Vertices contained in all or in no Minimum Liar's Dominating Sets of a Tree

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Abstract

Let $G = (V, E)$ be a graph having at least 3 vertices in each of its components. A set $L \subseteq V(G)$ is a liar's dominating set if (1) $|N_G[v] \cap L| \geq 2$ for all $v \in V(G)$ and (2) $|(N_G[u] \cup N_G[v]) \cap L| \geq 3$ for every pair $u, v \in V(G)$ of distinct vertices in G, where $N_G[x] = \{y \in V | xy \in E\} \cup \{x\}$ is the closed neighborhood of x in G. In this paper, we characterize the vertices that are contained in all or in no minimum liar's dominating sets in trees. Given a tree T, we also propose a polynomial time algorithm to compute the set of all vertices which are contained in every minimum liar's dominating set of T and the set of all vertices which are not contained in any minimum liar's dominating set of T.

Keywords: Liar's domination; Graph algorithm; Tree.

1 Introduction

Let $G = (V, E)$ be a graph. For each $v \in V$, let $N_G(v) = \{u | uv \in E(G)\}$ denote the neighborhood of v and let $N_G[v] = N_G(v) \cup \{v\}$ denote the closed neighborhood of v. A vertex $u \in V(G)$ is said to be dominated by a vertex $v \in V(G)$ if $u \in N_G[v]$. A set $D \subseteq V$ is called a *dominating set* of a graph $G = (V, E)$ if each vertex $v \in V$ is dominated by a vertex in D,

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i.e., $|N_G[v] \cap D| \geq 1$ for all $v \in V$. A set $D \subseteq V$ is called a *double-dominating set* of a graph $G = (V, E)$ if each vertex $v \in V$ is dominated by at least two vertices in D, i.e., $|N_G[v] \cap D| \geq 2$ for all $v \in V$. Domination and its variations have been extensively studied in the literature and are used to model many practical problems arising in computer networks and operations research (see [4, 5]).

Assume that a graph $G = (V, E)$ models a communication network where each vertex represents a communicating device and each edge represents a communication link between two of its end vertices. Assume that each vertex of the graph G is the possible location for an intruder such as a thief, a saboteur, a fire or some possible fault and there is exactly one intruder in the system represented by G. A protection device at a vertex v is assumed to be able to (1) detect the intruder at any vertex in its closed neighborhood $N_G[v]$, and (2) report the vertex $u \in N_G[v]$ at which the intruder is located. One is interested in deploying protection devices at minimum number of vertices so that the intruder can be detected and identified correctly. This can be solved by finding a minimum cardinality dominating set, say D, of G and deploying protection devices at all the vertices of D. If any one protection device can fail to detect the intruder, then to correctly detecting and identifying the intruder one needs to place the protection devices at all the vertices of a minimum cardinality double-dominating set of G. If, however, any one protection device in the closed neighborhood of the intruder vertex might (either deliberately or through a transmission error) misreport (lie) the location of an intruder in its closed neighborhood, then it has been shown by Slater [8] that to correctly detecting and identifying the intruder one needs to place the protection devices at all the vertices of a minimum cardinality set L satisfying (1) $|L \cap N_G[v]| \geq 2$ for every $v \in V(G)$, and (2) for every pair u, v of distinct vertices $|(N_G[u] \cup N_G[v]) \cap L| \geq 3$. Such a set L of vertices is called a liar's dominating set (see [8]).

A set $L \subseteq V(G)$ of a graph $G = (V, E)$ is called a *liar's dominating set* if (1) for all $v \in V(G)$, $|N_G[v] \cap L| \geq 2$ and (2) for every pair $u, v \in V(G)$ of distinct vertices, $|(N_G[u] \cup N_G[v]) \cap L| \geq 3$. The *liar's domination number*, denoted as $\gamma_{LR}(G)$, is the minimum cardinality of a liar's dominating set of G.

Liar's domination has been studied in [7, 8]. In [8], Slater has given some lower bounds for a general graph G having at least 3 vertices in terms of the number of vertices n, the number of edges m and the maximum degree $\Delta(G)$. He has also given an exact formula for the $\gamma_{LR}(G)$ if G is either a path P_n or a cycle C_n with n vertices. In the same paper he has proved that for a tree T with at least 3 vertices $(3/4)(n+1) \leq \gamma_{LR}(T) \leq n$ and also characterized the trees having $\gamma_{LR}(T) = n$. In [7] the trees having $\gamma_{LR}(T) = (3/4)(n+1)$ have been characterized. Slater also proved that the decision version of the minimum liar's dominating set problem is NP-complete [8] for general graphs.

It is interesting to study the characterization of the vertices of G that are contained in all or in no set with a certain property P. Indeed, Hammer *et al.* [3] have characterized the vertices which are contained in all or in no maximum stable sets in a graph. Mynhardt [6] has characterized the vertices that are contained in all minimum dominating set of trees. Cockayne *et al.* [2] have characterized the vertices contained in all or in no minimum total dominating set of trees and Blidia *et al.* [1] have characterized the vertices contained in all or in no minimum double dominating set of trees.

In this paper, we characterize the vertices belonging to all or to no minimum liar's dominating set in a tree. We also propose polynomial time algorithm to compute these sets of a tree.

The rest of this paper is organized as follows. In section 2, we give some pertinent definitions and state some known results which will be used in the rest of the paper. In Section 3, we describe *tree pruning* technique and *reduction* technique and give the characterization of the set of vertices contained in all or in no minimum liar's dominating set. In section 4, we show that computing the set of vertices contained in all or in no minimum liar's dominating set can be done in $O(n^3)$ time. Finally, Section 5 concludes this paper.

2 Preliminaries

Let $G = (V, E)$ be a graph. For $S \subseteq V$, let $G[S]$ denote the subgraph induced by G on S. The *distance* between two vertices u and v in a graph G is the number of edges in a shortest path between them and is denoted as $d_G(u, v)$. The *degree* of a vertex v, denoted by $deg_G(v)$, is the number of vertices adjacent to v. A *tree* is a connected graph which has no cycle. A tree T_r is called a *rooted tree* if one of its vertices, say r, has been designated as the *root*. In a rooted tree, the *parent* of a vertex is the vertex adjacent to it and is in the path from this vertex to the root. Every vertex v, except the root, has a unique parent and is denoted by $p(v)$. If a vertex v is the parent of a vertex u, then u is called a *child* of v. The degree of a vertex u in a rooted tree T_v is basically the degree of that vertex in the underlying tree. For a vertex u in a rooted tree T_v, let $C_{T_v}(u)$ and $D_{T_v}(u)$ denote the set of children and the set descendants of u, respectively. Let $D_{T_v}[u] = D_{T_v}(u) \cup \{u\}$. The *maximal subtree* of a rooted tree T_r at v is the rooted subtree $T_r[D_{T_r}[v]]$ of T_v induced by T_r on $D_{T_r}[v]$. A vertex v in T_r is called a *leaf* if v has no child in T_r. A *pendant vertex* is a vertex of degree one. A vertex y adjacent to a pendant vertex x is called a *support vertex* of x. Let $P(G)$ and $S(G)$ denote the set of all pendant vertices and the set of all support vertices of G, respectively. A vertex v in T_r which has at least two children is called a *branch vertex*. Let $B(T_r)$ denotes the set of all branch vertices of T_r. For a vertex w of T_r, we define $P^j(w)$ as the set of leaves $u \in D_{T_r}(w)$ such that $d_{T_r}(w,u) \equiv j(mod\ 4)$ with $j = 0, 1, 2, 3$, and every vertex in the path from w to $p(u)$ other than w has exactly one child. Let P_n denote the path having n vertices.

There is no explicit formula for $\gamma_{LR}(T)$ for a tree T. However, there is an explicit formula for $\gamma_{LR}(P_n), n \geq 3$.

Lemma 2.1. [8] *For a path $P_n, n \geq 3$, $\gamma_{LR}(P_n) = \lceil 3(n+1)/4 \rceil$.*

Definition 2.2. For a graph G, we define the sets $\mathcal{A}_{LR}(G)$ and $\mathcal{N}_{LR}(G)$ by

$$\mathcal{A}_{LR}(G) = \{v \in V(G) | v \text{ is in every minimum liar's dominating set of } G\}$$
and
$$\mathcal{N}_{LR}(G) = \{v \in V(G) | v \text{ is in no minimum liar's dominating set of } G\}.$$

The following proposition immediately follows from the definition of liar's dominating set.

Proposition 2.3. *Let G be a graph having at least three vertices. Then the following statements are true.*

(a) $P(G) \cup S(G) \subseteq \mathcal{A}_{LR}(G)$.

(b) If $u \in P(G)$ is adjacent to v and $N_G(v) = \{u, w\}$, then $w \in \mathcal{A}_{LR}(G)$.

(c) If $n \equiv 3(mod\ 4)$, then P_n has a unique minimum liar's dominating set. Moreover, if $n \equiv 3(mod\ 4)$ and $v_1, v_2, \ldots, v_n$ is an ordering of P_n such that $v_iv_{i+1} \in E(P_n), 1 \leq i \leq n-1$, then $\mathcal{A}_{LR}(P_n) = \{v_i | i \equiv j(mod\ 4)$, where $j \in \{1, 2, 3\}\}$ and $\mathcal{N}_{LR}(P_n) = \{v_i | i \equiv 0(mod\ 4)\}$.

Definition 2.4. Attaching a path P_k to x: Given a vertex x of a tree T, we say we attach a path P_k to x if we join x and a pendant vertex, say y, of P_k by the edge xy to obtain the tree T'. So $V(T') = V(T) \cup V(P_k)$ and $E(T') = E(T) \cup E(P_k) \cup \{xy\}$.

Lemma 2.5. *Let $T = (V, E)$ be a tree with at least three vertices. Let u be a pendant vertex of T and v be the support vertex of u. If T' is obtained from T by attaching a P_4 to u, then*

(a) $\gamma_{LR}(T') = \gamma_{LR}(T) + 3$.

(b) For all $w \in V(T)$, if either (i) $d_T(u, w) \geq 3$, or (ii) $d_T(u, w) = 2$ and $|N_T(v)| > 2$, then $w \in \mathcal{A}_{LR}(T)$ if and only if $w \in \mathcal{A}_{LR}(T')$.

(c) *For all* $w \in V(T)$, *if either* (i) $d_T(u,w) \geq 3$, *or* (ii) $d_T(u,w) = 2$ *and* $|N_T(v)| > 2$, *then* $w \in \mathcal{N}_{LR}(T)$ *if and only if* $w \in \mathcal{N}_{LR}(T')$.

Proof. Without loss of generality, let $V(T') = V(T) \cup \{a,b,c,d\}$ and $E(T') = E(T) \cup \{ua, ab, bc, cd\}$.

(a) Let L be a minimum liar's dominating set of T. By Proposition 2.3(a), $u, v \in L$. Let $L' = L \cup \{b,c,d\}$. Clearly L' is a liar's dominating set of T'. Hence $\gamma_{LR}(T') \leq \gamma_{LR}(T) + 3$.

Let L' be a minimum liar's dominating set of T'. By Proposition 2.3(b) $b, c, d \in L'$. If $a \notin L'$, then $L = L' \setminus \{b,c,d\}$ is clearly a liar's dominating set of T of size $|L'| - 3$. So assume that $a \in L'$. Since $deg_{T'}(u) = 2$, L' contains at least one of u and v. If exactly one of u and v belongs to L', then L'', which is obtained from L' by replacing a with u or v whichever is not present in L', is a minimum liar's dominating set of T' not containing a. So $L''' = L'' \setminus \{b,c,d\}$ is clearly a liar's dominating set of T of size $|L'| - 3$. So assume that $u, v \in L'$. If there exists a vertex $x \in N_T(v) \setminus \{u\}$ such that $x \in L'$, then $L' \setminus \{a\}$ is a liar's dominating set of T' of cardinality less than $|L'|$. This contradicts the minimality of L'. So $(N_T(v) \setminus \{u\}) \cap L' = \emptyset$. Now $L'' = (L' \setminus \{a\}) \cup \{x\}$, where $x \in N_T(w) \setminus \{u\}$, is a minimum liar's dominating set of T' not containing a. Now $L''' = L'' \setminus \{b,c,d\}$ is clearly a liar's dominating set of T of size $|L'| - 3$. Hence in all the cases $\gamma_{LR}(T) \leq \gamma_{LR}(T') - 3$.

Thus $\gamma_{LR}(T') = \gamma_{LR}(T) + 3$.

(b) Let $w \in V(T)$ be such that either (i) $d_T(u,w) \geq 3$, or (ii) $d_T(u,w) = 2$ and $|N_T(v)| > 2$.

Sufficiency: Suppose that $w \in \mathcal{A}_{LR}(T')$. If possible suppose that $w \notin \mathcal{A}_{LR}(T)$. So there exists a minimum liar's dominating set, say L, of T not containing w. Let $L' = L \cup \{b,c,d\}$. Clearly L' is a minimum liar's dominating set of T' not containing w. This is a contradiction to the fact that $w \in \mathcal{A}_{LR}(T')$. So, $w \in \mathcal{A}_{LR}(T)$.

Necessity: Let $w \in \mathcal{A}_{LR}(T)$. If possible suppose that $w \notin \mathcal{A}_{LR}(T')$. So there exists a minimum liar's dominating set, say L', of T' not containing w. By proposition 2.3(b) $b, c, d \in L'$. If $a \notin L'$, then $L = L' \setminus \{b,c,d\}$ is a minimum liar's dominating set of T not containing w. This is a contradiction to the fact that $w \in \mathcal{A}_{LR}(T)$. So assume that $a \in L'$. Since $deg_{T'}(u) = 2$, L' contains at least one of u and v. If exactly one of u and v belongs to L', then replace a in L' with u or v whichever is not present in L' to get the set L''. Clearly L'' is a minimum liar's dominating set of T' not containing a and $L''' = L'' \setminus \{b,c,d\}$ is a minimum liar's dominating set of T not containing w. This is a contradiction to that fact that $w \in \mathcal{A}_{LR}(T)$. So assume that L' contains both u and v. Note that if there exists a vertex $x \in N_T(v) \setminus \{u\}$ such that $x \in L'$, then deleting a from L' gives rise to a new liar's dominating set of T' of cardinality less than $|L'|$. This contradicts the minimality of L'. So assume that $(N_T(v) \setminus \{u\}) \cap L' = \emptyset$. Let $L'' = L' \setminus \{a\} \cup \{y\}$, where $y \in N_T(v) \setminus \{u, w\}$ if $d_T(u, w) = 2$ and $y \in N_T(v) \setminus \{u\}$ if $d_T(u, w) \geq 3$. Now L'' is a minimum liar's dominating set of T' not containing w. Let $L''' = L'' \setminus \{b, c, d\}$. Clearly L''' is a minimum liar's dominating set of T not containing w. This is a contradiction to the fact that $w \in \mathcal{A}_{LR}(T)$. Hence $w \in \mathcal{A}_{LR}(T')$.

(c) Let $w \in V(T)$ be such that either (i) $d_T(u, w) \geq 3$, or (ii) $d_T(u, w) = 2$ and $|N_T(v)| > 2$.

Sufficiency: Suppose that $w \in \mathcal{N}_{LR}(T')$. If possible suppose that $w \notin \mathcal{N}_{LR}(T)$. So there exists a minimum liar's dominating set, say L, of T containing w. Let $L' = L \cup \{b, c, d\}$. Clearly L' is a minimum liar's dominating set of T' containing w. This is a contradiction to the fact that $w \in \mathcal{N}_{LR}(T')$. So, $w \in \mathcal{N}_{LR}(T)$.

Necessity: Let $w \in \mathcal{N}_{LR}(T)$. If possible suppose that $w \notin \mathcal{N}_{LR}(T')$. So there exists a minimum liar's dominating set, say L', of T' containing w. By proposition 2.3(b) $b, c, d \in L'$. If $a \notin L'$, then $L = L' \setminus \{b, c, d\}$ is a minimum liar's dominating set of T containing w. This is a contradiction to that fact that $w \in \mathcal{N}_{LR}(T)$. So assume that $a \in L'$. Since $deg_{T'}(u) = 2$, L' contains at least one of u and v. If exactly one of u and v belongs to L', then replace a in L' with u or v whichever is not present in L' to get the set L''. Clearly L'' is a minimum liar's dominating set of T' not containing a and $L''' = L'' \setminus \{b, c, d\}$ is a minimum liar's dominating set of T containing w. This is a contradiction to that fact that $w \in \mathcal{N}_{LR}(T)$. So assume that L' contains both u and v. Note that if there exists a vertex $x \in N_T(v) \setminus \{u\}$ such that $x \in L'$, then deleting a from L' gives rise to a new liar's dominating set of T' of cardinality less than $|L'|$. This contradicts the minimality of L'. So if $d_T(u, w) = 2$, then L' cannot contain both u and v and if $d_T(u, w) \geq 3$, then assume that $(N_T(v) \setminus \{u\}) \cap L' = \emptyset$. Let $L'' = L' \setminus \{a\} \cup \{y\}$, where $y \in N_T(v) \setminus \{u\}$ if $d_T(u, w) \geq 3$. Now L'' is a minimum liar's dominating set of T' containing w. Let $L''' = L'' \setminus \{b, c, d\}$. Clearly L''' is a minimum liar's dominating set of T containing w. This is a contradiction to that fact $w \in \mathcal{N}_{LR}(T)$. Hence $w \in \mathcal{N}_{LR}(T')$. □

3 Characterization of $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ of a tree T

In this section, we characterize $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ of a tree T.

For a rooted tree T_v having root v, the sets $W^*(T_v)$ and $U^*(T_v)$ are defined by $W^*(T_v) = \{w \in C_{T_v}(v) | D_{T_v}(w) \cap B(T_v) = \emptyset, |P^3(w)| = 2$ and $P^0(w) \cup P^1(w) \cup P^2(w) = \emptyset\}$ and $U^*(T_v) = \{u \in C_{T_v}(v) | D_{T_v}(u) \cap B(T_v) = \emptyset, |P^1(u)| = 1, |P^0(u)| = 1$ and $P^2(u) \cup P^3(u) = \emptyset\}$.

We describe a class, $CT(v)$, of rooted trees having root v and a tree, denoted as T_v^*. This class of trees, and the tree T_v^*, will play an important role in our characterization.

Definition 3.1. The class $CT(v)$: A rooted tree $T_v \in CT(v)$ if T_v satisfies each of the following three conditions.

(*i*) T_v has at least three vertices.

(*ii*) The cardinality of $W^*(T_v)$ is at most one.

(*iii*) If $u \notin U^*(T_v) \cup W^*(T_v) \cup \{v\}$, then the subtree rooted at u is a path.

The tree T_v^* is a rooted tree having root v such that v has exactly two children, say w_1 and w_2, and $W^*(T_v^*) = \{w_1, w_2\}$.

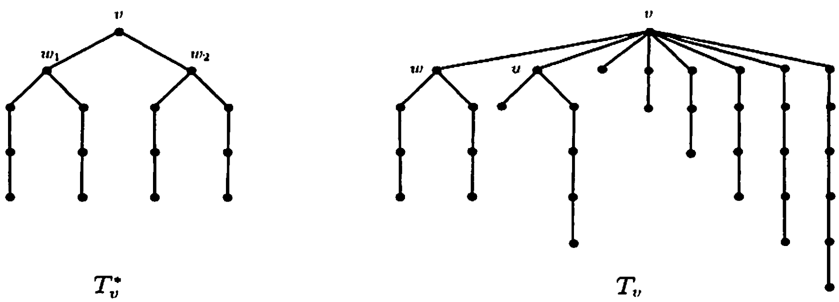

Figure 1: Tree T_v^* and T_v belonging to $CT(v)$.

The tree T_v in Figure 1 is a member of $CT(v)$. In the tree T_v of Figure 1, the nodes w and u are in $W^*(T_v)$ and $U^*(T_v)$, respectively. The tree T_v^* is shown in Figure 1.

3.1 Tree Pruning Technique

In this subsection, we describe a technique called *tree pruning* which will allow us to characterize the sets $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ of a tree T.

Let v be any vertex of T and let T_v be the rooted tree obtained from T by making v the root of T_v. The pruning is applied to a rooted tree T_v.

Definition 3.2. Tree Pruning at u:

Let u be a branch vertex of maximum distance from v in T_v such that $u \notin U^*(T_v) \cup W^*(T_v) \cup \{v\}$. Note that $|C_{T_v}(u)| \geq 2$ and $deg_{T_v}(x) \leq 2$ for $x \in D_{T_v}(u)$.

The **pruning of T_v at u** results in a rooted tree T_v' which is obtained from T_v as follows:

- If $|P^2(u)| \geq 1$, then delete $D_{T_v}(u)$ from T_v and attach a P_2 to u.
- If $|P^1(u)| \geq 2$ and $|P^2(u)| = 0$, then delete $D_{T_v}(u)$ from T_v and a attach P_2 to u.
- If $|P^1(u)| = 1$, $|P^2(u)| = 0$ and $|P^3(u)| \geq 1$, then delete $D_{T_v}(u)$ from T_v and attach a P_2 to u.
- If $|P^1(u)| = 1$, $|P^2(u)| = 0$ and $|P^3(u)| = 0$, then
  - If $|P^0(u)| \geq 2$ and $u \in C_{T_v}(v)$, then delete $D_{T_v}(u)$ from T_v and attach one P_1 and one P_4 to u.
  - If $|P^0(u)| \geq 1$ and $d_{T_v}(v,u) \geq 2$, then delete $D_{T_v}(u)$ from T_v and attach a P_1 to u.
- If $|P^1(u)| = 0$, $|P^2(u)| = 0$, $|P^3(u)| \geq 1$ and $|P^0(u)| \geq 1$, then delete $D_{T_v}(u)$ from T_v and attach a P_2 to u.
- If $|P^1(u)| = 0$, $|P^2(u)| = 0$ and $|P^0(u)| = 0$, then
  - If $|P^3(u)| \geq 3$ and $u \in C_{T_v}(v)$, then delete $D_{T_v}(u)$ from T_v and attach two P_3s to u.
  - If $|P^3(u)| \geq 2$ and $d_{T_v}(v,u) = 2$ and $p(u) \notin B(T_v)$, then delete $D_{T_v}(u)$ from T_v and attach a P_3 to u.
  - If $|P^3(u)| \geq 2$ and either $d_{T_v}(v,u) \geq 3$ or $p(u) \in B(T_v) \setminus \{v\}$, then delete $D_{T_v}[u]$ from T_v.
- If $|P^1(u)| = 0$, $|P^2(u)| = 0$, $|P^3(u)| = 0$ and $|P^0(u)| \geq 2$, then delete $D_{T_v}(u)$ from T_v and attach a P_4 to u.

Definition 3.3. Tree Pruning:
Repeat the **tree pruning** to T_v until a tree $\bar{T}_v$ is obtained so that for all $u \notin U^*(\bar{T}_v) \cup W^*(\bar{T}_v) \cup \{v\}$ $deg_{\bar{T}_v}(u) \leq 2$. The tree $\bar{T}_v$ is called the **pruning of** T_v.

We illustrate the tree pruning technique with the help of an example. Consider the rooted tree T_v^1 of Figure 2(a). The vertices p, q, r, w_1, w_1 and u are the branch vertices of T_v^1. The branch vertex p is at maximum distance from v and since $|P^1(p)| = 1$, $|P^2(p)| = |P^3(p)| = 0$, $|P^0(p)| \geq 1$ and $d_{T_v^1}(v,p) \geq 2$, we delete $D_{T_v^1}(p)$ and attach a path P_1 to p to obtain the tree T_v^2 which is shown in Figure 2(b). Now the only branch vertices of T_v^2 are q, r, w_1, w_2 and u and among these branch vertices q is the branch vertex at maximum distance from v. Since $|P^1(q)| = |P^2(q)| = 0$, $|P^3(q)| \geq 1$ and $|P^0(q)| \geq 1$, we delete $D_{T_v^2}(q)$ and attach a path P_2 to q to obtain the tree T_v^3 which is shown in Figure 2(c). The only branch vertices of T_v^3 are r, w_1, w_2 and u. Since r is the branch vertex at maximum distance from v and $|P^1(r)| \geq 2$ and $|P^2(r)| = 0$, we delete $D_{T_v^3}(r)$ and attach a path P_2 to r to obtain the tree T_v^4 which is shown in Figure 2(d). The only branch vertices of T_v^4 are w_1, w_2 and u. Since $w_1, w_2 \in W^*(T_v^4)$ and $u \in U^*(T_v^4)$, the pruning process can not be applied further. The pruning of T_v^1 is T_v^4 which is shown in Figure 2(d). So $\bar{T}_v^1 = T_v^4$.

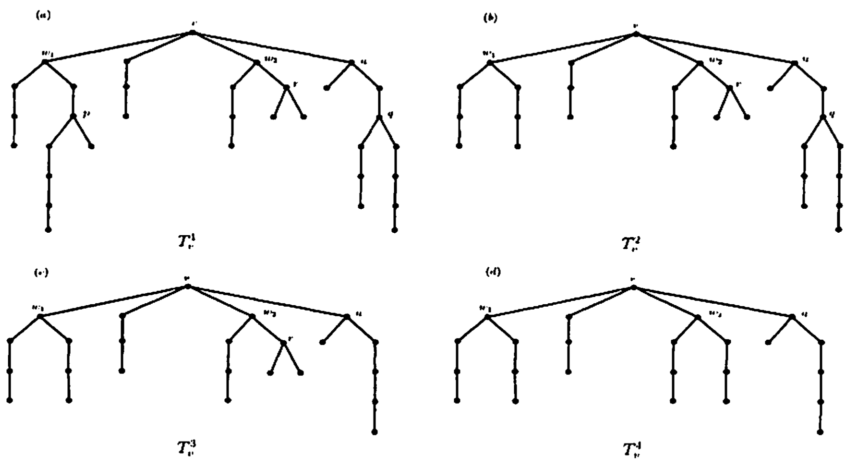

Figure 2: The pruning of the tree T_v^1 rooted at v.

The following lemma shows that the pruning of T_v at u preserves the property of v being in $\mathcal{A}_{LR}(T_v)$ (respectively, in $\mathcal{N}_{LR}(T_v)$).

Lemma 3.4. *Let T_v be a tree rooted at v and let u be a branch vertex of maximum distance from v in T_v such that $u \notin U^*(T_v) \cup W^*(T_v) \cup \{v\}$. Let $k_1 = |P^1(u)|$, $k_2 = |P^2(u)|$, $k_3 = |P^3(u)|$, and $k_4 = |P^0(u)|$. Suppose T_v' is the tree obtained from T_v as follows.*

(1) If $k_2 \geq 1$, then T_v' is obtained from T_v by deleting $D_{T_v}(u)$ and attaching a P_2 to u.

(2) If $k_1 \geq 2$ and $k_2 = 0$, then T_v' is obtained from T_v by deleting $D_{T_v}(u)$ and attaching a P_2 to u.

(3) If $k_1 = 1$, $k_2 = 0$ and $k_3 \geq 1$, then T_v' is obtained from T_v by deleting $D_{T_v}(u)$ and attaching a P_2 to u.

(4) If $k_1 = 1$, $k_2 + k_3 = 0$, $k_4 \geq 2$ and $u \in C_{T_v}(v)$, then T_v' is obtained from T_v by deleting $D_{T_v}(u)$ and attaching a P_1 and a P_4 to u.

(5) If $k_1 = 1$, $k_2 + k_3 = 0$, $k_4 \geq 1$ and $d_{T_v}(v, u) \geq 2$, then T_v' is obtained from T_v by deleting $D_{T_v}(u)$ and attaching a P_1 to u.

(6) If $k_1 + k_2 = 0$, $k_3 \geq 1$ and $k_4 \geq 1$, then T_v' is obtained from T_v by deleting $D_{T_v}(u)$ and attaching a P_2 to u.

(7) If $k_1 + k_2 + k_4 = 0$, $k_3 \geq 3$ and $u \in C_{T_v}(v)$, then T_v' is obtained from T_v by deleting $D_{T_v}(u)$ and attaching two P_3s to u.

(8) *If* $k_1+k_2+k_4=0$, $k_3\geq 2$, $d_{T_v}(v,u)=2$ *and* $p(u)\notin B(T_v)$, *then* T'_v *is obtained from* T_v *by deleting* $D_{T_v}(u)$ *and attaching a* P_3 *to* u.

(9) *If* $k_1+k_2+k_4=0$, $k_3\geq 2$ *and either* $d_{T_v}(v,u)\geq 3$ *or* $p(u)\in B(T_v)\setminus\{v\}$, *then* T'_v *is obtained from* T_v *by deleting* $D_{T_v}[u]$.

(10) *If* $k_1+k_2+k_3=0$ *and* $k_4\geq 2$, *then* T'_v *is obtained from* T_v *by deleting* $D_{T_v}(u)$ *and attaching a* P_4 *to* u.

Then the following statements are true.

(a) $v\in\mathcal{A}_{LR}(T_v)$ *if and only if* $v\in\mathcal{A}_{LR}(T'_v)$.

(b) $v\in\mathcal{N}_{LR}(T_v)$ *if and only if* $v\in\mathcal{N}_{LR}(T'_v)$.

Proof. Let u_1 be a child of u in T_v and let $x\in D_{T_v}(u_1)$ be a leaf vertex of T_v. If $d_{T_v}(u_1,x)\geq 4$, then let x_1,x_2, and x_3 be vertices in T_v such that $x_1=p(x), x_2=p(x_1)$, and $x_3=p(x_2)$. Let T_v^1 be the tree obtained from T_v by removing x,x_1,x_2, and x_3 from T_v. By Lemma 2.5, $v\in\mathcal{A}_{LR}(T_v)$ (respectively $v\in\mathcal{N}_{LR}(T_v)$) if and only if $v\in\mathcal{A}_{LR}(T_v^1)$ (respectively $v\in\mathcal{N}_{LR}(T_v^1)$). By repeating this process we can obtain a tree T_v^2 such that $d_{T_v^2}(u,x)\leq 4$ for all leaves x of T_v^2 which are descendant of u. So without loss of generality assume that $d_{T_v}(u,x)\leq 4$ for all leaves x of T_v which are descendant of u.

Let $P^1(u)=\{a_1,a_2,\ldots,a_{k_1}\}$, $P^2(u)=\{u_1,u_2,\ldots,u_{k_2}\}$, $P^3(u)=\{z_1,z_2,\ldots,z_{k_3}\}$ and $P^0(u)=\{s_1,s_2,\ldots,s_{k_4}\}$. Let $p(u_i)=t_i, 1\leq i\leq k_2$, $p(z_i)=y_i$, and $p(y_i)=x_i$, $1\leq i\leq k_3$, $p(s_i)=r_i$, $p(r_i)=q_i$, and $p(q_i)=p_i$, $1\leq i\leq k_4$.

Case 1: $k_2\geq 1$.

Let $X=(\cup_{i=1}^{i=k_1}\{a_i\})\cup(\cup_{j=2}^{j=k_2}\{t_j,u_j\})\cup(\cup_{k=1}^{k=k_3}\{x_k,y_k,z_k\})\cup(\cup_{l=1}^{l=k_4}\{q_l,r_l,s_l\})$. By Proposition 2.3, every minimum liar's dominating set of T_v contains X. Let $T'_v=T_v\setminus(D_{T_v}(u)\setminus\{t_1,u_1\})$. Let L' be a minimum liar's dominating set of T'_v. Clearly $L=L'\cup X$ is a liar's dominating set of T_v. Hence, $\gamma_{LR}(T_v)\leq\gamma_{LR}(T'_v)+k_1+2(k_2-1)+3k_3+3k_4$. On the other hand, let L be a minimum liar's dominating set of T_v. So $X\subset L$. Clearly $L'=L\setminus X$ is a liar's dominating set of T'_v. So $\gamma_{LR}(T'_v)\leq\gamma_{LR}(T_v)-k_1-2(k_2-1)-3k_3-3k_4$. Thus $\gamma_{LR}(T_v)=\gamma_{LR}(T'_v)+k_1+2(k_2-1)+3k_3+3k_4$. Hence L is a minimum liar's dominating set of T_v if and only if $L'=L\setminus X$ is a minimum liar's dominating set of T'_v.

(a) Suppose $v\in\mathcal{A}_{LR}(T'_v)$. Let L be an arbitrary minimum liar's dominating set of T_v. As seen above, $L'=L\setminus X$ is a minimum liar's dominating set of T'_v and so $v\in L'$. As $L'\subset L$, $v\in L$. Hence, $v\in\mathcal{A}_{LR}(T_v)$.

Conversely, suppose that $v\in\mathcal{A}_{LR}(T_v)$. Let L' be an arbitrary minimum liar's dominating set of T'_v. As seen above $L=L'\cup X$ is a minimum liar's dominating set of T_v and so $v \in L$. Since $v \notin X$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

(b) Suppose that $v \in \mathcal{N}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen above, $L' = L \setminus X$ is a minimum liar's dominating set of T'_v and so $v \notin L'$. Since $v \notin X$, $v \notin L$. Hence, $v \in \mathcal{N}_{LR}(T_v)$.

Conversely, suppose that $v \in \mathcal{N}_{LR}(T_v)$ and L' is an arbitrary minimum liar's dominating set of T'_v. Then $L = L' \cup X$ is a minimum liar's dominating set of T_v and so $v \notin L$. This implies that $v \notin L'$. Hence, $v \in \mathcal{N}_{LR}(T'_v)$.

Hence $v \in \mathcal{N}_{LR}(T_v)$ if and only if $v \in \mathcal{N}_{LR}(T'_v)$ in this case.

As the proof of item (b) is similar to item (a) as seen in **Case** 1, the proof of item (b) is omitted in the remaining cases.

Case 2: $k_1 \geq 2$ and $k_2 = 0$.

Let $X = (\cup_{i=1}^{i=k_1}\{a_i\}) \cup (\cup_{k=1}^{k=k_3}\{x_k, y_k, z_k\}) \cup (\cup_{l=1}^{l=k_4}\{q_l, r_l, s_l\})$. By Proposition 2.3, every minimum liar's dominating set of T_v contains X. Let T'_v be the tree obtained from T_v by deleting $D_{T_v}(u)$ and attaching a path $P_2 = xy$ to u so that y becomes a pendent vertex of T'_v. By Proposition 2.3, every minimum liar's dominating set of T'_v contains $\{x, y\}$. Let L' be a minimum liar's dominating set of T'_v. Clearly, $L = (L' \setminus \{x, y\}) \cup X$ is a liar's dominating set of T_v. Hence, $\gamma_{LR}(T_v) \leq \gamma_{LR}(T'_v) + k_1 + 3k_3 + 3k_4 - 2$. On the other hand, let L be a minimum liar's dominating set of T_v. Clearly, $L' = (L \setminus X) \cup \{x, y\}$ is a liar's dominating set of T'_v. Thus, $\gamma_{LR}(T'_v) \leq \gamma_{LR}(T_v) - k_1 - 3k_3 - 3k_4 + 2$. So $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + k_1 + 3k_3 + 3k_4 - 2$. Hence L is a minimum liar's dominating set of T_v if and only if $L' = (L \setminus X) \cup \{x, y\}$ is a minimum liar's dominating set of T'_v.

(a) Suppose that $v \in \mathcal{A}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen above, $L' = (L \setminus X) \cup \{x, y\}$ is a minimum liar's dominating set of T'_v and so $v \in L'$. Since $v \notin \{x, y\}$, $v \in L$. Hence, $v \in \mathcal{A}_{LR}(T)$.

Conversely, suppose that $v \in \mathcal{A}_{LR}(T_v)$ and L' is an arbitrary minimum liar's dominating set of T'_v. Then $L = (L' \cup X) \setminus \{x, y\}$ is a minimum liar's dominating set of T_v and so $v \in L$. Now since $v \notin X$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

Case 3: $k_1 = 1$ and $k_2 = 0$.

(i) $k_3 \geq 1$.

Let $X = \{a_1\} \cup \{z_1\} \cup (\cup_{k=2}^{k=k_3}\{x_k, y_k, z_k\}) \cup (\cup_{l=1}^{l=k_4}\{q_l, r_l, s_l\})$. By Proposition 2.3, every minimum liar's dominating set of T_v contains X. Let $T'_v = T_v \setminus (D_{T_v}(u) \setminus \{x_1, y_1\})$. Let L' be a minimum liar's dominating set of T'_v. Clearly $L = L' \cup X$ is a liar's dominating set of T_v. Hence, $\gamma_{LR}(T_v) \leq \gamma_{LR}(T'_v) + 3(k_3 - 1) + 3k_4 + 2$. On the other hand, let L be a minimum liar's dominating set of T_v. Clearly, $L' = L \setminus X$ is a liar's dominating set of T'_v. Thus, $\gamma_{LR}(T'_v) \leq \gamma_{LR}(T_v) - 3(k_3 - 1) - 3k_4 - 2$. So $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + 3(k_3 - 1) + 3k_4 + 2$. Hence L is a minimum liar's dominating set of T_v if and only if $L' = L \setminus X$ is a minimum liar's dominating set of T'_v.

(a) Suppose $v \in \mathcal{A}_{LR}(T'_v)$ and L be an arbitrary minimum liar's dominating set of T_v. As seen before, $L' = L \setminus X$ is a minimum liar's dominating set of T'_v and so $v \in L' \subset L$. Hence, $v \in \mathcal{A}_{LR}(T_v)$.

Conversely, suppose that $v \in \mathcal{A}_{LR}(T_v)$ and L' is an arbitrary minimum liar's dominating set of T'_v. Then $L = L' \cup X$ is a minimum liar's dominating set of T_v and so $v \in L$. Now since $v \notin X$, $v \in L'$. Hence, $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

(ii) $k_3 = 0$, $k_4 \geq 2$ and $u \in C_{T_v}(v)$.

Let $X = (\cup_{l=2}^{l=k_4}\{q_l, r_l, s_l\})$. By Proposition 2.3, every minimum liar's dominating set of T_v contains X. Let $T'_v = T_v \setminus (D_{T_v}(u) \setminus \{a_1, p_1, q_1, r_1, s_1\})$. Let L' be a minimum liar's dominating set of T'_v. Clearly, $L = L' \cup X$ is a liar's dominating set of T_v. Hence, $\gamma_{LR}(T_v) \leq \gamma_{LR}(T'_v) + 3(k_4 - 1)$. On the other hand, let L be a minimum liar's dominating set of T_v. By Proposition 2.3, $(\cup_{l=1}^{l=k_4}\{q_l, r_l, s_l\}) \subset L$. Note that, at most one of p_i, $1 \leq i \leq k_4$ can belong to L because otherwise by deleting p_t form L for some $1 \leq t \leq k_4$ such that $p_t \in L$ we get a new liar's dominating set whose cardinality is less than $|L|$. Now if $\{p_l | 1 \leq l \leq k_4\} \cap L = \emptyset$, then $L' = L \setminus X$ is a liar's dominating set of T'_v. So, without loss of generality, let p_1 be the only vertex such that $p_1 \in \{p_l | 1 \leq l \leq k_4\} \cap L$. In this case also $L' = L \setminus X$ is a liar's dominating set of T'_v. So $\gamma_{LR}(T'_v) \leq \gamma_{LR}(T_v) - 3(k_4 - 1)$. Thus $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + 3(k_4 - 1)$. Hence L is a minimum liar's dominating set of T_v if and only if $L' = L \setminus X$ is a minimum liar's dominating set of T'_v.

(a) Suppose that $v \in \mathcal{A}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen before, $L' = L \setminus X$ is a minimum liar's dominating set of T'_v and so $v \in L' \subset L$. Hence $v \in \mathcal{A}_{LR}(T_v)$.

Conversely, suppose that $v \in \mathcal{A}_{LR}(T_v)$ and L' is an arbitrary minimum liar's dominating set of T'_v. Then $L = L' \cup X$ is a minimum liar's dominating set of T_v and so $v \in L$. Now since $v \notin X$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

(iii) $k_3 = 0$, $k_4 \geq 1$ and $d_{T_v}(v, u) \geq 2$.

Let $X = (\cup_{l=1}^{l=k_4}\{q_l, r_l, s_l\})$. By Proposition 2.3, every minimum liar's dominating set of T_v contains X. Let $T' = T \setminus (D_{T_v}(u) \setminus \{a_1\})$. Let L' be a minimum liar's dominating set of T'_v. Clearly, $a_1, u, p(u) \in L'$ and $L = L' \cup X$ is a liar's dominating set of T_v. Hence, $\gamma_{LR}(T_v) \leq \gamma_{LR}(T'_v) + 3k_4$. On the other hand, let L be a minimum liar's dominating set of T_v. First note that, at most one of $\{p_l | 1 \leq l \leq k_4\}$ can be in L (say p_1). If $p_1 \notin L$, then clearly $L' = L \setminus X$ is a liar's dominating set of T'_v. Now, if $p_1, p(u) \in L$, then L is not a minimum liar's dominating set. So if $p_1 \in L$, then $p(u) \notin L$. Hence, clearly $L'' = L \setminus (X \cup \{p_1\}) \cup \{p(u)\}$ is a liar's dominating set of T'_v.

In any case, $\gamma_{LR}(T') \le \gamma_{LR}(T) - 3k_4$. Hence $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + 3k_4$.

(a) Suppose that $v \in \mathcal{A}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen above, depending on L, either L' or L'' is a minimum liar's dominating set of T'_v. Since $v \in L'$ and $v \in L''$ and $v \neq p(u)$, $v \in L$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Conversely, let $v \in \mathcal{A}_{LR}(T_v)$ and L' be an arbitrary minimum liar's dominating set of T'_v. Clearly $L = L' \cup X$ is a minimum liar's dominating set of T_v. Since $v \in L$ and $v \notin X$, $v \in L'$. So $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

Case 4. $k_1 + k_2 = 0$.

(i) $k_3 \ge 1$ and $k_4 \ge 1$.

Let $X = \{z_1\} \cup (\cup_{k=2}^{k=k_3}\{x_k, y_k, z_k\}) \cup (\cup_{l=1}^{l=k_4}\{q_l, r_l, s_l\})$. By Proposition 2.3, every minimum liar's dominating set of T_v contains X. Let $T'_v = T_v \setminus (D_{T_v}(u) \setminus \{x_1, y_1\})$. Let L' be a minimum liar's dominating set of T'_v. Clearly, $L = L' \cup X$ is a liar's dominating set of T_v. Hence $\gamma_{LR}(T_v) \le \gamma_{LR}(T'_v) + 3(k_3 - 1) + 3k_4 + 1$. On the other hand, let L be a minimum liar's dominating set of T_v. If $u \in L$, then clearly $L' = L \setminus X$ is a liar's dominating set of T'_v. If $u \notin L$, then by minimality we have $k_4 = 1$, and so $p_1 \in L$. In that case, $L'' = L \setminus (X \cup \{p_1\}) \cup \{u\}$ is a liar's dominating set of T'_v. Thus $\gamma_{LR}(T'_v) \le \gamma_{LR}(T'_v) - 3(k_3 - 1) - 3k_4 - 1$. Hence $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + 3(k_3 - 1) + 3k_4 + 1$.

(a) Suppose that $v \in \mathcal{A}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen above, depending on L, either L' or L'' is a minimum liar's dominating set of T'_v. Since $v \in L'$, $v \in L''$, and $v \neq u$, $v \in L$. Hence $v \in \mathcal{A}_{LR}(T_v)$.

Conversely, let $v \in \mathcal{A}_{LR}(T_v)$ and L' be an arbitrary minimum liar's dominating set of T'_v. Clearly $L = L' \cup X$ is a minimum liar's dominating set of T_v. Now, since $v \in L$ and $v \notin X$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

(ii) $k_3 \ge 3$, $k_4 = 0$ and $u \in C_{T_v}(v)$.

Let $X = (\cup_{k=3}^{k=k_3}\{x_k, y_k, z_k\})$. By Proposition 2.3, every minimum liar's dominating set of T_v contains X. Let $T'_v = T_v \setminus (D_{T_v}(u) \setminus \{x_1, y_1, z_1, x_2, y_2, z_2\})$. Let L' be a minimum liar's dominating set of T'_v. Clearly, $L = L' \cup X$ is a liar's dominating set of T_v. Hence, $\gamma_{LR}(T_v) \le \gamma_{LR}(T'_v) + 3(k_3 - 2)$. On the other hand, let L be a minimum liar's dominating set of T_v. Clearly $L' = L \setminus X$ is a liar's dominating set of T'_v. Thus, $\gamma_{LR}(T'_v) \le \gamma_{LR}(T_v) - 3(k_3 - 2)$. Hence, $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + 3(k_3 - 2)$.

(a) Suppose that $v \in \mathcal{A}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen before, $L' = L \setminus X$ is a minimum liar's dominating set of T'_v and so $v \in L' \subset L$. Hence $v \in \mathcal{A}_{LR}(T_v)$.

Conversely, suppose that, $v \in \mathcal{A}_{LR}(T_v)$ and L' be an arbitrary minimum liar's dominating set of T'_v. Then $L = L' \cup X$ is a minimum liar's dominating set of T_v and so $v \in L$. Now since $v \notin X$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

(iii) $k_3 \geq 2$, $k_4 = 0$, $d_{T_v}(v, u) = 2$ and $p(u) \notin B(T_v)$.

Let $X = (\cup_{k=2}^{k=k_3}\{x_k, y_k, z_k\})$. By Proposition 2.3, every minimum liar's dominating set of T_v contains X. Let $T'_v = T_v \setminus (D_{T_v}(u) \setminus \{x_1, y_1, z_1\})$. Let L' be a minimum liar's dominating set of T'_v. Clearly $L = L' \cup X$ is a liar's dominating set of T_v. Hence $\gamma_{LR}(T_v) \leq \gamma_{LR}(T'_v) + 3(k_3 - 1)$. On the other hand, let L be a minimum liar's dominating set of T_v. If $u \in L$, then clearly $L' = L \setminus X$ is a liar's dominating set of T'_v. And if $u \notin L$, then $p(u) \in L$. So, in this case also, L' is a liar's dominating set of T'_v. Thus $\gamma_{LR}(T'_v) \leq \gamma_{LR}(T_v) - 3(k_3 - 1)$. Hence $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + 3(k_3 - 1)$.

(a) Suppose that $v \in \mathcal{A}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen above, $L' = L \setminus X$ is a minimum liar's dominating set of T'_v. So $v \in L' \subset L$. Hence $v \in \mathcal{A}_{LR}(T_v)$.

Conversely, let $v \in \mathcal{A}_{LR}(T_v)$ and L' be an arbitrary minimum liar's dominating set of T'_v. Clearly $L = L' \cup X$ is a minimum liar's dominating set of T_v. Since $v \in L$ and $v \notin X$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

(iv) $k_3 \geq 2$, $k_4 = 0$ and either $d_{T_v}(v, u) \geq 3$ or $p(u) \in B(T_v) \setminus \{v\}$.

Let $T' = T \setminus D_{T_v}[u]$. Let L' be a minimum liar's dominating set of T'_v. Let $X = (\cup_{k=1}^{k=k_3}\{x_k, y_k, z_k\})$. Clearly $L = L' \cup X$ is a liar's dominating set of T_v. Hence $\gamma_{LR}(T_v) \leq \gamma_{LR}(T'_v) + 3k_3$. On the other hand, let L be a minimum liar's dominating set of T_v. If $u \notin L$, then $L' = L \setminus X$ is a liar's dominating set of T'_v. Now assume that $u \in L$. If $p(u) \notin L$, then $L'' = L \setminus (X \cup \{u\}) \cup \{p(u)\}$ is a liar's dominating set of T'_v. If $p(u) \in L$, then by minimality of L, $L''' = L \setminus (X \cup \{u\}) \cup \{x\}$ is a liar's dominating set of T'_v where $x \in N(p(u)) \setminus \{u\}$ and hence $\gamma_{LR}(T'_v) \leq \gamma_{LR}(T_v) - 3k_3$. Hence $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + 3k_3$.

(a) Suppose that $v \in \mathcal{A}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen above, depending on L, one of L', L'' and L''' is a minimum liar's dominating set of T'_v. Since $v \in L', v \in L'', v \in L'''$, and $v \notin \{p(u), x\}$, $v \in L$. Hence $v \in \mathcal{A}_{LR}(T_v)$.

Conversely, let $v \in \mathcal{A}_{LR}(T_v)$ and L' be an arbitrary minimum liar's dominating set of T'_v. Clearly, $L = L' \cup X$ is a minimum liar's dominating set of T_v. Since $v \in L$ and $v \notin X$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case.

(v) $k_3 = 0$ and $k_4 \geq 2$.

Let $T'_v = T_v \setminus (D_{T_v}(u) \setminus \{p_1, q_1, r_1, s_1\})$. Let L' be a minimum liar's dominating set of T'_v. Let $X = (\cup_{l=2}^{l=k_4}\{q_l, r_l, s_l\})$. If $u \in L'$, then $L_1 = L' \backslash X$ is a liar's dominating set of T_v. If $u \notin L$, $p_1, p(u) \in L'$ and then $L_2 = L' \cup (X \cup \{u\}) \backslash \{p_1\}$ is a liar's dominating set of T_v. Hence $\gamma_{LR}(T_v) \leq \gamma_{LR}(T'_v) + 3(k_4 - 1)$. On the other hand, let L be a minimum liar's dominating set of T_v. If $u \in L$, then without loss of generality, $p_l \notin L$ for $l \in \{2, \ldots, k_4\}$ and so $L' = L \setminus X$ is a liar's dominating set os T'_v. And if $u \notin L$, then by the minimality of L, $k_4 = 2$ and so $p_1, p_2 \in L$. Now $L'' = L \backslash \{p_2, q_2, r_2, s_2\} \cup \{u\}$ is a liar's dominating set of T'_v of cardinality $\gamma_{LR}(T_v) - 3(k_4 - 1)$ where $k_4 = 2$. It follows that $\gamma_{LR}(T'_v) \leq \gamma_{LR}(T_v) - 3(k_4 - 1)$. Hence $\gamma_{LR}(T_v) = \gamma_{LR}(T'_v) + 3(k_4 - 1)$.

(a) Suppose that $v \in \mathcal{A}_{LR}(T'_v)$ and L is an arbitrary minimum liar's dominating set of T_v. As seen above, depending on L, one of L' and L'' is a minimum liar's dominating set of T'_v. Since $v \in L', v \in L''$ and $v \neq u$, $v \in L$. Hence $v \in \mathcal{A}_{LR}(T_v)$.

Conversely, let $v \in \mathcal{A}_{LR}(T_v)$ and L' be an arbitrary minimum liar's dominating set of T'_v. Again as above, depending on L', one of L_1 and L_2 is a minimum liar's dominating set of T_v. Since $v \in L_1, L_2$ and $v \notin X \cup \{u\}$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(T'_v)$.

Hence $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(T'_v)$ in this case. □

Since $\bar{T}_v$ is obtained from T_v by applying the pruning process a finite number of times, the following corollary follows from Lemma 3.4.

Corollary 3.5. *Let T_v be a tree rooted at v and $\bar{T}_v$ be the pruning of T_v. Then*

(a) $v \in \mathcal{A}_{LR}(T_v)$ if and only if $v \in \mathcal{A}_{LR}(\bar{T}_v)$.

(b) $v \in \mathcal{N}_{LR}(T_v)$ if and only if $v \in \mathcal{N}_{LR}(\bar{T}_v)$.

3.2 The Tree Reduction Technique

If neither $\bar{T}_v = T^*_v$ nor $\bar{T}_v \in CT(v)$, then we apply reduction technique which will produce a tree $\tilde{T}_v$ rooted at v such that either $\tilde{T}_v = T^*_v$ or $\tilde{T}_v \in CT(v)$. The tree $\tilde{T}_v$ is called the **reduction of $\bar{T}_v$**.

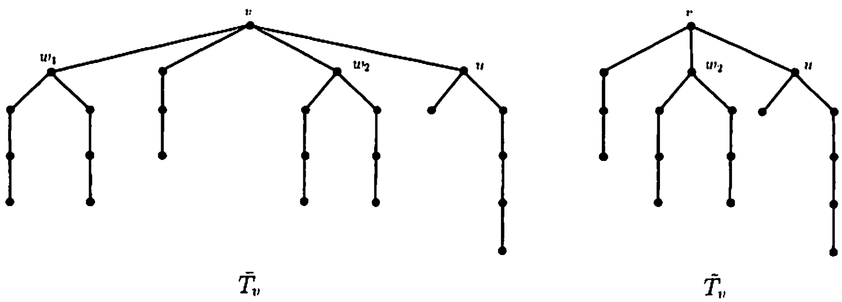

Figure 3: The reduction of $\bar{T}_v$.

In Figure 3, the tree $\bar{T}_v$ rooted at v contains $w_1, w_2 \in W^*(\bar{T}_v)$. Hence we reduce $\bar{T}_v$ to obtain a tree $\tilde{T}_v$ rooted at v which contains only one vertex $w_2 \in W^*(\tilde{T}_v)$.

Lemma 3.6. *Suppose that neither $\bar{T}_v = T_v^*$ nor $\bar{T}_v \in CT(v)$ but $deg_{\bar{T}_v}(u) \leq 2$ for all $u \notin U^*(\bar{T}_v) \cup W^*(\bar{T}_v) \cup \{v\}$. If $C_{\bar{T}_v}(v) \setminus W^*(\bar{T}_v) \neq \emptyset$, then let $S = W^*(\bar{T}_v) \setminus \{w_1\}$, where $w_1 \in W^*(\bar{T}_v)$. If $C_{\bar{T}_v}(v) \setminus W^*(\bar{T}_v) = \emptyset$, then let $S = W^*(\bar{T}_v) \setminus \{w_1, w_2\}$, where $w_1, w_2 \in W^*(\bar{T}_v)$. Let $\tilde{T}_v = \bar{T}_v[V(\bar{T}_v) \setminus D_{\bar{T}_v}[S]]$. Then $v \in \mathcal{A}_{LR}(\bar{T}_v)$ (respectively $v \in \mathcal{N}_{LR}(\bar{T}_v)$) if and only if $v \in \mathcal{A}_{LR}(\tilde{T}_v)$ (respectively $v \in \mathcal{N}_{LR}(\tilde{T}_v)$).*

Proof. By Lemma 2.5 we can assume without loss of generality that all the leaves of $\bar{T}_v$ in $D_{\bar{T}_v}(w)$ are at distance 3 for each $w \in W^*(\bar{T}_v)$.

Case 1: $C_{\bar{T}_v}(v) \setminus W^*(\bar{T}_v) \neq \emptyset$.

Let $X = \cup_{w \in S} D(w)$. Let L' be a minimum liar's dominating set of $\tilde{T}_v$. Clearly $L = L' \cup X$ is a liar's dominating set of $\bar{T}_v$. Hence $\gamma_{LR}(\bar{T}_v) \leq \gamma_{LR}(\tilde{T}_v) + |X|$. On the other hand, let L be a minimum liar's dominating set of $\bar{T}_v$. Note that at most two vertices from $W^*(\bar{T}_v)$ can belong to L since L is a minimum liar's dominating set. If $|W^*(\bar{T}_v) \cap L| \leq 1$, then clearly $L' = L \setminus X$ is a liar's dominating set of $\tilde{T}_v$. So assume that $|W^*(\bar{T}_v) \cap L| = 2$. Let $W^*(\bar{T}_v) \cap L = \{x_1, x_2\}$. If there exists $u \in C_{\bar{T}_v}(v) \setminus W^*(\bar{T}_v)$ such that $u \in L$, then $L \setminus \{x_2\}$ is a liar's dominating set of $\bar{T}_v$ of cardinality less than $|L|$ which is a contradiction to the minimality of L. So $(C_{\bar{T}_v}(v) \setminus W^*(\bar{T}_v)) \cap L = \emptyset$. Now $L'' = (L \setminus (X \cup \{x_2\})) \cup \{u\}$ is a liar's dominating set of $\tilde{T}_v$, where $u \in (C_{\bar{T}_v}(v) \setminus W^*(\bar{T}_v))$. So $\gamma_{LR}(\tilde{T}_v) \leq \gamma_{LR}(\bar{T}_v) - |X|$. Hence $\gamma_{LR}(\bar{T}_v) = \gamma_{LR}(\tilde{T}_v) + |X|$.

Let $v \in \mathcal{A}_{LR}(\tilde{T}_v)$ and L be an arbitrary minimum liar's dominating set of $\bar{T}_v$. As we have seen above, either L' or L'' is a minimum liar's dominating set of $\tilde{T}_v$. Since $v \in L'$, $v \in L''$, and $v \neq u$, $v \in L$. This implies that $v \in \mathcal{A}_{LR}(\bar{T}_v)$. On the other hand, let $v \in \mathcal{A}_{LR}(\bar{T}_v)$ and L' be an arbitrary minimum liar's dominating set of $\tilde{T}_v$. Then $L = L' \cup X$ is a minimum liar's dominating set of $\bar{T}_v$ and so $v \in L$. Now since $v \notin X$, $v \in L'$. Hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Thus $v \in \mathcal{A}_{LR}(\bar{T}_v)$ if and only if $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Let $v \in \mathcal{N}_{LR}(\tilde{T}_v)$ and L be an arbitrary minimum liar's dominating set of $\bar{T}_v$. As we have seen above, either L' or L'' is a minimum liar's dominating set of $\tilde{T}_v$. Now $v \notin L'$ and $v \notin L''$. Since neither $v = u$ nor $v \in X$, $v \notin L$. This implies that $v \in \mathcal{N}_{LR}(\bar{T}_v)$. On the other hand, let $v \in \mathcal{N}_{LR}(\bar{T}_v)$ and let L' be an arbitrary minimum liar's dominating set of $\tilde{T}_v$. Now $L = L' \cup X$ is a minimum liar's dominating set of $\bar{T}_v$ and so $v \notin L$. Since $L' \subset L$, $v \notin L'$. Hence $v \in \mathcal{N}_{LR}(\tilde{T}_v)$.

Thus $v \in \mathcal{N}_{LR}(\bar{T}_v)$ if and only if $v \in \mathcal{N}_{LR}(\tilde{T}_v)$.

Case 2: $C_{\bar{T}_v}(v) \setminus W^*(\bar{T}_v) = \emptyset$.

In this case $\tilde{T}_v = T_v^*$. Let $X = V(\bar{T}_v) \setminus (\{v\} \cup C_{\bar{T}_v}(v))$. By Proposition 2.3, $X \subseteq L$ for every minimum liar's dominating set L of $\bar{T}_v$. By definition of liar's dominating set, $|N_{\bar{T}_v}[v] \cap L| \geq 2$ for every minimum liar's dominating set L of $\bar{T}_v$. So $X \cup \{v, w_1\}$ is a minimum liar's dominating set of $\bar{T}_v$, where

$w_1 \in N_{\bar{T}_v}(v)$. Similarly $X \cup \{w_1, w_2\}$ is a minimum liar's dominating set of $\bar{T}_v$, where $w_1, w_2 \in N_{\bar{T}_v}(v)$. Hence $v \notin \mathcal{A}_{LR}(\bar{T}_v)$ and $v \notin \mathcal{N}_{LR}(\bar{T}_v)$. Using the similar arguments it can be shown that $v \notin \mathcal{A}_{LR}(\tilde{T}_v)$ and $v \notin \mathcal{N}_{LR}(\tilde{T}_v)$.

Thus $v \in \mathcal{A}_{LR}(\bar{T}_v)$ if and only if $v \in \mathcal{A}_{LR}(\tilde{T}_v)$ and $v \in \mathcal{N}_{LR}(\bar{T}_v)$ if and only if $v \in \mathcal{N}_{LR}(\tilde{T}_v)$. □

3.3 Characterization

Let T be a tree having at least three vertices. Let T_v be the rooted tree obtained from T by making a vertex v of T the root of T_v. By Corollary 3.5 and by Lemma 3.6, $v \in \mathcal{A}_{LR}(T_v)$(respectively $v \in \mathcal{N}_{LR}(T_v)$) if and only if $v \in \mathcal{A}_{LR}(\tilde{T}_v)$(respectively $v \in \mathcal{N}_{LR}(\tilde{T}_v)$). So we first find the characterization for $v \in \mathcal{A}_{LR}(\tilde{T}_v)$ and for $v \in \mathcal{N}_{LR}(\tilde{T}_v)$. We then characterize the sets $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ for a tree T.

Theorem 3.7. *Let T be a tree having at least three vertices and v be any vertex of T. Let $\bar{T}_v$ be the pruning of T_v and let $\tilde{T}_v$ be the reduction of $\bar{T}_v$. Then*

(a) $v \in \mathcal{A}_{LR}(\tilde{T}_v)$ if and only if at least one of the following conditions is satisfied:

(1) $deg_{\tilde{T}_v}(v) = 1$.

(2) $C_{\tilde{T}_v}(v)$ *contains a leaf of* $\tilde{T}_v$.

(3) $C_{\tilde{T}_v}(v)$ *contains a vertex x such that* $C_{\tilde{T}_v}(x) = \{y\}$ *and y is a leaf of* $\tilde{T}_v$.

(4) $|P^0(v)| \geq 3$.

(5) $|P^1(v)| \geq 2$.

(6) $|P^2(v)| \geq 2$.

(7) $|P^0(v)| = 2$ *and* $|P^3(v)| \geq 1$.

(8) $|P^0(v)| = 2$ *and* $|P^1(v)| = 1$.

(9) $|P^0(v)| \in \{1, 2\}$ *and* $|P^2(v)| = 1$.

(10) $|P^1(v)| = |P^2(v)| = 1$.

(11) $|P^0(v)| = |P^1(v)| = 1$ *and* $|P^3(v)| \geq 1$.

(12) $|U^*(\tilde{T}_v)| \geq 2$.

(13) $|U^*(\tilde{T}_v)| = 1$ *and* $|P^0(v) \cup P^1(v) \cup P^2(v)| \geq 1$.

(14) $|P^0(v) \cup P^1(v) \cup P^2(v) \cup P^3(v)| = 0$ *and* $|U^*(\tilde{T}_v)| = 1$ *and* $|W^*(\tilde{T}_v)| = 1$.

(15) $|P^2(v)| = 1$, $|W^*(\tilde{T}_v)| = 1$ *and* $|P^3(v)| = 0$.

(b) $v \in \mathcal{N}_{LR}(\tilde{T}_v)$ *if and only if* $|P^3(v)| \geq 2$ *and* $P^0(v) \cup P^1(v) \cup P^2(v) \cup U^*(\tilde{T}_v) = \emptyset$.

Proof. If any of the conditions (1), (2) and (3) is true, then by Proposition 2.3, $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$. So let us assume that v is neither a pendant vertex nor a support vertex nor there exists a pendant vertex y with $N_{\tilde{T}_v}(y) = \{x\}$ such that $d_{\tilde{T}_v}(y, v) = 2$ and $deg_{\tilde{T}_v}(x) = 2$. Let b be a leaf vertex of $\tilde{T}_v$. If $b \in P^i(v)$ for $0 \leq i \leq 2$, then replace the path from v to b in $\tilde{T}_v$ with a path from v to b of length $i + 4$. If $b \in P^3(v)$, then replace the path from v to b in $\tilde{T}_v$ with a path from v to b of length 3. Likewise for every $w \in W^*(\tilde{T}_v)$ we replace the path from w to b with a path from w to b of length 3 if $b \in P^3(w)$ and for every $u \in U^*(\tilde{T}_v)$ we replace the path from u to b with a path from u to b of length 1 if $b \in P^1(u)$ and we replace the path from u to b with a path from u to b of length 4 if $b \in P^0(u)$. Let the tree obtained using the above replacement be $\tilde{T}_v^*$. So every leaf of $\tilde{T}_v^*$ is at a distance 2,3,4,5 or 6 from v and if a leaf x is at a distance two from v then $deg_{\tilde{T}_v^*}(p(x)) > 2$. By Lemma 2.5, $v \in \mathcal{A}_{LR}(\tilde{T}_v)$ if and only if $v \in \mathcal{A}_{LR}(\tilde{T}_v^*)$ and $v \in \mathcal{N}_{LR}(\tilde{T}_v)$ if and only if $v \in \mathcal{N}_{LR}(\tilde{T}_v^*)$. Hence without loss of generality we can assume that every leaf of $\tilde{T}_v$ is at a distance $2, 3, 4, 5$ or 6 from v and if a leaf x is at a distance two from v then $deg_{\tilde{T}_v}(p(x)) > 2$.

Sufficiency:

Sufficiency for (a)**:**

Let L be an arbitrary minimum liar's dominating set of $\tilde{T}_v$.

Case 1. $|P^0(v)| \geq 3$.

Let x, y and z be any three vertices of $P^0(v)$. Let $P_x = v, x_1, x_2, x_3, x$, $P_y = v, y_1, y_2, y_3, y$, and $P_z = v, z_1, z_2, z_3, z$ be the paths from v to x, y, and z, respectively. By Proposition 2.3, $\{x, x_3, x_2, y, y_3, y_2, z, z_3, z_2\} \subseteq L$. If $v \notin L$, then $\{x_1, y_1, z_1\} \subseteq L$. In this case, $L' = L \cup \{v\} \setminus \{y_1, z_1\}$ is a liar's dominating set of $\tilde{T}_v$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 2. $|P^1(v)| \geq 2$.

Let x and y be two vertices of $P^1(v)$. Let $P_x = v, x_1, x_2, x_3, x_4, x$ and $P_y = v, y_1, y_2, y_3, y_4, y$ be the paths from v to x and y in $\tilde{T}_v$, respectively. By Proposition 2.3, $\{x, x_4, x_3, y, y_4, y_3\} \subseteq L$. If $v \notin L$, then $\{x_1, x_2, y_1, y_2\} \subseteq L$. In this case, $L' = (L \cup \{v\}) \setminus \{x_2, y_2\}$ is a liar's dominating set of $\tilde{T}_v$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 3. $|P^2(v)| \geq 2$.

Let x and y be two vertices of $P^2(v)$. Let $P_x = v, x_1, x_2, x_3, x_4, x_5, x$ and $P_y = v, y_1, y_2, y_3, y_4, y_5, y$ be the paths from v to x and y in $\tilde{T}_v$, respectively. By Proposition 2.3, $\{x, x_5, x_4, y, y_5, y_4\} \subseteq L$. If $v \notin L$, then $\{x_1, x_2, x_3, y_1, y_2, y_3\} \subseteq L$. In this case, $L' = (L \cup \{v\}) \setminus \{x_3, y_3\}$ is a liar's dominating set of $\tilde{T}_v$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 4. $|P^0(v)| = 2$ and $|P^3(v)| \geq 1$.

Let $P^0(v) = \{x, y\}$ and let $z \in P^3(v)$. Let $P_x = v, x_1, x_2, x_3, x$, $P_y = v, y_1, y_2, y_3, y$ and $P_z = v, z_1, z_2, z$ be the paths from v to x, y and z in $\tilde{T}_v$, respectively. By Proposition 2.3, $\{x, x_3, x_2, y, y_3, y_2, z, z_2, z_1\} \subseteq L$. If $v \notin L$, then $\{x_1, y_1\} \subseteq L$. In this case, $L' = L \cup \{v\} \setminus \{x_1, y_1\}$ is a liar's dominating set of $\tilde{T}_v$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 5. $|P^0(v)| = 2$ and $|P^1(v)| = 1$

Let $P^0(v) = \{x, y\}$ and $P^1(v) = \{z\}$. Let $P_x = v, x_1, x_2, x_3, x$, $P_y = v, y_1, y_2, y_3, y$, and $P_z = v, z_1, z_2, z_3, z_4, z$ be the paths from v to x, y and z in $\tilde{T}_v$, respectively. By Proposition 2.3, $\{x, x_3, x_2, y, y_3, y_2, z, z_4, z_3\} \subseteq L$. If $v \notin L$, then $\{x_1, y_1, z_1, z_2\} \subseteq L$. In this case, $L' = L \cup \{v\} \setminus \{x_1, y_1\}$ is a liar's dominating set of $\tilde{T}_v$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 6. $|P^0(v)| \in \{1, 2\}$ and $|P^2(v)| = 1$.

Let $x \in P^0(v)$ and $y \in P^2(v)$. Let $P_x = v, x_1, x_2, x_3, x$ and $P_y = v, y_1, y_2, y_3, y_4, y_5, y$ be the paths from v to x and y in $\tilde{T}_v$, respectively. By Proposition 2.3, $\{x, x_3, x_2, y, y_5, y_4\} \subseteq L$. If $v \notin L$, then $\{x_1, y_1, y_2, y_3\} \subseteq L$. In this case, $L' = (L \cup \{v\}) \setminus \{x_1, y_3\}$ is a liar's dominating set of $\tilde{T}_v$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 7. $|P^1(v)| = |P^2(v)| = 1$.

Let $P^1(v) = \{x\}$ and $P^2(v) = \{y\}$. Let $P_x = v, x_1, x_2, x_3, x_4, x$ and $P_y = v, y_1, y_2, y_3, y_4, y_5, y$ be the paths from v to x and y in $\tilde{T}_v$, respectively. By Proposition 2.3, $\{x, x_4, x_3, y, y_5, y_4\} \subseteq L$. If $v \notin L$, then $\{x_1, x_2, y_1, y_2, y_3\} \subseteq L$. In this case, $L' = (L \cup \{v\}) \setminus \{x_2, y_3\}$ is a liar's dominating set of $\tilde{T}_v$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 8. $|P^0(v)| = |P^1(v)| = 1$ and $|P^3(v)| \geq 1$.

Let $P^0(v) = \{x\}$, $P^1(v) = \{y\}$, and $z \in P^3(v)$. Let $P_x = v, x_1, x_2, x_3, x$, $P_y = v, y_1, y_2, y_3, y_4, y$, and $P_z = v, z_1, z_2, z$ be the paths from v to x, y and z in $\tilde{T}_v$, respectively. By Proposition 2.3, $\{x, x_3, x_2, y, y_4, y_3, z, z_2, z_1\} \subseteq L$. If $v \notin L$, then $\{x_1, y_1, y_2\} \subseteq L$. In that case, $L' = (L \cup \{v\}) \setminus \{x_1, y_2\}$ is a liar's dominating set of $\tilde{T}_v$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 9. $|U^*(\tilde{T_v})| \geq 2$.

Let u_1 and u_2 be any two vertices of $U^*(\tilde{T_v})$. Let $P_{a_1} = u_1, a_1$ and $P_{b_1} = u_1, b_{11}, b_{12}, b_{13}, b_1$ be the paths from u_1 to a_1 and b_1 in $\tilde{T_v}$, respectively and $P_{a_2} = u_2, a_2$ and $P_{b_2} = u_2, b_{21}, b_{22}, b_{23}, b_2$ be the paths from u_2 to a_2 and b_2 in $\tilde{T_v}$, respectively. By Proposition 2.3, $\{a_1, u_1, b_1, b_{13}, b_{12}, a_2, u_2, b_2, b_{23}, b_{22}\} \subseteq L$. If $v \notin L$, then $\{b_{11}, b_{21}\} \subseteq L$. In this case, $L' = (L \cup \{v\}) \setminus \{b_{11}, b_{21}\}$ is a liar's dominating set of $\tilde{T_v}$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T_v})$.

Case 10. $|U^*(\tilde{T_v})| = 1$ and $|P^0(v) \cup P^1(v) \cup P^2(v)| \geq 1$.

Let $U^*(\tilde{T_v}) = \{u\}$ and $x \in P^0(v) \cup P^1(v) \cup P^2(v)$. Let $P_a = u, a$ and $P_b = u, b_1, b_2, b_3, b$ be the paths from u to a and b in $\tilde{T_v}$, respectively. If $x \in P^0(v)$, then let $P_x = v, x_1, x_2, x_3, x$. If $x \in P^1(v)$, then let $P_x = v, x_1, x_2, x_3, x_4, x$. If If $x \in P^2(v)$, then let $P_x = v, x_1, x_2, x_3, x_4, x_5, x$. Now, if $v \notin L$, then $\{b_1, x_1\} \subseteq L$. In this case, $L' = (L \cup \{v\}) \setminus \{b_1, x_1\}$ is a liar's dominating set of $\tilde{T_v}$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. So $v \in L$ and hence $v \in \mathcal{A}_{LR}(\tilde{T_v})$.

Case 11. $|P^0(v) \cup P^1(v) \cup P^2(v) \cup P^3(v)| = 0$ and $|U^*(\tilde{T_v})| = 1$ and $|W^*(\tilde{T_v})| = 1$.

Let $U^*(\tilde{T_v}) = \{u\}$ and $W^*(\tilde{T_v}) = \{w\}$. Let $P_a = u, a$ and $P_b = u, b_1, b_2, b_3, b$ be the paths from u to a and b in $\tilde{T_v}$, respectively. Let $P_c = w, c_1, c_2, c$ and $P_d = w, d_1, d_2, d$ be the paths from w to c and d in $\tilde{T_v}$, respectively. By Proposition 2.3, $a, u, b, b_3, b_2, c, c_2, c_1, d, d_2, d_1 \in L$. If $v \notin L$, then $\{w, b_1\} \subseteq L$. In that case, $L' = (L \cup \{v\}) \setminus \{w, b_1\}$ is a liar's dominating set of $\tilde{T_v}$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. Thus $v \in L$ and so $v \in \mathcal{A}_{LR}(\tilde{T_v})$.

Case 12. $|P^2(v)| = 1$, $|W^*(\tilde{T_v})| = 1$ and $|P^3(v)| = 0$.

Let $P^1(v) = \{x\}$ and $W^*(\tilde{T_v}) = \{w\}$. Let $P_x = v, x_1, x_2, x_3, x_4, x_5, x$ be the path from v to x in $\tilde{T_v}$ and $P_a = w, a_1, a_2, a$ and $P_b = w, b_1, b_2, b$ be the paths from w to a and b in $\tilde{T_v}$, respectively. By Proposition 2.3, $\{x, x_5, x_4, a, a_2, a_1, b, b_2, b_1\} \subseteq L$. If $v \notin L$, then $\{x_1, x_2, x_3, w\} \subseteq L$. In this case, $L' = (L \cup \{v\}) \setminus \{w, x_3\}$ is a liar's dominating set of $\tilde{T_v}$ of cardinality $|L| - 1$. This is a contradiction to the minimality of L. Thus $v \in L$ and so $v \in \mathcal{A}_{LR}(\tilde{T_v})$.

Sufficiency for (b):

Case 13. $|P^3(v)| \geq 2$ and $P^0(v) \cup P^1(v) \cup P^2(v) \cup U^*(\tilde{T_v}) = \emptyset$.

Since $|P^3(v)| \geq 2$ and $P^0(v) \cup P^1(v) \cup P^2(v) \cup U^*(\tilde{T_v}) = \emptyset$, every liar's dominating set of $\tilde{T_v}$ must contain $V(\tilde{T_v}) \setminus (\{v\} \cup W^*(\tilde{T_v}))$. Since $V(\tilde{T_v}) \setminus (\{v\} \cup W^*(\tilde{T_v}))$ is a liar's dominating set of $\tilde{T_v}$, $V(\tilde{T_v}) \setminus (\{v\} \cup W^*(\tilde{T_v}))$ is the only minimum liar's dominating set of $\tilde{T_v}$. Thus $v \in \mathcal{N}_{LR}(\tilde{T_v})$.

Necessity:

Necessity for (a):

Let $v \in \mathcal{A}_{LR}(\tilde{T_v})$. We will show that at least one of the conditions (1) to (15) of Theorem 3.7 will be satisfied. If possible let us assume that none of the conditions (1) to (15) of Theorem 3.7 are satisfied. It can be seen easily that at least one of the following conditions will be true.

- $|P^3(v)| \geq 2$ and $P^0(v) \cup P^1(v) \cup P^2(v) \cup U^*(\tilde{T}_v) = \emptyset$.
- $|P^0(v)| = 2$ and $|P^1(v)| = |P^2(v)| = |P^3(v)| = |U^*(\tilde{T}_v)| = 0$.
- $|P^0(v) \cup P^1(v)| = 1$, $P^2(v) = P^3(v) = U^*(\tilde{T}_v) = \emptyset$ and $|W^*(\tilde{T}_v)| = 1$.
- $|P^0(v) \cup P^1(v) \cup P^2(v)| = 1$, $|P^3(v)| \geq 1$ and $U^*(\tilde{T}_v) = \emptyset$.
- $|P^0(v)| = |P^1(v)| = 1$ and $P^2(v) = P^3(v) = U^*(\tilde{T}_v) = \emptyset$.
- $|P^3(v)| = 1$, $P^0(v) = P^1(v) = P^2(v) = U^*(\tilde{T}_v) = \emptyset$ and $|W^*(\tilde{T}_v)| = 1$.
- $P^0(v) = P^1(v) = P^2(v) = \emptyset$, $|P^3(v)| \geq 1$ and $|U^*(\tilde{T}_v)| = 1$.
- $|W^*(\tilde{T}_v)| = 2$ and $P^0(v) = P^1(v) = P^2(v) = P^3(v) = U^*(\tilde{T}_v) = \emptyset$.

We show that in all of these above cases $v \notin \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 1. $|P^3(v)| \geq 2$ and $P^0(v) \cup P^1(v) \cup P^2(v) \cup U^*(\tilde{T}_v) = \emptyset$.

Under this condition, we have already seen in the sufficiency of (*b*) that $v \notin \mathcal{A}_{LR}(\tilde{T}_v)$.

Case 2. $|P^0(v)| = 2$ and $|P^1(v)| = |P^2(v)| = |P^3(v)| = |U^*(\tilde{T}_v)| = 0$.

Let $P^0(v) = \{x, y\}$. Let $P_x = v, x_1, x_2, x_3, x$ and $P_y = v, y_1, y_2, y_3, y$ be the paths from v to x and y in $\tilde{T}_v$, respectively. If $W^*(\tilde{T}_v) = \emptyset$ then $\tilde{T}_v = P_x \cup P_y$. By Proposition 2.3, $\gamma_{LR}(\tilde{T}_v) = 8$. Clearly $\{x, x_3, x_2, x_1, y_1, y_2, y_3, y\}$ and $\{x, x_3, x_2, v, y_1, y_2, y_3, y\}$ are two minimum liar's dominating set of $\tilde{T}_v$. Hence, $v \notin \mathcal{A}_{LR}(\tilde{T}_v) \cup \mathcal{N}_{LR}(\tilde{T}_v)$. So let $W^*(\tilde{T}_v) = \{w\}$. Then $V(\tilde{T}_v) \setminus \{x_1, y_1\}$ and $V(\tilde{T}_v) \setminus \{v, w\}$ are minimum liar's dominating sets of $\tilde{T}_v$. Thus $v \notin \mathcal{A}_{LR}(\tilde{T}_v) \cup \mathcal{N}_{LR}(\tilde{T}_v)$.

Case 3. $|P^0(v) \cup P^1(v)| = 1$, $P^2(v) = P^3(v) = U^*(\tilde{T}_v) = \emptyset$, and $|W^*(\tilde{T}_v)| = 1$.

Let $W^*(\tilde{T}_v) = \{w\}$. Since $|P^0(v) \cup P^1(v)| = 1$, either $P^1(v) = \emptyset$ or $P^0(v) = \emptyset$.

First suppose that $P^1(v) = \emptyset$. Let $P^0(v) = \{x\}$. Since $V(\tilde{T}_v) \setminus \{v\}$ and $V(\tilde{T}_v) \setminus \{w\}$ are minimum liar's dominating sets, $v \notin \mathcal{A}_{LR}(\tilde{T}_v) \cup \mathcal{N}_{LR}(\tilde{T}_v)$.

Next suppose that $P^0(v) = \emptyset$. Let $P^1(v) = \{x\}$. Since $V(\tilde{T}_v) \setminus \{v\}$ and $V(\tilde{T}_v) \setminus \{w\}$ are minimum liar's dominating sets, $v \notin \mathcal{A}_{LR}(\tilde{T}_v) \cup \mathcal{N}_{LR}(\tilde{T}_v)$.

Case 4. $|P^0(v) \cup P^1(v) \cup P^2(v)| = 1$, $|P^3(v)| \geq 1$, and $U^*(\tilde{T}_v) = \emptyset$.

Since $|P^0(v) \cup P^1(v) \cup P^2(v)| = 1$, exactly two of the sets $P^0(v)$, $P^1(v)$, and $P^2(v)$ are empty.

Let $P^1(v) = P^2(v) = \emptyset$ and $P^0(v) = \{x\}$. Let $P_x = v, x_1, x_2, x_3, x$ be the path from v to the leaf vertex x. Then $V(\tilde{T}_v) \setminus (W^*(\tilde{T}_v) \cup \{x_1\})$ and $V(\tilde{T}_v) \setminus (W^*(\tilde{T}_v) \cup \{v\})$ are minimum liar's dominating sets of $\tilde{T}_v$. Thus $v \notin \mathcal{A}_{LR}(\tilde{T}_v) \cup \mathcal{N}_{LR}(\tilde{T}_v)$.

Let $P^0(v) = P^2(v) = \emptyset$ and $P^1(v) = \{x\}$. Let $P_x = v, x_1, x_2, x_3, x_4, x$ be the path from v to the leaf vertex x. Then $V(\tilde{T}_v) \setminus (W^*(\tilde{T}_v) \cup \{x_2\})$ and

$V(\tilde{T_v}) \setminus (W^*(\tilde{T_v}) \cup \{v\})$ are minimum liar's dominating sets of $\tilde{T_v}$. Thus $v \notin \mathcal{A}_{LR}(\tilde{T_v}) \cup \mathcal{N}_{LR}(\tilde{T_v})$.

Let $P^0(v) = P^1(v) = \emptyset$ and $P^2(v) = \{x\}$. Let $P_x = v, x_1, x_2, x_3, x_4, x_5, x$ be the path from v to the leaf vertex x. Then $V(\tilde{T_v}) \setminus (W^*(\tilde{T_v}) \cup \{x_3\})$ and $V(\tilde{T_v}) \setminus (W^*(\tilde{T_v}) \cup \{v\})$ are minimum liar's dominating sets of $\tilde{T_v}$. Thus $v \notin \mathcal{A}_{LR}(\tilde{T_v}) \cup \mathcal{N}_{LR}(\tilde{T_v})$.

Case 5. $|P^0(v)| = |P^1(v)| = 1$ and $P^2(v) = P^3(v) = U^*(\tilde{T_v}) = \emptyset$.

Let $P^0(v) = \{x\}$ and $P^1(v) = \{y\}$. Let $P_x = v, x_1, x_2, x_3, x$ and $P_y = v, y_1, y_2, y_3, y_4, y$ be the paths from v to the leaf vertices x and y, respectively. Then $V(\tilde{T_v}) \setminus (W^*(\tilde{T_v}) \cup \{x_1\})$ and $V(\tilde{T_v}) \setminus (W^*(\tilde{T_v}) \cup \{v\})$ are minimum liar's dominating sets of $\tilde{T_v}$. Thus $v \notin \mathcal{A}_{LR}(\tilde{T_v}) \cup \mathcal{N}_{LR}(\tilde{T_v})$.

Case 6. $|P^3(v)| = 1$, $P^0(v) = P^1(v) = P^2(v) = U^*(\tilde{T_v}) = \emptyset$, and $|W^*(\tilde{T_v})| = 1$.

Let $P^3(v) = \{x\}$ and $W^*(\tilde{T_v}) = \{w\}$. Then $V(\tilde{T_v}) \setminus \{w\}$ and $V(\tilde{T_v}) \setminus \{v\}$ are minimum liar's dominating sets of $\tilde{T_v}$. Thus $v \notin \mathcal{A}_{LR}(\tilde{T_v}) \cup \mathcal{N}_{LR}(\tilde{T_v})$.

Case 7. $P^0(v) = P^1(v) = P^2(v) = \emptyset$, $|P^3(v)| \geq 1$, and $|U^*(\tilde{T_v})| = 1$.

Let $U^*(\tilde{T_v}) = \{u\}$. Let $P_a = u, a$ and $P_b = u, b_1, b_2, b_3, b$ be the paths from u to leaf vertices a and b. Then $V(\tilde{T_v}) \setminus (W^*(\tilde{T_v}) \cup \{b_1\})$ and $V(\tilde{T_v}) \setminus (W^*(\tilde{T_v}) \cup \{v\})$ are minimum liar's dominating sets of $\tilde{T_v}$. Thus $v \notin \mathcal{A}_{LR}(\tilde{T_v}) \cup \mathcal{N}_{LR}(\tilde{T_v})$.

Case 8. $|W^*(\tilde{T_v})| = 2$ and $P^0(v) = P^1(v) = P^2(v) = P^3(v) = U^*(\tilde{T_v}) = \emptyset$.

In this case, $\tilde{T_v} = T_v^*$. Let $W^*(T) = \{w_1, w_2\}$. Then $V(\tilde{T_v}) \setminus \{w_1\}$ and $V(\tilde{T_v}) \setminus \{v\}$ are minimum liar's dominating sets of $\tilde{T_v}$. Hence $v \notin \mathcal{A}_{LR}(\tilde{T_v}) \cup \mathcal{N}_{LR}(\tilde{T_v})$.

Necessity for (b):

Let $v \in \mathcal{N}_{LR}(\tilde{T_v})$. If possible suppose that either $|P^3(v)| \leq 1$ or $P^0(v) \cup P^1(v) \cup P^2(v) \cup U^*(\tilde{T_v}) \neq \emptyset$. Then one of the following conditions must hold.

(i) $deg_{\tilde{T_v}}(v) = 1$.

(ii) $C_{\tilde{T_v}}(v)$ contains a leaf of $\tilde{T_v}$.

(iii) $C_{\tilde{T_v}}(v)$ contains a vertex x such that $C_{\tilde{T_v}}(x) = \{y\}$ and y is a leaf of $\tilde{T_v}$.

(iv) $|P^0(v)| \geq 3$.

(v) $|P^1(v)| \geq 2$.

(vi) $|P^2(v)| \geq 2$.

(vii) $|P^0(v)| = 2$ and $|P^3(v)| \geq 1$.

(viii) $|P^0(v)| = 2$ and $|P^1(v)| = 1$.

(ix) $|P^0(v)| \in \{1, 2\}$ and $|P^2(v)| = 1$.

(x) $|P^1(v)| = |P^2(v)| = 1$.

(xi) $|P^0(v)| = |P^1(v)| = 1$ and $|P^3(v)| \geq 1$.

(xii) $|U^*(\tilde{T_v})| \geq 2$.

(xiii) $|U^*(\tilde{T_v})| = 1$ and $|P^0(v) \cup P^1(v) \cup P^2(v)| \geq 1$.

(xiv) $|P^0(v) \cup P^1(v) \cup P^2(v) \cup P^3(v)| = 0$ and $|U^*(\tilde{T_v})| = 1$ and $|W^*(\tilde{T_v}) = 1$.

(xv) $|P^2(v)| = 1$, $|W^*(\tilde{T_v})| = 1$ and $|P^3(v)| = 0$.

(xvi) $|P^0(v)| = 2$ and $|P^1(v)| = |P^2(v)| = |P^3(v)| = |U^*(\tilde{T_v})| = 0$.

(xvii) $|P^0(v) \cup P^1(v)| = 1$, $P^2(v) = P^3(v) = U^*(\tilde{T_v}) = \emptyset$ and $|W^*(\tilde{T_v})| = 1$.

(xviii) $|P^0(v) \cup P^1(v) \cup P^2(v)| = 1$, $|P^3(v)| \geq 1$ and $U^*(\tilde{T_v}) = \emptyset$.

(xix) $|P^0(v)| = |P^1(v)| = 1$ and $P^2(v) = P^3(v) = U^*(\tilde{T_v}) = \emptyset$.

(xx) $|P^3(v)| = 1$, $P^0(v) = P^1(v) = P^2(v) = U^*(\tilde{T_v}) = \emptyset$ and $|W^*(\tilde{T_v})| = 1$.

(xxi) $P^0(v) = P^1(v) = P^2(v) = \emptyset$, $|P^3(v)| \geq 1$ and $|U^*(\tilde{T_v})| = 1$.

(xxii) $|W^*(\tilde{T_v})| = 2$ and $P^0(v) = P^1(v) = P^2(v) = P^3(v) = U^*(\tilde{T_v}) = \emptyset$.

We have already seen in the proof of the sufficiency part of (a) and in the necessity part of (a) that under each of the above condition, $v \notin \mathcal{N}_{LR}(\tilde{T_v})$. Hence necessity of (b) is proved. □

By Corollary 3.5, Lemma 3.6, and Theorem 3.7, we have the following characterization.

Theorem 3.8. *Let v be a vertex of a tree T having at least three vertices. Let $\bar{T_v}$ be the pruning of T_v and let $\tilde{T_v}$ be the reduction of $\bar{T_v}$. Then*

- *$v \in \mathcal{A}_{LR}(T)$ if and only if $v \in \mathcal{A}_{LR}(\tilde{T_v})$.*
- *$v \in \mathcal{N}_{LR}(T)$ if and only if $v \in \mathcal{N}_{LR}(\tilde{T_v})$.*

4 Computation of $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ of a tree T

In this section, we propose a polynomial time algorithm to compute $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ of a tree T. Let T be a tree having at least three vertices. By Proposition 2.3, $(P(T) \cup S(T)) \subseteq \mathcal{A}_{LR}(T)$. Let $V' = V \setminus (P(T) \cup S(T))$. Let $v \in V'$. We construct a rooted tree T_v rooted at v from the given tree T. Next we compute the sets $W^*(T_v)$ and $U^*(T_v)$. If there exists a vertex u such that $u \notin U^*(T_v) \cup W^*(T_v) \cup \{v\}$ having $deg_{T_v}(u) \geq 3$, we apply tree pruning on T_v to get $\bar{T_v}$. If neither $\bar{T_v} = T_v^*$ nor $\bar{T_v} \in CT(v)$, we apply the reduction technique on $\bar{T}_v$ to get $\tilde{T}_v$. We apply Theorem 3.7 to decide whether $v \in \mathcal{A}_{LR}(T)$ or $v \in \mathcal{N}_{LR}(T)$. We repeat this process for all $v \in V'$ to compute the sets $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$. The steps for computing $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ are summarized in Algorithm 1.

Algorithm 1: ALL_OR_NO

```
Input: A tree T = (V, E) having at least three vertices.
Output: The sets A_LR(T) and N_LR(T)
A_LR(T) ← (P(T) ∪ S(T));
N_LR(T) ← ∅;
begin
    for for all v ∈ V \ (P(T) ∪ S(T)) do
        Construct the rooted tree T_v;
        Apply the tree pruning technique on T_v to obtain T̄_v;
        if neither T̄_v = T*_v nor T̄_v ∈ CT(v) then
            Apply the tree reduction technique on T̄_v to obtain T̃_v;
        if one of the conditions of Theorem 3.7(a) is satisfied then
            A_LR(T) = A_LR(T) ∪ {v};
        else if the condition of Theorem 3.7(b) is satisfied then
            N_LR(T) = N_LR(T) ∪ {v};
    Output((A_LR(T), N_LR(T)));
```

We show that Algorithm 1 can be implemented in $O(n^3)$ time. Assume that the tree $T = (V, E)$ is given in adjacency list representation. The sets $P(T)$ and $S(T)$ can be computed in $O(n)$ time. Let $V' = V \setminus (P(T) \cup S(T))$ and $v \in V'$. Use BFS (breadth first search) at v to make T a rooted tree T_v. We then partition the set of vertices according to their distance from v such that all the vertices with same distance from the root v are in the same class. These two steps can be done in $O(n)$ time using BFS. Next we compute the sets $W^*(T_v)$ and $U^*(T_v)$. If there exists a vertex u such that $u \notin U^*(T_v) \cup W^*(T_v) \cup \{v\}$ having $deg_{T_v}(u) \geq 3$, we apply tree pruning on T_v at u. At this point, $deg_{T_v}(x) \leq 2$ for all $x \in D_{T_v}(u)$. Now we count the number of leaves which are the descendants of u and having distance $i(mod\ 4)$ from u for each $i = 0, 1, 2$ and 3. Depending upon the cases of the pruning technique we form a new tree T_v' rooted at v. This can be done in $O(n)$ time. Now we check that whether $deg_{T_v'}(u) \leq 2$ for all $u \notin U^*(T_v') \cup W^*(T_v') \cup \{v\}$. If $deg_{T_v'}(u) \leq 2$ for all $u \notin U^*(T_v') \cup W^*(T_v') \cup \{v\}$ then $T_v' = \bar{T}_v$; otherwise we repeat the process at most $O(n)$ time to obtain $\bar{T}_v$. Hence the pruning of T_v, $\bar{T}_v$, can be computed in $O(n^2)$ time. Next we verify whether $\bar{T}_v = T_v^*$ or $\bar{T}_v \in CT(v)$ or not. Since $deg_{\bar{T}_v}(u) \leq 2$ for all $u \notin U^*(\bar{T}_v) \cup W^*(\bar{T}_v) \cup \{v\}$, this verification will take $O(n)$ time. If neither $\bar{T}_v = T_v^*$ nor $\bar{T}_v \in CT(v)$, we apply the reduction technique on $\bar{T}_v$. Again clearly this reduction of $\bar{T}_v$ can be done in $O(n)$ time to obtain $\tilde{T}_v$.

After having the rooted tree $\tilde{T}_v$, we are now in a position to check the conditions of Theorem 3.7 to decide whether $v \in \mathcal{A}_{LR}(\tilde{T}_v)$ or $v \in \mathcal{N}_{LR}(\tilde{T}_v)$

or $v \notin \mathcal{A}_{LR}(\tilde{T}_v) \cup \mathcal{N}_{LR}(\tilde{T}_v)$. Note that the conditions of Theorem 3.7 can be checked in $O(n)$ time. Since we have to repeat this step for all the vertices of V', the overall complexity of algorithm 1 is $O(n^3)$ time. The proof of correctness of algorithm 1 follows from Proposition 2.3, Lemma 3.4, Corollary 3.5, Lemma 3.6, Theorem 3.7, and Theorem 3.8.

In view of the above discussions, we have the following theorem.

Theorem 4.1. *For a tree T having at least three vertices, the sets $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ can be computed in $O(n^3)$ time.*

After computing $\mathcal{A}_{LR}(T)$, it can be checked in $O(n^2)$ time whether $\mathcal{A}_{LR}(T)$ is a liar's dominating set of T. If it is, then T has a unique liar's dominating set. Hence, we have the following theorem.

Theorem 4.2. *Recognizing whether a tree T has a unique liar's dominating set can be done in $O(n^3)$ time.*

5 Conclusion

In this paper, we have characterized the set, $\mathcal{A}_{LR}(T)$, of vertices of a tree T that are present in all minimum liar's dominating set of T. Similarly, we have also characterized the set, $\mathcal{N}_{LR}(T)$, of all vertices which are not present in any of the minimum liar's dominating set of T. We have shown that $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ can be computed in $O(n^3)$ time. If $\mathcal{A}_{LR}(T)$ becomes a liar's dominating set of T, then T has a unique liar's dominating set of T. We have shown that trees having unique minimum liar's dominating set can be recognized in $O(n^3)$ time. It would be interesting to design a linear time algorithm to compute $\mathcal{A}_{LR}(T)$ and $\mathcal{N}_{LR}(T)$ of a tree.

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