A note on the security number of grid-like graphs*

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ABSTRACT. For any graph $G = (V, E)$, a non-empty set $S \subseteq V$ is *secure* if and only if $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$. The cardinality of a minimum secure set in G is the *security number* of G. In this note we give a new proof for the security number of grid-like graphs.

Keywords: Security number; Cartesian product; Grid.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *degree* $d_G(v)$ of a vertex v is the number of edges incident to the vertex. We denote by $\delta(G)$ the *minimum degree* of the vertices of G. The *open neighborhood* $N_G(v)$ of a vertex v is the set of all vertices adjacent to v, and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. Similarly, we can define an *open* and *closed neighborhood* of a set $X \subseteq V$, i.e., $N_G(X) = \bigcup_{v \in X} N_G(v) \backslash X$ and $N_G[X] = \bigcup_{v \in X} N_G[v]$. We write $N(v)$, $N[v]$, $N(X)$ and $N[X]$ for short if G is clear from the context. In a graph G with at least one cycle, the length of a shortest cycle is called its *girth*. A vertex v in G is a *cut vertex* if $G - v$ has more connected components than G.

Brigham et al. [1] introduced the concept of a secure set as a generalization of a defensive alliance [6]. To better serve our purpose, we use the following characterization as our working definition in lieu of the formal definition given in [1].

Theorem 1 ([1]). *A non-empty set $S \subseteq V$ of a graph $G = (V, E)$ is secure if and only if $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$.*

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For a subset S of vertices of G, if the vertices in S are regarded as *defenders*, then the vertices in $N[S]-S$ are called *attackers* of S. If S is not a secure set of G, then there exists a $X \subseteq S$ such that $|N[X] \cap S| < |N[X]-S|$ by Theorem 1, such a set X of vertices is called a *witness set* of S.

The cardinality of a minimum secure set in G is the *security number* of G and is denoted $s(G)$. A secure set of cardinality $s(G)$ is called an *$s(G)$-set*. By Theorem 1, it is easy to see that the induced subgraph by a minimum secure set of G is connected. Previous work on secure sets in graphs can be found in [1, 2, 3, 4, 5].

Let P_n and C_m, respectively, denote a path on n vertices and a cycle on m vertices. The *Cartesian product* of graphs G_1 and G_2 is denoted $G_1 \Box G_2$, where $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \Box G_2) = \{(v_i, u_i)(v_j, u_j) : v_i = v_j \text{ and } u_i u_j \in E(G_2) \text{ or } v_i v_j \in E(G_1) \text{ and } u_i = u_j\}$. The class of graphs which contains exactly $P_m \Box P_n$, $P_m \Box C_n$ and $C_m \Box C_n$ is the class of *grid-like* graphs.

Brigham et al. [1] established the following upper bounds on the security number of grid-like graphs, and conjectured that the equalities hold.

Proposition 2 ([1]). *For grid-like graphs, we have*
(a) $s(P_m \Box P_n) = \min\{m, n, 3\}$.
(b) $s(P_m \Box C_n) \leq \min\{2m, n, 6\}$.
(c) $s(C_3 \Box C_3) = 4$ *and* $s(C_m \Box C_n) \leq \min\{2m, 2n, 12\}$ for $\max\{m, n\} \geq 4$.

In 2009, Kozawa et al. [5] proved that the conjecture is true. That is, the equalities hold in Proposition 2. In this note we give a new proof for the result.

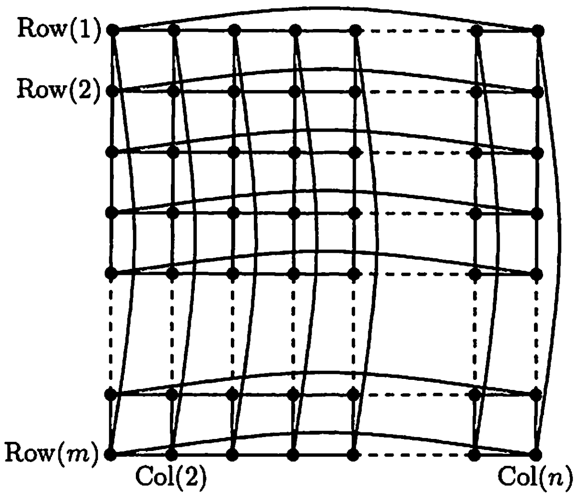

Figure 1: $C_m \Box C_n$

2 New Proof

For convenience, we say that the graph $C_m \Box C_n$ has m rows and n columns, which is shown in Fig. 1. Clearly, each row has n vertices and each column has m vertices. We begin to prove our main result.

Theorem 3. *$s(C_m \Box C_n) = \min\{2m, 2n, 12\}$ for $\max\{m, n\} \geq 4$.*

Proof. We write $G = C_m \Box C_n$ for short in this proof. By Proposition 2, we only need to show that $s(G) \geq \min\{2m, 2n, 12\}$. The proof is by contradiction. Suppose that $s(G) < \min\{2m, 2n, 12\}$. Let S be an $s(G)$-*set* of G and the subgraph induced by S is denoted by H. As mentioned before, H is connected. Since G is 4-regular, each vertex in H has at least two neighbors in S by Theorem 1, that is, $\delta(H) \geq 2$. This implies that H contains at least a cycle. To complete the proof, we first consider the structure of H.

Claim 3.1. *S could meet neither all rows nor all columns of G.*

Suppose not, without loss of generality, let S meet the m rows of G. If S meets at most $n-2$ columns of G, then clearly each row of G contains at least two attackers of S. Hence $|N[S] - S| \geq 2m > |S|$, contradicting our assumption that S is a secure set.

If S meets $n-1$ columns of G. Since H is connected, then $m+n-2 \leq |S| < \min\{2m, 2n, 12\} \leq \min\{m+n, 12\}$. So $|S| = m+n-2$ or $m+n-1$. For the latter case, we have $m+n-1 = |S| \leq \min\{2m-1, 2n-1, 11\}$, which leads to $m = n$, $4 \leq m \leq 6$. However, for both cases of S, it is routine to verify that there is a pendant vertex, a contradiction.

If S meets n columns of G. Similarly, we have $|S| = m+n-1$ and $m = n$, $4 \leq m \leq 6$. The only case S without pendant vertex is depicted in Fig. 2. However, $|N[S]-S| = 2(m-1)+2(m-3) = 4m-8 > 2m-1 = |S|$, again a contradiction. □

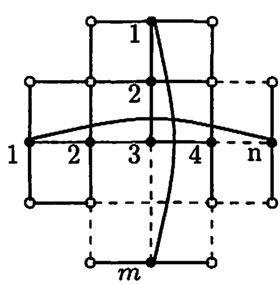

Figure 2: S meets m rows and n columns of G without pendant vertex ($\bullet \in S$, $\circ \in N[S] - S$).

Note that each cycle of odd length in H must meet either all rows or all columns of G. Then H doesn't contain any cycle of odd length by Claim 3.1.

Claim 3.2. *H contains no cut vertices.*

Suppose not, let u be a cut vertex of H. If $H - u$ has at least three components, then $d_H(u) \geq 3$. Since $\delta(H) \geq 2$, H contains at least three vertex-disjoined cycles. Note that the girth of H is at least 4. This implies that $|V(H)| \geq 12$, contradicting our assumption that $|V(H)| = s(G) < 12$.

If $H - u$ has exactly two components. Let H_1 and H_2 be two components of $H - u$ and $|V(H_1)| \leq |V(H_2)|$. So $|V(H_1)| \leq 5$ as $|S| < 12$.

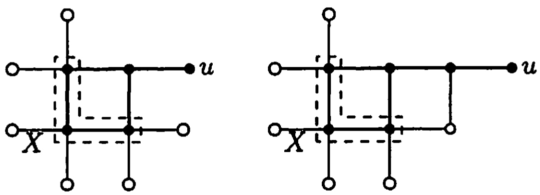

Figure 3: The graphs H_1 and $H_1 + u$ when $d_H(u) = 2$ ($\bullet \in S$, $\circ \in N[S] - S$).

If $d_H(u) = 2$, then each H_i contains at most one vertex of degree one of $H - u$, which is the neighbor of u in H. Hence each H_i contains an induced cycle of length at least 4. Since $|V(H_1)| \leq 5$, H_1 is isomorphic to either C_4 or the graph obtained from C_4 by attaching a pendant edge (see Fig. 3). Hence $|S| \geq 2|V(H_1)| + 1 \geq 9$. This implies that $m, n \geq 5$ as $|S| < \min\{2m, 2n, 11\}$. Denote by X the set of three vertices of degree two of H_1 that are not neighbors of u in H. Clearly $|N[X] \cap S| = 4 < 6 = |N[X] - S|$, contradicting the fact that S is a secure set of G.

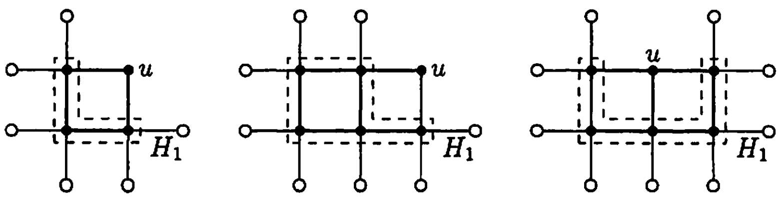

Figure 4: The graphs H_1 and $H_1 + u$ when $d_H(u) \geq 3$ ($\bullet \in S$, $\circ \in N[S] - S$).

If $d_H(u) \geq 3$, then u lies in a cycle of H, since the removal of u produces exactly two components. So $H_1 + u$ is isomorphic to either C_4 or C_6 (see Fig. 4) other than the cases shown in Fig. 3. If $H_1 + u$ is the graph as described in Fig. 3, we then obtain a contradiction as before. If $H_1 + u$ is the graph as described in Fig. 4, then $|S| \geq 2|V(H_1)| \geq 8$ and thus $m, n \geq 5$. Let $X = V(H_1)$. Since X has the unique neighbor u in S, it is easy to see that $|N[X] \cap S| < |N[X] - S|$, which is a contradiction. □

Claim 3.3. *Each vertex in H lies in a cycle C_4.*

As mentioned above, H contains no cycles of odd length. By Claim 3.2, each vertex in H lies in a cycle of even length of H. Suppose that there exists a vertex $w \in H$ that is not contained in any cycle C_4 of H. By Claim 3.1, H has no induced cycle C_6. Thus w must lie in an induced cycle C_{2k} $(4 \leq k \leq 5)$ of H by the restriction of $s(G) < 12$. If $\min\{m, n\} < 5$, our assumption implies that $s(G) < 8$, then H contains no induced cycle C_8 or C_{10}, a contradiction.

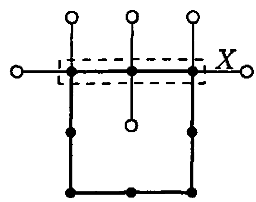

Figure 5 ($\bullet \in S$, $\circ \in N[S] - S$)

So $\min\{m, n\} \geq 5$. For the cycle C_{10} of H. By Claim 3.1, the embedding of induced C_{10} in G is obviously unique, which is a 3×4 rectangle. Then $|N(C_{10})| \geq 13$. Furthermore, S can contain at most one vertex from $N(C_{10})$ by the restriction of $s(G) < 12$. Thus, $|N[C_{10}] - S| \geq 13 - (11 - |C_{10}|) = 12$, so the cycle C_{10} itself is a witness set, a contradiction. For the cycle C_8 of H. As before, the embedding of induced C_8 is a unique 3×3 rectangle. By $\delta(H) \geq 2$, the other at most three possible vertices could be adjacent to at most three sides of the rectangle, leaving three consecutive vertices, say x_1, x_2, x_3, lying on a side of the rectangle C_8 that are exposed to six attackers (see Fig. 5). Let $X = \{x_1, x_2, x_3\}$. Thus, $|N[X] \cap S| = 5 < 6 = |N[X] - S|$, so X is a witness set of S, a contradiction. Therefore, each vertex in H lies in a cycle C_4, as claimed. □

All the possible induced subgraphs of $C_m \square C_n$ with less than 12 vertices and satisfying with Claims 1-3 are exhibited in Fig. 6. Thus H must be one of the Tetris graphs. By our assumption $|S| = |V(H)| < \min\{2m, 2n, 12\}$, it is verified that $|S| < |N[S] - S|$ by a simple calculation in each case, a contradiction. Therefore, $s(G) \geq \min\{2m, 2n, 12\}$. □

For the security number of $P_m \square C_n$, we need the following result due to [5].

Lemma 4 ([5]). $s(C_{2m}\Box C_n) \le 2s(P_m\Box C_n)$.

By Lemma 4 and Theorem 3, $s(P_m\Box C_n) = \min\{2m, n, 6\}$ can be easily derived.

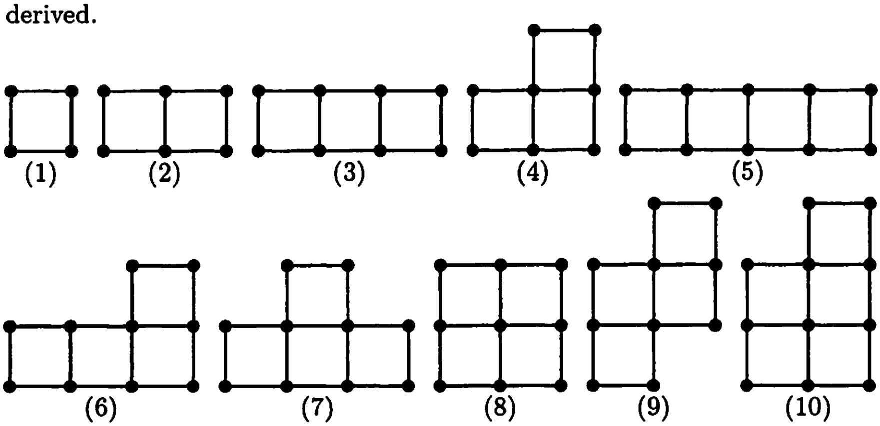

Figure 6: Tetris graphs

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