Rainbow k-Connection in Dense Graphs

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Abstract

An edge-coloured path is *rainbow* if the colours of its edges are distinct. For a positive integer k, an edge-colouring of a graph G is *rainbow k-connected* if any two vertices of G are connected by k internally vertex-disjoint rainbow paths. The *rainbow k-connection number* $rc_k(G)$ is defined to be the minimum integer t such that there exists an edge-colouring of G with t colours which is rainbow k-connected. We consider $rc_2(G)$ when G has fixed vertex-connectivity. We also consider $rc_k(G)$ for large complete bipartite and multipartite graphs G with equipartitions. Finally, we determine sharp threshold

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functions for the properties $rc_k(G) = 2$ and $rc_k(G) = 3$, where G is a random graph. Related open problems are posed.

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1 Introduction

In this paper, unless otherwise stated, all graphs are finite, simple and undirected. For basic terminology in graph theory, we refer to the book by Bollobás [1]. An edge-coloured path is *rainbow* if the colours of its edges are distinct. For $k \in \mathbb{N}$, an edge-colouring of a graph G is *rainbow k-connected* if any two vertices of G are connected by k internally vertex-disjoint rainbow paths. The *rainbow k-connection number* $rc_k(G)$ is defined to be the minimum integer t such that, there exists an edge-colouring of G with t colours which is rainbow k-connected. We write $rc(G) = rc_1(G)$. Note that, by Menger's Theorem [19], a graph is k-connected if and only if any two vertices are connected by k internally vertex-disjoint paths. Hence, $rc_k(G)$ is well-defined for k-connected graphs G.

The function $rc_k(G)$ was introduced by Chartrand et al. [4, 5] in 2008. They studied $rc_k(G)$ for many graphs, notably when G is complete, and complete bipartite and multipartite. They also introduced the *strong rainbow connection number* $src(G)$, and considered some relationships between $rc(G)$ and $src(G)$. An application to secure data transfer was presented as well. The subject has since attracted considerable interest. These include the study of $rc_k(G)$ when G satisfies some condition on its minimum degree, or forbidden subgraphs, or diameter; or when G is regular. The computational complexity of $rc_k(G)$ has also been studied. Further related functions have been introduced, such as the *rainbow vertex connection number* $rvc(G)$, and the *k-rainbow index* $rx_k(G)$. See for example, Caro et al. [2], Chartrand et al. [6], and Krivelevich and Yuster [12]. Recently, Li et al. [14], and Li and Sun [17], published a survey and a book on the current status of rainbow connection.

We continue the study of $rc_k(G)$. First, we consider $rc_k(G)$ for graphs G with given vertex-connectivity. The case $k = 1$ was asked by Hajo Broersma (at the IWOCA workshop 2009). Let G have order n. For $k = 1$, Caro et al. [2] proved that $rc(G) \leq \frac{2n}{3}$ and $rc(G) \leq \frac{n}{2} + O(\sqrt{n})$ if G is 2-connected. Li and Shi [13] proved that $rc(G) \leq \frac{3(n+1)}{5}$ if G is 3-connected. Chandran et al. [3] proved that, if G has minimum degree δ, then $rc(G) \leq \frac{3n}{\delta+1} + 3$. Hence, if G is ℓ-connected, then $rc(G) \leq \frac{3n}{\ell+1} + 3$. Here, we prove the following result for the case $k = 2$.

Theorem 1.1 *If $\ell \geq 2$ and G is an ℓ-connected graph on $n \geq \ell+1$ vertices, then $rc_2(G) \leq \frac{(\ell+1)n}{\ell}$.*

In the case $\ell = 2$, we can do better if G is *series-parallel*. A *2-connected series-parallel graph* is a (simple) graph which can be obtained from a K_3, and then repeatedly applying a sequence of operations, each of which is a subdivision, or replacement of an edge by a double edge. These graphs are a well-known sub-family of the 2-connected graphs.

Theorem 1.2 *If G is a 2-connected series-parallel graph on $n \geq 3$ vertices, then $rc_2(G) \leq n$.*

Note that $rc_2(C_n) = n$ if C_n is the cycle on n vertices. More generally, Theorem 1.3 below shows that Theorem 1.2 is tight when G is a *generalised Θ-graph*. That is, $G = \Theta_{q_1,\dots,q_t}$ is the union of $t \geq 2$ paths with lengths $q_1 \geq \cdots \geq q_t \geq 1$, where $q_{t-1} \geq 2$, and the paths are pairwise internally vertex-disjoint with the same two end-vertices.

Theorem 1.3 *If $G = \Theta_{q_1,\dots,q_t}$ is a generalised Θ-graph on n vertices, then*

$$rc_2(G) = \begin{cases} n & \text{if } t = 2, \\ n-1 \text{ or } n-2 & \text{if } t \geq 3. \end{cases}$$

These results naturally lead to the following question.

Problem 1.4 *What is the minimum constant $c > 0$ such that for all 2-connected graphs G on n vertices, we have $rc_2(G) \leq cn$?*

By Theorems 1.1 and 1.3, we have $1 \leq c \leq \frac{3}{2}$. Theorem 1.2 suggests that $c = 1$ may possibly be the correct answer.

Theorem 1.2 has an instant corollary. A result of Elmallah and Colbourn [8] says that any 3-connected planar graph contains a 2-connected series-parallel spanning subgraph.

Corollary 1.5 *If G is a 3-connected planar graph on $n \geq 4$ vertices, then $rc_2(G) \leq n$.* □

Our next aim is to continue the study of $rc_k(G)$ when G is a complete bipartite or multipartite graph. For $1 \leq n_1 \leq \cdots \leq n_t$ with $t \geq 2$, let $K_{n_1,\dots,n_t}$ denote the complete multipartite graph with class sizes $n_1, \dots, n_t$. Chartrand et al. [4] determined $rc(K_{n_1,\dots,n_t})$ exactly, as follows. If $\sum_{i=1}^{t-1} n_i = m$ and $n_t = n$, then

$$rc(K_{n_1,\dots,n_t}) = \begin{cases} n & \text{if } t = 2 \text{ and } n_1 = 1, \\ \min(\lceil \sqrt[m]{n}\,\rceil, 4) & \text{if } t = 2 \text{ and } 2 \leq n_1 \leq n_2, \\ 1 & \text{if } t \geq 3 \text{ and } n_t = 1, \\ 2 & \text{if } t \geq 3,\ n_t \geq 2 \text{ and } m > n, \\ \min(\lceil \sqrt[m]{n}\,\rceil, 3) & \text{if } t \geq 3 \text{ and } m \leq n. \end{cases}$$

Chartrand et al. [5] also proved that $rc_k(K_{n,n}) = 3$ if $k \geq 2$ and $n = 2k\lceil\frac{k}{2}\rceil$. They asked if for every $k \geq 2$, there is a function $f(k)$ such that for every $n \geq f(k)$, we have $rc_k(K_{n,n}) = 3$. Li and Sun [16] proved that this is the case, when $f(k) = 2k\lceil\frac{k}{2}\rceil$. Both of these results considered explicit colourings. With a random method, we are able to improve the result to $f(k) = 2k + o(k)$, as follows.

Theorem 1.6 *Let* $0 < \varepsilon < \frac{1}{2}$ *and* $k \geq \frac{1}{2}(\theta - 1)(1 - 2\varepsilon) + 2$, *where* $\theta = \theta(\varepsilon)$ *is the largest solution of* $2x^2e^{-\varepsilon^2(x-2)} = 1$. *If* $n \geq \frac{2k-4}{1-2\varepsilon} + 1$, *then* $rc_k(K_{n,n}) = 3$.

For example, if we set $\varepsilon = \frac{1}{6}$ so that $\theta \approx 469.94$, this result shows that for $k \geq 159$ and $n \geq 3k - 5$, we have $rc_k(K_{n,n}) = 3$.

On the other hand, how small can the function $f(k)$ be? The next result shows that the best we can hope for is approximately $f(k) \geq \frac{3k}{2}$.

Theorem 1.7 *For any 3-colouring of the edges of* $K_{n,n}$, *there exist* $u, v \in V(K_{n,n})$ *where the number of internally vertex-disjoint rainbow* $u - v$ *paths is at most* $\frac{2n^2}{3(n-1)}$.

We can extend this to complete multipartite graphs with equipartitions. Let $K_{t\times n}$ denote the complete multipartite graph with $t \geq 3$ classes of size n. For $k \geq 2$, when considering bipartite graphs $K_{n,n}$, we cannot achieve $rc_k(K_{n,n}) = 2$. However, we may hope for $rc_k(K_{t\times n}) = 2$. Using a similar random method, we have the following.

Theorem 1.8 *Let* $0 < \varepsilon < \frac{1}{2}$, $t \geq 3$, *and* $k \geq \frac{1}{2}\theta(t - 2)(1 - 2\varepsilon) + 1$, *where* $\theta = \theta(\varepsilon, t)$ *is the largest solution of* $\frac{1}{2}t^2x^2e^{-(t-2)\varepsilon^2 x} = 1$. *If* $n \geq \frac{2k-2}{(t-2)(1-2\varepsilon)}$, *then* $rc_k(K_{t\times n}) = 2$.

For example, if we set $t = 3$ and $\varepsilon = \frac{1}{6}$ so that $\theta \approx 501.86$, this result shows that for $k \geq 169$ and $n \geq 3k - 3$, we have $rc_k(K_{3\times n}) = 2$.

Again, going in the other direction, the following result shows that the best lower bound for n would be approximately $n \geq \frac{2k}{t-1}$.

Theorem 1.9 *Let* $t \geq 3$. *For any 2-colouring of the edges of* $K_{t\times n}$, *there exist* $u, v \in V(K_{t\times n})$ *where the number of internally vertex-disjoint rainbow* $u - v$ *paths is at most* $\frac{(t-1)n^2}{2(n-1)}$.

The related problem of considering $rc_k(G)$ when G is a complete graph has already been well studied. Obviously, we have $rc(K_n) = 1$ for $n \geq 2$, and $rc_k(K_n) \geq 2$ for $n > k \geq 2$. Chartrand et al. [5] proved that, for $k \geq 2$, if $n \geq (k + 1)^2$, then $rc_k(K_n) = 2$. The bound on n was later improved by Li and Sun [15] to $n \geq ck^{3/2} + o(k^{3/2})$ (for some constant c), and then by Dellamonica et al. [7] to $n \geq 2k + o(k)$. The latter bound is also asymptotically the best possible.

Our last aim is to study $rc_k(G)$ when G is some *random graph model.* In this direction, Caro et al. [2] considered $rc(G_{n,p})$, where $G_{n,p}$ is the random graph on n vertices with edge probability p. Recall that, if Q is a graph property and $p = p(n)$, then $G_{n,p}$ satisfies Q *almost surely (a.s.)* if $\mathbb{P}(G_{n,p}$ satisfies $Q) \to 1$ as $n \to \infty$. A function $f(n)$ is a *sharp threshold function* for Q if there are constants $c, C > 0$ such that, $G_{n,cf(n)}$ does not satisfy Q a.s., and $G_{n,p}$ satisfies Q a.s. for all $p \geq Cf(n)$. Caro et al. [2] proved that $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc(G_{n,p}) \leq 2$. This was generalised by He and Liang [10], who proved that if $d \geq 2$ and $k \leq O(\log n)$, then $p = \frac{(\log n)^{1/d}}{n^{(d-1)/d}}$ is a sharp threshold function for the property $rc_k(G_{n,p}) \leq d$. Here, we prove the following result.

Theorem 1.10 *$p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc_k(G_{n,p}) \leq 2$ for all $k \geq 1$.*

We can consider other random graph models. Let $G_{n,m,p}$ be the random bipartite graph with class sizes n and m, and edge probability p. Let $G_{n,M}$ be the random graph on n vertices with M edges, endowed with the uniform probability distribution. We can analogously define sharp threshold functions for these models. We have the following results.

Theorem 1.11 *$p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc_k(G_{n,n,p}) \leq 3$ for all $k \geq 1$.*

Theorem 1.12 *$M = \sqrt{n^3 \log n}$ is a sharp threshold function for the property $rc_k(G_{n,M}) \leq 2$ for all $k \geq 1$.*

This paper will be organised as follows. We prove Theorems 1.1 to 1.3 in Section 2; Theorems 1.6 to 1.9 in Section 3; and Theorems 1.10 to 1.12 in Section 4. In Section 5, we present some related open problems. Throughout, we say that an edge-coloured graph is *rainbow* if its edges have distinct colours. Unless otherwise stated, we simply say that the paths $Q_1, Q_2, \ldots$ are *disjoint* if they are internally vertex-disjoint. Recall that a graph G is *minimally k-connected* if G is k-connected, but $G - e$ is not k-connected for every $e \in E(G)$.

2 Graphs with given Connectivity

In this section, we prove Theorems 1.1 to 1.3. For Theorem 1.1, we first recall the Fan Lemma, which is a consequence of Menger's Theorem.

Theorem 2.1 (Fan Lemma, [19]) *Let G be a k-connected graph. Then, for any vertex $u \in V(G)$ and any set $X \subset V(G-u)$ with $|X| \geq k$, there are k paths from u to X such that for any two paths, their only common vertex is u.*

Proof of Theorem 1.1. We shall in fact prove a stronger result, as follows.

Theorem 2.2 *If $\ell \geq 2$, and G is an ℓ-connected graph on $n \geq \ell + 1$ vertices, then there exists an edge-colouring of G with at most $\frac{(\ell+1)n}{\ell}$ colours satisfying the following.*

(a) For any two vertices $u, v \in V(G)$, there are two disjoint rainbow $u-v$ paths.

(b) For any vertex $u \in V(G)$ and any set $X \subset V(G)$ with $|X| = 2$, there are two rainbow $u - X$ paths whose only common vertex is u.

(c) For any two sets $X, Y \subset V(G)$ with $|X|, |Y| = 2$, there are two rainbow $X - Y$ paths which do not intersect.

Note that in (b), if $u \in X$, then one of the paths is taken to be the vertex u. Likewise, in (c), if X and Y intersect, then a suitable path is a vertex in $X \cap Y$.

Clearly Theorem 1.1 follows from Theorem 2.2, so we prove Theorem 2.2.

Proof of Theorem 2.2. We first find subgraphs $H_0 \subset H_1 \subset \cdots \subset H_t \subset G$, for some $t \geq 0$, with $V(H_t) = V(G)$, as follows. Firstly, let H_0 be a cycle of G on at least ℓ vertices (The fact that any ℓ-connected graph contains a cycle on at least ℓ vertices is well-known. For example, this is an exercise in [1], Ch. III.6). Now suppose that we have found the graphs $H_0, \ldots, H_{i-1}$ for some $i \geq 1$. If $V(H_{i-1}) = V(G)$, then set $H_{i-1} = H_t$. Otherwise, there exists a vertex $v_i \in V(G) \setminus V(H_{i-1})$. By Theorem 2.1, v_i sends ℓ paths to H_{i-1}, with each pair of paths meeting only at v_i, i.e., the union of these ℓ paths is a subdivided $K_{1,\ell}$. Let H_i be the union of H_{i-1} with these ℓ paths. Repeat this process until it terminates.

We prove inductively that for every $0 \leq i \leq t$, there is an edge-colouring of H_i with at most $\frac{(\ell+1)|V(H_i)|}{\ell}$ colours such that properties (a) to (c) in Theorem 2.2 hold, with H_i in place of G. Then, setting $i = t$ implies that Theorem 2.2 holds for G.

To proceed, define an edge-colouring of each H_i as follows. Firstly, give H_0 a rainbow colouring. Then, for $1 \leq i \leq t$, suppose that we have an edge-colouring for H_{i-1}, with colours $1, \ldots, m$. The graph H_i is obtained by attaching a subdivided $K_{1,\ell}$ to H_{i-1}, where the ℓ paths meet at v_i. Let the paths be $Q_1, \ldots, Q_\ell$, with $e(Q_1) \geq \cdots \geq e(Q_\ell) \geq 1$. For the case $\ell = 2$, we may assume that $e(Q_1) \geq e(Q_2) = 1$. Let $F = Q_1 \cup \cdots \cup Q_\ell$. For each $1 \leq j \leq \ell$, let $w_j \in V(H_{i-1})$ be the other end-vertex of Q_j. We call the edges of Q_j incident to v_i and w_j the *first edge* and *last edge* of Q_j, respectively.

Case 1. $\ell + 1 \leq |V(F)| \leq 2\ell - 1$, $e(Q_1) = |V(F)| - \ell$ and $e(Q_2) = \cdots = e(Q_\ell) = 1$.

Give Q_1 a rainbow colouring with colours $m+1, \ldots, m+|V(F)|-\ell$. In view of $|V(F)|-\ell \leq \ell-1$, we colour $Q_2, \ldots, Q_\ell$ with colours $m+1, \ldots, m+|V(F)| - \ell$ in such a way that each colour appears at least once. We have used $|V(F)| - \ell$ new colours in total.

Case 2. $|V(F)| \geq 2\ell$, $e(Q_1) = |V(F)| - \ell$ and $e(Q_2) = \cdots = e(Q_\ell) = 1$.

Give Q_1 a rainbow colouring with colours $m + 1, \ldots, m + |V(F)| - \ell$. Colour all of $Q_2, \ldots, Q_\ell$ with colour $m + |V(F)| - \ell + 1$. We have used $|V(F)| - \ell + 1$ new colours in total.

Case 3. $e(Q_2) \geq 2$.

Let $1 \leq s \leq \frac{1}{2}\ell$ be the largest integer such that $e(Q_1), \ldots, e(Q_{2s}) \geq 2$. For each $1 \leq j \leq s$, colour the first edge of Q_{2j-1} and the last edge of Q_{2j} with colour $m+2j-1$, and the last edge of Q_{2j-1} and the first edge of Q_{2j} with colour $m+2j$. Colour the last edge of Q_{2s+1} and all of $Q_{2s+2}, \ldots, Q_\ell$ (each having length 1) with colour $m+2s+1$. Colour the remaining edges of F with further new, distinct colours. We have used $|V(F)| - \ell$ new colours in total.

Repeating inductively, we have a colouring for H_i, for every $0 \leq i \leq t$. We prove inductively that for every $0 \leq i \leq t$, the colouring for H_i satisfies all of our requirements. Certainly, for H_0, we have used $|V(H_0)| < \frac{(\ell+1)|V(H_0)|}{\ell}$ colours, and properties (a) to (c) hold. Now for $1 \leq i \leq t$, suppose that in the colouring for H_{i-1}, at most $\frac{(\ell+1)|V(H_{i-1})|}{\ell}$ colours are used, and properties (a) to (c) hold. Let $F, Q_1, \ldots, Q_\ell$ be defined as before. Note that $|V(H_{i-1})| = |V(H_i)| - |V(F)| + \ell$. For Cases 1 and 3, since $|V(F)| \geq \ell + 1$, the total number of colours used by H_i is at most $\frac{(\ell+1)|V(H_{i-1})|}{\ell} + |V(F)| - \ell < \frac{(\ell+1)|V(H_i)|}{\ell}$. For Case 2, since $|V(F)| \geq 2\ell$, the total number of colours used by H_i is at most $\frac{(\ell+1)|V(H_{i-1})|}{\ell} + |V(F)| - \ell + 1 \leq \frac{(\ell+1)|V(H_i)|}{\ell}$.

Next, we show that in Cases 1 to 3, properties (a) to (c) hold for H_i. For (a), we are done by the inductive hypothesis if $u, v \in V(H_{i-1})$. Similarly when $\{u\} \cup X \subset V(H_{i-1})$ for (b), and when $X, Y \subset V(H_{i-1})$ for (c). We consider the other possibilities. Since this involves a lengthy case by case analysis, we will only sketch the arguments.

Let $A = V(F) \setminus V(H_{i-1})$, $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. For two (possibly equal) vertices $x, y \in V(F)$, write xFy for the unique $x - y$

path in F. We make the following simple observation, which we will use repeatedly.

Observation 2.3 *For the edge-colouring of F in Cases 1 to 3, the following hold.*

(A) For any $x, y \in A$, the path xFy is rainbow.

(B) For any $1 \leq a \leq \ell$ and $x, y \in V(Q_a)$ with $x \in V(yQ_av_i - y)$, there exists $1 \leq b \leq \ell$, $b \neq a$, such that Q_b does not use the colours in the paths xQ_av_i and yQ_aw_a. □

Now, the following arguments apply in each of Cases 1 to 3. Observation 2.3 and the inductive hypothesis will be applied repeatedly, so we abbreviate these to Ob 2.3(A) and H_{i-1}(a), etc. Let $\mathcal{H}$ be the set of all rainbow subpaths of H_{i-1}.

(a) Without loss of generality, we have the following cases. In each case, we find two disjoint rainbow $u - v$ paths in H_i.

- $u \in A$ and $v \in V(H_{i-1})$. Let $u \in V(Q_a)$ for some a. By Ob 2.3(B), there exists $b \neq a$ such that uFw_b is rainbow. By H_{i-1}(b), there are $w_a - v$ and $w_b - v$ paths $R, R' \in \mathcal{H}$ which meet only at v. Take the paths uQ_aw_aRv and $uFw_bR'v$.

- $u, v \in A$. If $u, v \in V(Q_a)$ for some a, assume that $u \in V(vQ_av_i)$. By Ob 2.3(B) and H_{i-1}(a), there exist $b \neq a$ and a $w_b - w_a$ path $R \in \mathcal{H}$ such that $uFw_bRw_aQ_av$ is rainbow. Take uQ_av and $uFw_bRw_aQ_av$. If $u \in V(Q_a - v_i)$ and $v \in V(Q_b - v_i)$ for some $a \neq b$, then by H_{i-1}(a), there is a $w_a - w_b$ path $R \in \mathcal{H}$. Take uFv and $uQ_aw_aRw_bQ_bv$ (uFv is rainbow by Ob 2.3(A)).

(b) Without loss of generality, we have the following cases. In each case, we find two rainbow $u - X$ paths in H_i, meeting only at u.

- $u \in A$ and $x_1, x_2 \in V(H_{i-1})$. This is similar to the first item in (a) above. We apply H_{i-1}(c) instead of H_{i-1}(b) to obtain two suitable disjoint rainbow $u - X$ paths.

- $x_1 \in A$ and $u, x_2 \in V(H_{i-1})$. Let $x_1 \in V(Q_a)$ for some a. If $x_2 \neq w_a$, take $uRw_aQ_ax_1$ and $uR'x_2$, for some $R, R' \in \mathcal{H}$ meeting only at u (by H_{i-1}(b)). If $x_2 = w_a$, take uRw_bFx_1 and $uR'x_2$, for some $b \neq a$ and $R, R' \in \mathcal{H}$ meeting only at u (by Ob 2.3(B), H_{i-1}(b)).

- $u, x_1 \in A$ and $x_2 \in V(H_{i-1})$. Let $u \in V(Q_a)$ for some a. If $x_1 \notin V(uQ_aw_a)$, take uFx_1 and $uQ_aw_aRx_2$, for some $R \in \mathcal{H}$ (by Ob 2.3(A), H_{i-1}(a)). If $x_1 \in V(uQ_aw_a)$, take uQ_ax_1 and uFw_bRx_2 for some $b \neq a$ and $R \in \mathcal{H}$ (by Ob 2.3(B), H_{i-1}(a)).

- $x_1, x_2 \in A$ and $u \in V(H_{i-1})$. If $x_1, x_2 \in V(Q_a)$ for some a with $x_2 \in V(x_1 Q_a v_i)$, take $uRw_aQ_ax_1$ and $uR'w_bFx_2$, for some $b \neq a$ and $R, R' \in \mathcal{H}$ meeting only at u (by Ob 2.3(B), H_{i-1}(b)). If $x_1 \in V(Q_a - v_i)$ and $x_2 \in V(Q_b - v_i)$ for some $a \neq b$, take $uRw_aQ_ax_1$ and $uR'w_bQ_bx_2$, for some $R, R' \in \mathcal{H}$ meeting only at u (by H_{i-1}(b)).

- $u, x_1, x_2 \in A$. Assume that $x_2 \notin V(uFx_1)$. Take uFx_1 and uFx_2 if they meet only at u (by Ob 2.3(A)). Otherwise, let $u \in V(Q_a)$ for some a. If $u, x_1, x_2 \in V(Q_a)$, then $x_1 \in V(uQ_ax_2)$. Take uQ_ax_1, and $uFw_bRw_aQ_ax_2$ or $uQ_aw_aRw_bFx_2$, for some $b \neq a$ and $R \in \mathcal{H}$ (by Ob 2.3(B), H_{i-1}(a)). If $x_2 \in V(Q_c)$ for some $c \neq a$, take uFx_1 and $uQ_aw_aRw_cQ_cx_2$, for some $R \in \mathcal{H}$ (by Ob 2.3(A), H_{i-1}(a)).

(c) Without loss of generality, we have the following cases. In each case, we find two disjoint rainbow $X - Y$ paths in H_i.

- $x_1 \in A$ and $x_2, y_1, y_2 \in V(H_{i-1})$, or $x_1, x_2 \in A$ and $y_1, y_2 \in V(H_{i-1})$. These are similar to the second and fourth items in (b) above, respectively. For both, apply H_{i-1}(c) instead of H_{i-1}(b) to obtain two suitable disjoint rainbow $X - Y$ paths.

- $x_1, y_1 \in A$ and $x_2, y_2 \in V(H_{i-1})$. Take x_1Fy_1 and x_2Ry_2, for some $R \in \mathcal{H}$ (by Ob 2.3(A), H_{i-1}(a)).

- $x_1, x_2, y_1 \in A$ and $y_2 \in V(H_{i-1})$. Assume that $x_2 \notin V(x_1Fy_1)$. Let $x_2 \in V(Q_a)$ for some a. If x_1Fy_1 and $x_2Q_aw_a$ are disjoint, take x_1Fy_1 and $x_2Q_aw_aRy_2$, for some $R \in \mathcal{H}$ (by Ob 2.3(A), H_{i-1}(a)). Otherwise, we have $x_1, y_1 \in V(x_2Q_aw_a)$. Take $x_1Q_ay_1$ and $x_2Fw_bRy_2$, for some $b \neq a$ and $R \in \mathcal{H}$ (by Ob 2.3(B), H_{i-1}(a)).

- $x_1, x_2, y_1, y_2 \in A$. Assume that $x_2, y_2 \notin V(x_1Fy_1)$. Take x_1Fy_1 and x_2Fy_2 if they are disjoint (by Ob 2.3(A)). Otherwise, let $x_2 \in V(Q_a)$ and $y_2 \in V(Q_b)$ for some a, b. If $a \neq b$ and $x_2, y_2 \neq v_i$, take x_1Fy_1 and $x_2Q_aw_aRw_bQ_by_2$, for some $R \in \mathcal{H}$ (by Ob 2.3(A), H_{i-1}(a)). If $a = b$, then $x_1, y_1 \in V(x_2Q_ay_2)$. Take $x_1Q_ay_1$, and $x_2Fw_cRw_aQ_ay_2$ or $x_2Q_aw_aRw_cFy_2$, for some $c \neq a$ and $R \in \mathcal{H}$ (by Ob 2.3(B), H_{i-1}(a)).

We have now proved that properties (a) to (c) hold for H_i. This completes the proof of Theorem 2.2, and hence of Theorem 1.1. □□

Remark. The authors originally found a shorter proof of Theorem 1.1 in the case $\ell = 2$. The proof considered an edge-colouring of a minimally 2-connected spanning subgraph of a given 2-connected graph. A sketch of this proof can be found in [9].

Proof of Theorem 1.2. By the definition of a 2-connected series-parallel graph, it is easy to see (and well-known) that G can be constructed as follows. There are graphs $G_0 \subset G_1 \subset \cdots \subset G_t = G$ for some $t \geq 0$. $G_0 = Q_0$

is a cycle. For $1 \le i \le t$, G_i is obtained from G_{i-1} by attaching a path Q_i of length at least 1 to G_{i-1}, by identifying the end-vertices of Q_i with two distinct vertices $x, y \in V(G_{i-1})$. x and y must be the end-vertices of some path $P \subset Q_j$ for some $0 \le j < i$, and if an end-vertex of some Q_ℓ $(1 \le \ell < i,\ \ell \ne j)$ is an internal vertex of P, then P must contain both end-vertices of Q_ℓ. Note that every G_i $(0 \le i \le t)$ is a 2-connected series-parallel graph.

We successively embed each G_i into the plane as a plane graph, and define an orientation for G_i (we can think of this as turning G_i into a directed graph), as follows. Fix $z \in V(G_0)$. Firstly, embed the cycle G_0 into the plane and orient it clockwise. Now for $1 \le i \le t$ and each $0 \le j < i$, suppose that we have defined an orientation and a plane embedding for G_j, with the exterior cycle of G_j oriented clockwise, and containing z. Moreover, for each $1 \le j < i$ (if $i \ge 2$), assume that the path Q_j is embedded into the exterior face of G_{j-1}. If we can embed Q_i into the exterior face of G_{i-1}, then we do so, and orient Q_i so that, z remains on the new exterior cycle (of G_i), and the cycle remains directed clockwise. In addition, if $z \in V(Q_i)$, then choose the embedding and orientation for Q_i so that z is the tail vertex of Q_i. Note that this embedding and orientation of Q_i is unique, up to homotopism.

Otherwise, delete $Q_{i-1}, Q_{i-2}, \ldots$ successively until we reach the first j $(0 \le j < i-1)$ such that we can embed Q_i into the exterior face of G_j. Embed Q_i and orient it in the same way as above. Note that Q_i is embedded into the bounded face created by Q_{j+1} when it was embedded into the exterior face of G_j. Hence, we can re-embed and re-orient $Q_{j+1}, \ldots, Q_{i-1}$, in this order, to achieve an embedding and orientation for G_i. Finally, re-label $Q_i, Q_{j+1}, \ldots, Q_{i-1}$ with, respectively, $Q_{j+1}, Q_{j+2}, \ldots, Q_i$.

Repeat for each i until we reach $G_t = G$. For $0 \le i \le t$, let H_i be the exterior cycle of G_i, so that we have $z \in V(H_i)$ for every i.

Next, for each G_i, we define an edge-colouring with $|V(G_i)|$ colours, inductively as follows. Firstly, give G_0 a rainbow colouring, so that $|V(G_0)|$ colours are used. For $1 \le i \le t$, suppose that G_{i-1} is coloured with $|V(G_{i-1})|$ colours. Q_i is embedded into the exterior face of G_{i-1}. Colour the head edge of $\overrightarrow{Q_i}$ with the colour of the edge of $E(H_{i-1}) \setminus E(H_i)$ incident with the tail edge of $\overrightarrow{Q_i}$. Give the other edges of $\overrightarrow{Q_i}$ new and distinct colours. Then G_i is coloured with $|V(G_i)|$ colours. Note that by induction, H_i is rainbow coloured for every i.

We claim that for each $0 \le i \le t$, the colouring for G_i is rainbow 2-connected, which implies Theorem 1.2. Proceed by induction. Initially, the colouring for G_0 is rainbow 2-connected. For $1 \le i \le t$, suppose that the colouring for G_{i-1} is rainbow 2-connected. Let $u, v \in V(G_i)$. We want to find two disjoint rainbow $u - v$ paths in G_i. G_i is obtained by embedding Q_i into the exterior face of G_{i-1}. If $u, v \in V(G_{i-1})$, then we are done by induction, and if $u, v \in V(H_i)$, then we are also done, since H_i is rainbow coloured.

It remains to consider the case $u \in V(G_{i-1}) \setminus V(H_i)$ and $v \in V(Q_i) \setminus V(G_{i-1})$. For this, we aim to find two rainbow paths from u to H_i, meeting only at u, with both having 'almost' no colours in common with H_i. Proceed by deleting $Q_i, Q_{i-1}, Q_{i-2}, \ldots$ until we reach the first j $(0 \leq j < i)$ such that $u \in V(H_j)$. Re-embed Q_{j+1}. Then u becomes an interior vertex of G_{j+1}. Let Q_{j+1} be attached to H_j at x_1 and y_1, with x_1 directed towards u, then y_1, along H_j. Let $\overrightarrow{R_1}$ and $\overrightarrow{S_1}$ be the $x_1 - u$ and $u - y_1$ directed paths, respectively. Since H_j is rainbow coloured, so are $\overrightarrow{R_1}$ and $\overrightarrow{S_1}$. Moreover, between $\overrightarrow{R_1} \cup \overrightarrow{S_1}$ and H_{j+1}, the only edges that have the same colour are the tail edge of $\overrightarrow{R_1}$ and the head edge of $\overrightarrow{Q_{j+1}}$. Note also that $z \in V(\overrightarrow{y_1 H_{j+1} x_1})$. Now, we prove that for every $1 \leq \ell \leq i - j$, there are distinct $x_\ell, y_\ell \in V(H_{j+\ell})$ such that in $G_{j+\ell}$, the following hold.

(a) There are rainbow $x_\ell - u$ and $u - y_\ell$ directed paths $\overrightarrow{R_\ell}$ and $\overrightarrow{S_\ell}$, meeting only at u.

(b) Among all edges of $\overrightarrow{R_\ell} \cup \overrightarrow{S_\ell}$ and $H_{j+\ell}$, the only possible edges with the same colour are the tail edge of $\overrightarrow{R_\ell}$ and the head edge of $\overrightarrow{x_\ell H_{j+\ell} y_\ell}$. Hence, the directed cycle $\overrightarrow{x_\ell R_\ell u S_\ell y_\ell H_{j+\ell} x_\ell}$ is rainbow.

(c) If a and b are internal vertices of $\overrightarrow{x_\ell H_{j+\ell} y_\ell}$ and $\overrightarrow{y_\ell H_{j+\ell} x_\ell}$ respectively, then for all $0 \leq m \leq j + \ell$, we have $\{a, b\} \not\subset V(Q_m)$.

(d) $z \in V(\overrightarrow{y_\ell H_{j+\ell} x_\ell})$.

We proceed inductively. For $\ell = 1$, (a) to (d) all hold. For $2 \leq \ell \leq i - j$, suppose that they hold for $G_{j+\ell-1}$. We have the vertices $x_{\ell-1}, y_{\ell-1}$, and by (a), the directed paths $\overrightarrow{R_{\ell-1}}, \overrightarrow{S_{\ell-1}}$. By (d), let $\overrightarrow{T_1}, \overrightarrow{T_2}$ and $\overrightarrow{T_3}$ be the $x_{\ell-1} - y_{\ell-1}$, $y_{\ell-1} - z$ and $z - x_{\ell-1}$ directed paths along $H_{j+\ell-1}$. Embed and orient $Q_{j+\ell}$ as before. Let x and y be the tail and head vertices of $\overrightarrow{Q_{j+\ell}}$, respectively. By (c), x and y must both belong to $\overrightarrow{T_1}$ or $\overrightarrow{T_2} \cup \overrightarrow{T_3}$, since $G_{j+\ell}$ is a 2-connected series-parallel graph. We consider two cases. Note that the following arguments apply even if we have $z \in \{x_{\ell-1}, y_{\ell-1}\}$.

- $x, y \in V(\overrightarrow{T_1})$, or $x, y \in V(\overrightarrow{T_2} - z)$, or $x, y \in V(\overrightarrow{T_3})$.

  We have $x_{\ell-1}, y_{\ell-1} \in V(H_{j+\ell})$. Set $x_\ell = x_{\ell-1}$, $y_\ell = y_{\ell-1}$, $\overrightarrow{R_\ell} = \overrightarrow{R_{\ell-1}}$ and $\overrightarrow{S_\ell} = \overrightarrow{S_{\ell-1}}$. Clearly $G_{j+\ell}$ satisfies (a), (c) and (d). For (b), note that $\overrightarrow{R_\ell} \cup \overrightarrow{S_\ell} = \overrightarrow{R_{\ell-1}} \cup \overrightarrow{S_{\ell-1}}$, and to get from the set of colours of $H_{j+\ell-1}$ to that of $H_{j+\ell}$, we remove some colours of $H_{j+\ell-1}$ and add some new colours, unused in $G_{j+\ell-1}$. Hence, among the edges of

$\overrightarrow{R_\ell} \cup \overrightarrow{S_\ell}$ and $H_{j+\ell}$, the only ones that can have the same colour will still be the tail edge of $\overrightarrow{R_\ell}$ and the head edge of $\overrightarrow{x_\ell H_{j+\ell} y_\ell}$. They will actually have different colours if $x, y \in V(\overrightarrow{T_1})$, $y = y_{\ell-1}$, and x, y are non-adjacent in $H_{j+\ell-1}$.

- $x \in V(\overrightarrow{T_3})$, $y \in V(\overrightarrow{T_2} - z)$ and $\{x, y\} \neq \{x_{\ell-1}, y_{\ell-1}\}$.

  Here, $x_{\ell-1}, y_{\ell-1}$ can possibly be interior vertices of $G_{j+\ell}$. Along $\overrightarrow{H_{j+\ell-1}}$, $y_{\ell-1}$ is directed towards $y, z, x, x_{\ell-1}$, in this order. Let $x_\ell = x$, $y_\ell = y$. Then $G_{j+\ell}$ clearly satisfies (c) and (d). Let $\overrightarrow{R_\ell} = \overrightarrow{x H_{j+\ell-1} x_{\ell-1} R_{\ell-1} u}$, $\overrightarrow{S_\ell} = \overrightarrow{u S_{\ell-1} y_{\ell-1} H_{j+\ell-1} y}$. We have $\overrightarrow{R_\ell}$, $\overrightarrow{S_\ell}$ are rainbow, since $\overrightarrow{R_\ell} \cup \overrightarrow{S_\ell} \subset \overrightarrow{x_{\ell-1} R_{\ell-1} u S_{\ell-1} y_{\ell-1} H_{j+\ell-1} x_{\ell-1}}$, the latter of which is rainbow by (b) for $G_{j+\ell-1}$. Hence, $G_{j+\ell}$ satisfies (a). For (b), note that $\overrightarrow{H_{j+\ell}} = \overrightarrow{x Q_{j+\ell} y H_{j+\ell-1} x}$. Again by (b) for $G_{j+\ell-1}$, $\overrightarrow{R_\ell} \cup \overrightarrow{S_\ell}$ has no colours in common with $\overrightarrow{y H_{j+\ell-1} x}$. Also, by the construction of the colouring of $\overrightarrow{Q_{j+\ell}}$, among the edges of $\overrightarrow{R_\ell} \cup \overrightarrow{S_\ell}$ and $\overrightarrow{Q_{j+\ell}}$, the only ones that can have the same colour are the tail edge of $\overrightarrow{R_\ell}$ and the head edge of $\overrightarrow{Q_{j+\ell}} = \overrightarrow{x_\ell H_{j+\ell} y_\ell}$. They will actually have different colours if $x = x_{\ell-1}$ and $x_{\ell-1}, y_{\ell-1}$ are non-adjacent in $H_{j+\ell-1}$. This proves (b) for $G_{j+\ell}$.

Hence, properties (a) to (d) hold for G_i. By (a), we have the vertices $x_i, y_i \in V(H_i)$ and the directed paths $\overrightarrow{R_i}, \overrightarrow{S_i}$. If $v \in V(\overrightarrow{x_i H_i y_i}) \setminus \{x_i, y_i\}$, take $u\overleftarrow{R_i}x_i\overrightarrow{H_i}v$ and $u\overrightarrow{S_i}y_i\overleftarrow{H_i}v$. If $v \in V(\overrightarrow{y_i H_i x_i})$, take $\overleftarrow{u R_i x_i H_i v}$ and $\overrightarrow{u S_i y_i H_i v}$. By (b), we have two disjoint rainbow $u - v$ paths in G_i, in both cases. The induction on i is complete, and Theorem 1.2 follows. □

Proof of Theorem 1.3. Let $Q_1, \ldots, Q_t$ be the t paths, and x, y be their common end-vertices. The case $t = 2$ is clear (G is a cycle). Now, let $t \geq 3$.

Firstly, since $e(G - y) = n - 2$, if we colour G with fewer than $n - 2$ colours, then for some $u, v \in \Gamma(y) \setminus \{x\}$, the unique $u - v$ path P in $G - y$ is not rainbow. But in G, there is only one pair of disjoint $u - v$ paths, and P is one of the paths. Hence, $rc_2(G) \geq n - 2$.

Secondly, colour G as follows. For $1 \leq i \leq t - 1$, colour the edges of Q_i incident to x and y with colours i and $i + 1$ (modulo $t - 1$) respectively. Colour the other edges with further distinct colours. Then we have used $n - 1$ colours for G, and this colouring is rainbow 2-connected, since $t \geq 3$. Hence, $rc_2(G) \leq n - 1$. □

3 Complete Bipartite and Multipartite Graphs

In this section, we prove Theorems 1.6 to 1.9. We will only sketch the proofs of Theorems 1.8 and 1.9, since these are similar to the proofs of Theorems 1.6 and 1.7.

Proof of Theorem 1.6. Colour the edges of a perfect matching of $K_{n,n}$ with colour 1, and randomly and independently colour the other edges with colours 2 and 3. For $u, v \in V(K_{n,n})$, let $E_{u,v}$ be the event that there are fewer than k disjoint rainbow $u - v$ paths. If $\mathbb{P}(\bigcup_{u,v} E_{u,v}) < 1$, then a suitable 3-colouring of $K_{n,n}$ exists, and Theorem 1.6 follows.

Fix u and v in the same class of $K_{n,n}$, and let $\mathcal{P}$ be a set of $n-2$ disjoint $u - v$ paths of length 2, with no path using colour 1. The probability that each path of $\mathcal{P}$ is rainbow is $\frac{1}{2}$, and these probabilities are independent. Let X be the number of rainbow $u - v$ paths of $\mathcal{P}$. We have $X \sim \mathrm{Bi}(n-2, \frac{1}{2})$. Using the Chernoff bound (see, for example, [11] Ch. 2.1; note that $k - 3 < \frac{1}{2}(n-2)$, so the bound applies), and the fact that $n \geq \frac{2k-4}{1-2\varepsilon} + 1$ implies $\frac{n-2k+4}{n-2} \geq 2\varepsilon$,

$$\begin{aligned}\mathbb{P}(E_{u,v}) &\leq \mathbb{P}(X \leq k-3) = \mathbb{P}\left(X \leq \frac{1}{2}(n-2)\left(1 - \frac{n-2k+4}{n-2}\right)\right)\\ &< e^{-\varepsilon^2(n-2)}.\end{aligned}$$

Now, fix u and v in different classes of $K_{n,n}$, and let $\mathcal{P}$ be a set of $n-1$ disjoint $u-v$ paths of length 3, each of which having one edge with colour 1. The probability that each path of $\mathcal{P}$ is rainbow is $\frac{1}{2}$, and these probabilities are independent. Let X be the number of rainbow $u - v$ paths in $\mathcal{P}$, so that $X \sim \mathrm{Bi}(n-1, \frac{1}{2})$. A similar calculation (note that $k - 2 < \frac{1}{2}(n-1)$ and $\frac{n-2k+3}{n-1} \geq 2\varepsilon$) gives

$$\begin{aligned}\mathbb{P}(E_{u,v}) &\leq \mathbb{P}(X \leq k-2) = \mathbb{P}\left(X \leq \frac{1}{2}(n-1)\left(1 - \frac{n-2k+3}{n-1}\right)\right)\\ &< e^{-\varepsilon^2(n-1)}.\end{aligned}$$

By the union bound, $\mathbb{P}(\bigcup_{u,v} E_{u,v}) < 2n^2 e^{-\varepsilon^2(n-2)}$. On $[0, \infty)$, the function $2x^2 e^{-\varepsilon^2(x-2)}$ is eventually decreasing, and tends to 0 as $x \to \infty$. If $\theta = \theta(\varepsilon)$ is the largest solution of $2x^2 e^{-\varepsilon^2(x-2)} = 1$, then $\mathbb{P}(\bigcup_{u,v} E_{u,v}) < 1$ for $n \geq \theta$. Hence, the result holds for $n \geq \frac{2k-4}{1-2\varepsilon} + 1$ with $k \geq \frac{1}{2}(\theta - 1)(1 - 2\varepsilon) + 2$. □

Proof of Theorem 1.7. Let A and B be the classes of $K_{n,n}$. We prove that there exist $u, v \in A$ which will work for the theorem. For

$u, v \in A$, any rainbow $u - v$ path must have length 2. Let $Z(\{u,v\})$ be the number of monochromatic $u - v$ paths of length 2. Note that $\sum_{u,v \in A} Z(\{u,v\}) = \sum_{b \in B} Y(b)$, where $Y(b)$ is the number of monochromatic paths of length 2 with middle vertex b. For $b \in B$ and $i \in \{1,2,3\}$, let $d_i(b)$ be the number of edges of colour i at b. Then, by the convexity of $\binom{x}{2}$ (for $x \in \mathbb{R}$),

$$\begin{aligned}
\mathbb{E}Z &= \frac{1}{\binom{n}{2}} \sum_{u,v \in A} Z(\{u,v\}) = \frac{1}{\binom{n}{2}} \sum_{b \in B} Y(b) \\
&= \frac{1}{\binom{n}{2}} \sum_{b \in B} \left(\binom{d_1(b)}{2} + \binom{d_2(b)}{2} + \binom{d_3(b)}{2} \right) \\
&\geq \frac{1}{\binom{n}{2}} \sum_{b \in B} 3 \binom{\frac{1}{3}(d_1(b) + d_2(b) + d_3(b))}{2} \\
&= \frac{1}{\binom{n}{2}} \cdot 3n \binom{\frac{1}{3}n}{2} = \frac{n^2 - 3n}{3(n-1)}.
\end{aligned}$$

Hence, there exist $u, v \in A$ such that the number of rainbow $u - v$ paths is at most $n - \frac{n^2-3n}{3(n-1)} = \frac{2n^2}{3(n-1)}$. □

Proof of Theorem 1.8. Colour the edges of $K_{t \times n}$ randomly and independently with 2 colours. For $u, v \in V(K_{t \times n})$, let $E_{u,v}$ be the event that there are fewer than k disjoint rainbow $u - v$ paths.

First, let u and v be in the same partite set, and let X be the number of rainbow $u - v$ paths of length 2, so that $X \sim \mathrm{Bi}((t-1)n, \frac{1}{2})$. By the Chernoff bound (note that $k - 1 < \frac{1}{2}(t-1)n$, and $n \geq \frac{2k-2}{(t-2)(1-2\varepsilon)}$ implies $\frac{(t-1)n-2k+2}{(t-1)n} \geq 2\varepsilon$),

$$\begin{aligned}
\mathbb{P}(E_{u,v}) &= \mathbb{P}(X \leq k-1) = \mathbb{P}\left(X \leq \frac{1}{2}(t-1)n \left(1 - \frac{(t-1)n - 2k + 2}{(t-1)n} \right) \right) \\
&< e^{-(t-1)\varepsilon^2 n}.
\end{aligned}$$

Next, let u and v be in different partite sets, and let X be the number of rainbow $u - v$ paths of length 2, so that $X \sim \mathrm{Bi}((t-2)n, \frac{1}{2})$. As before (note that $k - 2 < \frac{1}{2}(t-2)n$ and $\frac{(t-2)n-2k+4}{(t-2)n} \geq 2\varepsilon$),

$$\begin{aligned}
\mathbb{P}(E_{u,v}) &= \mathbb{P}(X \leq k-2) = \mathbb{P}\left(X \leq \frac{1}{2}(t-2)n \left(1 - \frac{(t-2)n - 2k + 4}{(t-2)n} \right) \right) \\
&< e^{-(t-2)\varepsilon^2 n}.
\end{aligned}$$

We have $\mathbb{P}(\bigcup_{u,v} E_{u,v}) < \frac{1}{2}t^2n^2e^{-(t-2)\varepsilon^2 n}$. If $\theta = \theta(\varepsilon, t)$ is the largest solution of $\frac{1}{2}t^2x^2e^{-(t-2)\varepsilon^2 x} = 1$, then $\mathbb{P}(\bigcup_{u,v} E_{u,v}) < 1$ for $n \geq \theta$. Hence, the result holds for $n \geq \frac{2k-2}{(t-2)(1-2\varepsilon)}$ with $k \geq \frac{1}{2}\theta(t-2)(1-2\varepsilon)+1$. □

Proof of Theorem 1.9. Let A be a class of $K_{t\times n}$ and $B = V(K_{t\times n}) \setminus A$. For $u, v \in A$, any rainbow $u-v$ path must have length 2. Let $Z(\{u,v\})$ be the number of monochromatic $u-v$ paths of length 2. Then, by a similar calculation as in Theorem 1.7,

$$\mathbb{E}Z = \frac{1}{\binom{n}{2}} \sum_{u,v\in A} Z(\{u,v\}) \geq \frac{1}{\binom{n}{2}} \cdot 2(t-1)n\binom{\frac{1}{2}n}{2} = \frac{(t-1)(n^2-2n)}{2(n-1)}.$$

Hence, there exist $u, v \in A$ such that the number of rainbow $u-v$ paths is at most $(t-1)n - \frac{(t-1)(n^2-2n)}{2(n-1)} = \frac{(t-1)n^2}{2(n-1)}$. □

4 Random Graphs

We first prove Theorem 1.10. We recall the result for $rc(G_{n,p})$ by Caro et al.

Theorem 4.1 ([2]) *$p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc(G_{n,p}) \leq 2$.*

The proof of Theorem 1.10 will be similar to that of Theorem 4.1. A key result used in the proof of Theorem 4.1 is Theorem 4.2 below. We generalise this in Theorem 4.3.

Theorem 4.2 ([2]) *For any non-complete graph G on n vertices, with minimum degree $\delta(G) \geq \frac{n}{2} + \log_2 n$, we have $rc(G) = 2$.*

Theorem 4.3 *For all $k \geq 2$ and $\varepsilon > 0$, there exists an integer $N = N(k,\varepsilon)$ so that, for all graphs G on $n \geq N$ vertices with minimum degree $\delta(G) \geq \frac{n}{2} + (1+\varepsilon)\log_2 n$, we have $rc_k(G) = 2$.*

Proof. Colour the edges of G with 2 colours, randomly and independently. Let $u, v \in V(G)$, and E be the event that we have fewer than k disjoint rainbow $u-v$ paths. We have $|\Gamma(u) \cap \Gamma(v)| \geq t$, where $t = 2(1+\varepsilon)\log_2 n$ (we may assume that $t \in \mathbb{N}$). Let $\mathcal{P}$ be a set of t disjoint $u-v$ paths of length 2, and X be the number of rainbow paths in $\mathcal{P}$. For $P \in \mathcal{P}$, $\mathbb{P}(P$ is rainbow$) = \frac{1}{2}$, so $X \sim \mathrm{Bi}(t, \frac{1}{2})$. For $0 \leq \ell \leq t$, $\mathbb{P}(X \leq \ell) = (\frac{1}{2})^t \sum_{r=0}^{\ell} \binom{t}{r} \leq (\frac{1}{2})^t (1+t)^\ell$. For fixed $k \geq 2$ and $\varepsilon > 0$, $\lim_{n\to\infty} \frac{(1+2(1+\varepsilon)\log_2 n)^{k-1}}{n^{2\varepsilon}} = 0$, so

$$\mathbb{P}(E) \leq \mathbb{P}(X \leq k-1) \leq \left(\frac{1}{2}\right)^{2(1+\varepsilon)\log_2 n} (1+2(1+\varepsilon)\log_2 n)^{k-1} = o\left(\frac{1}{n^2}\right).$$

There are $\binom{n}{2}$ pairs u, v, so from the union bound, with positive probability when n is sufficiently large, any two vertices in G have at least k

disjoint rainbow paths connecting them. Hence, there exists a 2-colouring of G which is rainbow k-connected. □

Proof of Theorem 1.10. By Theorem 4.1, it suffices to consider $k \geq 2$, and to show that there is a constant $C > 0$ such that, for $p = C\sqrt{\log n/n}$, we have $rc_k(G_{n,p}) = 2$ a.s. By Theorem 4.3 with $\varepsilon = 1$, it suffices to show that any two vertices of $G_{n,p}$ have at least $4\log_2 n$ common neighbours, a.s. In the proof of Theorem 4.1 [2], it was shown that there is a constant $C' > 0$ such that, for $p = C'\sqrt{\log n/n}$, any two vertices of $G_{n,p}$ have at least $2\log_2 n$ common neighbours, a.s. Here, we take a larger constant for C. □

Next, we prove Theorem 1.11. We first prove Theorem 4.4, which is the analogue of Theorem 4.3 for bipartite graphs. The version for $k = 1$ was also proved by Caro et al.

Theorem 4.4 *Let $c = 1/\log \frac{9}{7}$. For all $k \geq 2$ and $\varepsilon > 0$, there exists an integer $N = N(k, \varepsilon)$ so that, for all bipartite graphs G on $n \geq N$ vertices, where any two vertices in the same class have at least $2c(1+\varepsilon)\log n$ common neighbours, we have $rc_k(G) = 3$.*

Theorem 4.5 ([2]) *Let $c = 1/\log \frac{9}{7}$. If G is a non-complete bipartite graph on n vertices, and any two vertices in the same class have at least $2c\log n$ common neighbours, then $rc(G) = 3$.*

Proof of Theorem 4.4. Colour the edges of G with 3 colours, randomly and independently. Let $u, v \in V(G)$, $t = 2c(1+\varepsilon)\log n$ (assume that $t \in \mathbb{N}$), and E be the event that we have fewer than k disjoint rainbow $u-v$ paths.

First, let u and v be in the same class, so $|\Gamma(u)\cap\Gamma(v)| \geq t$. Let $\mathcal{P}$ be a set of t disjoint $u-v$ paths of length 2, and X be the number of rainbow paths in $\mathcal{P}$. For $P \in \mathcal{P}$, $\mathbb{P}(P \text{ is rainbow}) = \frac{2}{3}$, so $X \sim \mathrm{Bi}(t, \frac{2}{3})$. For $0 \leq \ell \leq t$, $\mathbb{P}(X \leq \ell) = \sum_{r=0}^{\ell}(\frac{1}{3})^{t-r}(\frac{2}{3})^r\binom{t}{r} \leq (\frac{7}{9})^t(1+t)^\ell$. For fixed $k \geq 2$ and $\varepsilon > 0$, $\mathbb{P}(E) \leq P(X \leq k-1) \leq (\frac{7}{9})^{2c(1+\varepsilon)\log n}(1 + 2c(1+\varepsilon)\log n)^{k-1} = o(\frac{1}{n^2})$.

Next, let u and v be in different classes. We claim that there is a set $\mathcal{P}$ of $t-1$ disjoint $u-v$ paths of length 3. Let $B = \Gamma(u) \setminus \{v\}$, and note that $|B| \geq t-1$. For each vertex $x \in B$, let $A_x = (\Gamma(v) \cap \Gamma(x)) \setminus \{u\}$, and note that $|A_x| \geq t-1$. For the sets A_x, take a system of distinct representatives $\{y_x\}_{x\in B}$. Then we can take $\mathcal{P} = \{uxy_xv : x \in B\}$, and the claim holds. Now, let X be the number of rainbow paths of $\mathcal{P}$. For $P \in \mathcal{P}$, $\mathbb{P}(P \text{ is rainbow}) = \frac{2}{9}$, so $X \sim \mathrm{Bi}(t-1, \frac{2}{9})$. For $0 \leq \ell \leq t-1$, $\mathbb{P}(X \leq \ell) = \sum_{r=0}^{\ell}(\frac{7}{9})^{t-1-r}(\frac{2}{9})^r\binom{t-1}{r} \leq (\frac{7}{9})^t(1+t)^\ell$. As before, $\mathbb{P}(E) \leq \mathbb{P}(X \leq k-1) = o(\frac{1}{n^2})$.

There are $\binom{n}{2}$ pairs u, v, so applying the union bound, there exists an edge-colouring of G with 3 colours which is rainbow k-connected. □

Proof of Theorem 1.11. For the first part, we prove that for $C_1 >$

6, if $p \geq C_1\sqrt{\log n/n}$, then $rc_k(G_{n,n,p}) = 3$, a.s. By Theorem 4.5, and Theorem 4.4 with $\varepsilon = 1$, it suffices to show that any two vertices in the same class of $G_{n,n,p}$ have at least $4c\log(2n)$ common neighbours, a.s., where $c = 1/\log\frac{9}{7}$. Fix two vertices u, v in one class. For a vertex w in the other class, the probability that w is a common neighbour of u and v is $C_1^2\log n/n$. If X be the number of common neighbours of u and v, then $X \sim \text{Bi}(n, C_1^2\log n/n)$ and $\mathbb{E}X = C_1^2\log n$. By Chernoff's inequality, $\mathbb{P}(X \leq \frac{1}{2}C_1^2\log n) \leq \exp(-\frac{1}{8}C_1^2\log n) = o(\frac{1}{n^2})$. There are $2\binom{n}{2}$ pairs u, v. By the union bound, any two vertices in the same class of $G_{n,n,p}$ have at least $\frac{1}{2}C_1^2\log n > 4c\log(2n)$ common neighbours in the other class, a.s.

For the second part, we prove that for $0 < C_2 < \frac{1}{2}$, $G_{n,n,p}$ has diameter at least 4 a.s., if $p = C_2\sqrt{\log n/n}$. Let A, B be two disjoint sets in one class, with $|A| = |B| = \frac{n}{2}$ (assume that $\frac{n}{2} \in \mathbb{N}$). Let $A = \{a_1, \dots, a_{n/2}\}$ and $B = \{b_1, \dots, b_{n/2}\}$. The probability of the event E_i that a_i and b_i have a common neighbour in the other class is $1 - \left(1 - \frac{C_2^2\log n}{n}\right)^n$. The events E_i are independent, so the probability that all pairs a_i, b_i have a common neighbour is $\left(1 - \left(1 - \frac{C_2^2\log n}{n}\right)^n\right)^{n/2} = o_n(1)$. Hence, there is a pair of vertices in one class with no common neighbour, a.s., so they have distance at least 4 between them. □

We finish with the proof of Theorem 1.12, which is similar to the previous proofs.

Proof of Theorem 1.12. For the first part, we prove that for $C_1 > 3$, if $M \geq C_1\sqrt{n^3\log n}$, then $rc_k(G_{n,M}) = 2$, a.s. By Theorem 4.2, and Theorem 4.3 with $\varepsilon = 1$, it suffices to show that any two vertices of $G_{n,M}$ have at least $4\log_2 n$ common neighbours, a.s. Let $N = \binom{n}{2}$, and fix two vertices u, v. For another vertex w, the probability that w is a common neighbour of u and v is $\binom{N-2}{M-2}/\binom{N}{M} = \frac{(M-1)M}{(N-1)N}$. If X is the number of common neighbours of u and v, then $X \sim \text{Bi}\left(n-2, \frac{(M-1)M}{(N-1)N}\right)$ and $\mathbb{E}X = (n-2)\frac{(M-1)M}{(N-1)N} \geq 2C_1^2\log n$. By Chernoff's inequality, $\mathbb{P}(X \leq C_1^2\log n) \leq \exp(-\frac{1}{8}\mathbb{E}X) \leq \exp(-\frac{1}{4}C_1^2\log n) = o(\frac{1}{n^2})$. There are $\binom{n}{2}$ pairs u, v. By the union bound, any two vertices in $G_{n,M}$ have at least $C_1^2\log n > 4\log_2 n$ common neighbours, a.s.

For the second part, we prove that if $0 < C_2 < \frac{1}{5}$, then $G_{n,M}$ has diameter at least 3 a.s., if $M = C_2\sqrt{n^3\log n}$. Let A, B be two disjoint sets of vertices, with $|A| = |B| = n^{1/2}$ (assume that $n^{1/2} \in \mathbb{N}$). The probability that $A \cup B$ is an independent set is

$$\begin{aligned}\frac{\binom{N-\binom{2n^{1/2}}{2}}{M}}{\binom{N}{M}} &= \frac{\left(N - \binom{2n^{1/2}}{2} - C_2\sqrt{n^3\log n} + 1\right)\cdots\left(N - C_2\sqrt{n^3\log n}\right)}{\left(N - \binom{2n^{1/2}}{2} + 1\right)\cdots N} \\ &= 1 - o_n(1).\end{aligned}$$

Let $A = \{a_1, \ldots, a_{n^{1/2}}\}$, $B = \{b_1, \ldots, b_{n^{1/2}}\}$. The probability of the event E_i that a_i and b_i have a common neighbour is $1-\left(1-\frac{M(M-1)}{N(N-1)}\right)^{n-2n^{1/2}}$. The events E_i are independent, so the probability that all pairs a_i, b_i have a common neighbour is

$$\left(1-\left(1-\frac{M(M-1)}{N(N-1)}\right)^{n-2n^{1/2}}\right)^{n^{1/2}} \leq \left(1-\left(1-\frac{M(M-1)}{N(N-1)}\right)^{n}\right)^{n^{1/2}}$$
$$= o_n(1),$$

since $M = C_2\sqrt{n^3 \log n}$ and $0 < C_2 < \frac{1}{5}$.

It follows that there is a non-adjacent pair a_i, b_i with no common neighbour a.s., and such a_i and b_i have distance at least 3 between them. □

5 Open Problems

In this section, we pose some open problems which are related to the results of this paper.

We can ask the following extension of Problem 1.4: *If $1 \leq k \leq \ell$, derive a sharp upper bound on $rc_k(G)$ for every ℓ-connected graph G.* A result of Mader [18] implies that any minimally k-connected graph on n vertices has at most kn edges. If G is ℓ-connected on n vertices, then by considering a minimally k-connected spanning subgraph of G, we have $rc_k(G) \leq kn$. Therefore, we ask the following question.

Problem 5.1 *Let $1 \leq k \leq \ell$. Find the least constant $c = c(k, \ell)$, where $0 < c \leq k$, such that for all ℓ-connected graphs G on n vertices, we have $rc_k(G) \leq cn$.*

We have already asked the question of whether or not do we have $c(2, 2) = 1$.

For Theorems 1.6 and 1.7, if k is sufficiently large, we still do not know the best function $f(k)$ such that, if $n \geq f(k)$, then $rc_k(K_{n,n}) = 3$ (unlike the analogous situation for complete graphs). We only know that the answer lies between $\frac{3k}{2}$ and $2k + o(k)$.

Problem 5.2 *For k sufficiently large, is there a constant $\frac{3}{2} \leq c \leq 2$ such that, if $n \geq ck$, then $rc_k(K_{n,n}) = 3$?*

In relation to Theorems 1.6 and 1.8, for complete bipartite and multipartite graphs G, all known results for $rc_k(G)$ with $k \geq 2$ only concern those graphs with equipartitions. Chartrand et al. [5] asked the question of whether for each $k \geq 2$, there exists a function $f(k)$ such that for all $f(k) \leq m \leq n$, we have $rc_k(K_{m,n}) = 3$. We can see that the answer to this question is negative, since the same authors ([4], Th. 2.6) proved that if $n > 3^m$, then $rc(K_{m,n}) = 4$. For complete multipartite graphs, they also proved that for $t \geq 3$, $n_1 \leq \cdots \leq n_t$ and $m = n_1 + \cdots + n_{t-1}$, we have $rc(K_{n_1,\ldots,n_t}) = 3$ if $n_t > 2^m$ ([4], Th. 2.7). Therefore, we pose the following problem.

Problem 5.3 *For $k, t \geq 2$ and $n_1 \leq \cdots \leq n_t$, is there a function $f(k,t)$ such that, if $n_1 \geq f(k,t)$, then*

$$rc_k(K_{n_1,\ldots,n_t}) = \begin{cases} 3 \text{ or } 4 & \text{if } t = 2, \\ 2 \text{ or } 3 & \text{if } t \geq 3? \end{cases}$$

Moreover, when precisely do we have $rc_k(K_{n_1,n_2}) = 3$? When precisely do we have $rc_k(K_{n_1,\ldots,n_t}) = 2$ if $t \geq 3$?

Finally, for random graphs, we can obviously ask the following.

Problem 5.4 *For $d \geq 2$, determine a sharp threshold function for the property $rc_k(G) \leq d$, where G is another random graph model.*

In particular, an answer for random regular graphs would be interesting.

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