}(G) > k-2$, and let P be a path in G, where $P = v_1, \ldots, v_r$. If $r \leq k-2$, then a vertex in $\alpha(P, v_1)$ hits $\lfloor \frac{1}{2}(k-r) \rfloor$ vertices outside of $V(P)$.*

Proof. By Corollary 7, an r-path beginning with vertex v_1 exists. Let $W = \alpha(P, v_1)$. For each $v_i \in W$, we have $e(v_i, P) \leq |W|$ (by Lemma 5). To the contrary, suppose $e(v_i, G-P) \leq \lfloor \frac{1}{2}(k-r) \rfloor - 1$ for each $v_i \in W$. Thus for each $v_i \in W$, we have

$$d(v_i) = e(v_i, G-P) + e(v_i, P) \leq \frac{1}{2}(k-r) - 1 + |W|.$$

By Corollary 3(*ii*), we have

$$\begin{aligned} \sum_{v \in W} d(v) &> |W|(k-2) - [e(W, G-P) + e(W, P-W)] \\ &\geq |W|(k-2) - \left[|W|(\frac{1}{2}(k-r) - 1) + |W|(r - |W|)\right] \\ &= |W|(\frac{1}{2}(k-r) - 1 + |W|) \end{aligned}$$

which implies a vertex v in W has degree greater than $\frac{1}{2}(k-r) - 1 + |W|$, a contradiction. □

For natural numbers r, ℓ_1 and ℓ_2, the double-broom $DB(r, \ell_1, \ell_2)$ consists of an r-path $a_1, \ldots, a_r$ and ℓ_1 additional vertices adjacent to a_1 and ℓ_2 additional vertices adjacent to a_r. We now prove Theorem 1.

Theorem 1. *If G is a graph with $\bar{d}(G) > k-2$, then G contains every double-broom on k vertices.*

Proof. Let G' be a subgraph of G that is minimal with $\bar{d}(G') > k-2$. If G' contains every double-broom on k vertices, then so does G. For this reason, we will simply assume that G is minimal with $\bar{d}(G) > k-2$. For natural numbers r, ℓ_1 and ℓ_2, let T be the double-broom $DB(r, \ell_1, \ell_2)$, where $\ell_1 \geq \ell_2$ and $r + \ell_1 + \ell_2 = k$. It follows that $\ell(a_r) \leq \lfloor \frac{1}{2}(k-r) \rfloor$. Let T' be the tree obtained from T by removing the leaf neighbors of a_r (the tree T' is called a *broom*).

Since $\bar{d}(G) > k-2$, there is a vertex $v_1 \in V(G)$ with degree least $k-1$. Let P be a path in G, where $P = v_1, \ldots, v_r$ and v_r hits at least $\lfloor \frac{1}{2}(k-r) \rfloor$ vertices outside of $V(P)$ (possible by Corollary 7 and Lemma 8). Map the path $a_1, \ldots, a_r$ in T' to $v_1, \ldots, v_r$, respectively, in G. Since $e(v_r, V(G-P)) \geq \lfloor \frac{1}{2}(k-r) \rfloor$, this mapping can be extended to an embedding of T' into G. Finally, since $d(v_1) \geq k-1$, this embedding of T' into G can be extended to an embedding of T into G. □

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