Decomposition of complete tripartite graphs into gregarious 3-paths and 6-cycles

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Abstract

In this paper, we refer to a decomposition of a tripartite graph into paths of length 3, or into 6-cycles, as gregarious if each subgraph has at least one vertex in each of the three partite sets. For a tripartite graph to have a 6-cycle decomposition it is straightforward to see that all three parts must have even size. Here we provide a gregarious decomposition of a complete tripartite graph $K(r, s, t)$ into paths of length 3, and thus of $K(2r, 2s, 2t)$ into gregarious 6-cycles, in all possible cases, when the straightforward necessary conditions on r, s, t are satisfied.

1 Introduction and necessary conditions

A complete tripartite graph $K(r, s, t)$ has $r + s + t$ vertices which are partitioned into three sets of sizes r, s, t such that any pair of vertices in different parts has an edge joining them, whereas any pair of vertices in the same part has no edge joining them.

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A 6-cycle is a 2-regular simple connected graph with six vertices and six edges. If the vertices are $x_1, x_2, \ldots, x_6$ and the edges are $x_i x_{i+1}$ for $1 \leqslant i \leqslant 5$ and $x_1 x_6$, then we denote such a 6-cycle by $(x_1, x_2, x_3, x_4, x_5, x_6)$ (or any cyclic permutation thereof).

We shall call a path of length three a 3-path. If such a path has edges ab, bc, cd then we shall denote the path by $[a, b, c, d]$ (or, equivalently, $[d, c, b, a]$).

A *decomposition* of a graph G into copies of a graph H, where H is some subgraph of G, is an edge-disjoint partition of the edge-set of G into copies of the graph H.

Various papers have investigated complete tripartite graph decompositions, into subgraphs such as cycles. The paper [2] was the first to impose a particular constraint upon the decomposition, requiring that every subgraph (a 4-cycle in [2]) should have at least one vertex in all three of the partite sets of the tripartite graph.

The smallest simple cycle is of course a 3-cycle, and it is well-known that there exists a 3-cycle decomposition of $K(r, s, t)$ if and only if $r = s = t$; moreover, any latin square of order r yields a 3-cycle decomposition of $K(r, r, r)$.

In the case that the subgraph is a 5-cycle, as in the case of 3-cycles, the constraint that every cycle should have at least one vertex in each of the three parts occurs naturally, because every odd length cycle in a tripartite graph must hit all three parts. The problem of verifying that some fairly straightforward necessary conditions for existence of a 5-cycle decomposition of $K(r, s, t)$ are sufficient is still open. In [7], a small reward is offered for its solution, and work towards this solution appears in [5], [6] and [1]; the case when the part sizes r, s, t (satisfying some necessary conditions) are all odd and different, remains largely open.

So with the requirement that every cycle in a tripartite graph decomposition has at least one vertex in every partite set, the next case to consider is that of 6-cycles. Here, as in [2], we shall mis-use the term "gregarious" to mean that each 6-cycle has at least one vertex in each of the three partite sets. Apart from [2], all subsequent papers on gregarious decompositions, such as [3, 4, 8], have required the number of partite sets in the complete multipartite graph being decomposed into cycles to be at least as great as the cycle length.

Suppose that there exists a gregarious 6-cycle decomposition of $K(r, s, t)$. Then this graph must have even degree, so $r + s$, $r + t$, $s + t$ are all even, implying that r, s and t all have the same parity. Moreover, the total number of edges must be 0 (mod 6), that is, $rs + rt + st \equiv 0 \pmod 6$; so r, s and t must all be *even*.

Following on from the idea of a cycle in a tripartite graph being gregarious if it has vertices in all three parts ([2]), we shall call a 3-path in a tripartite graph *gregarious* if it has its four vertices belonging to all three parts, and not just two of the parts. So now suppose that $[a, b, c, b']$ is such a gregarious 3-path in some tripartite graph (where the notation implies that b and b' belong to the same part, with a and c in the other two parts). If we expand each vertex two-fold (so from a vertex x, two vertices x_1 and x_2 are taken), we double the total number of vertices, and we obtain two 6-cycles from this one 3-path (see Figure 1):

$$(a_1, b_1, c_1, b_1', c_2, b_2) \quad \text{and} \quad (a_2, b_1, c_2, b_2', c_1, b_2).$$

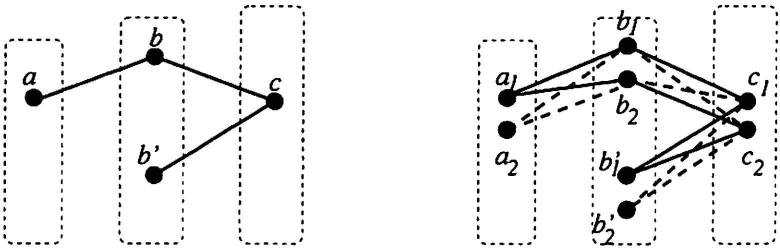


Figure 1: Two gregarious 6-cycles from a gregarious 3-path.

So, we will find gregarious 3-path decompositions when conditions allow, then we will use these 3-path decompositions to create gregarious 6-cycle decompositions as indicated in Figure 1. Of course a particular gregarious decomposition of $K(2r, 2s, 2t)$ into 6-cycles does not have to have come from one of $K(r, s, t)$ into 3-paths, but we only require *existence* of some gregarious 6-cycle decomposition for each possible case.

Since the graph $K(1, 1, 1)$ is just a triangle and cannot be decomposed into a gregarious 3-path, we look separately at decomposing $K(2, 2, 2)$ into gregarious 6-cycles. Suppose the vertex set of $K(2, 2, 2)$ consists of the three partite sets $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$; then the graph can be decomposed into the two 6-cycles $(a_1, b_1, c_1, b_2, a_2, c_2)$ and $(a_1, b_2, c_2, b_1, a_2, c_1)$.

We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set, respectively, of a graph G. We also use $G \setminus H$, where H is a subgraph of G, to denote the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$.

In subsequent sections, by finding gregarious 3-path decompositions of $K(r, s, t)$, we shall prove the following main result of this paper:

Main Theorem: *There exists a gregarious 6-cycle decomposition of the complete tripartite graph $K(\rho, \sigma, \tau)$ if and only if the following two conditions hold:*

(a) ρ, σ, τ *are all even, say* $\rho = 2r$, $\sigma = 2s$ *and* $\tau = 2t$, *and*

   (i) $r \equiv s \equiv t \equiv 1 \pmod 3$; *or*

   (ii) $r \equiv s \equiv t \equiv 2 \pmod 3$; *or*

   (iii) *at least two of* r, s, t *must be congruent to* $0 \pmod 3$;

(b) *if* $r \leqslant s \leqslant t$, *then* $s \leqslant 4r$ *and* $t(s-2r) \leqslant 2rs$.

2 Gregarious 3-paths

First, we provide some useful methods for extending a gregarious 3-path decomposition when we increase the size of the partite sets.

Lemma 2.1 *The graph* $K(x,x,y+3) \setminus K(x,x,y)$ *has a decomposition into gregarious 3-paths.*

PROOF. Let the vertices of the first two partite sets be $\{a_1, a_2, \ldots, a_x\}$ and $\{b_1, b_2, \ldots, b_x\}$, and let the three additional vertices in the last set be c_1, c_2, c_3. Use $2x$ paths:

$$[a_i, c_1, b_i, c_2],\ [c_2, a_i, c_3, b_i], \quad \text{for all } i = 1, \ldots, x.$$

□

Lemma 2.2 *The graph* $K(x,2x,y+2) \setminus K(x,2x,y)$ *has a decomposition into gregarious 3-paths.*

PROOF. Let the vertices of the first two partite sets be $\{a_1, a_2, \ldots, a_x\}$ and $\{b_1, b_2, \ldots, b_{2x}\}$, and let the two additional vertices in the last set be c_1, c_2. Use the following $2x$ paths:

$$[a_i, c_1, b_{2i-1}, c_2],\ [a_i, c_2, b_{2i}, c_1], \quad \text{for all } i = 1, \ldots, x.$$

□

From these two results, we obtain the following lemma.

Lemma 2.3 *If* $K(x,y,z)$ *has a decomposition into gregarious 3-paths, where* $x \leqslant y \leqslant 2x$, *then* $K(x,y,z+6)$ *can also be decomposed into gregarious 3-paths.*

PROOF. Partition the first partite set X into sets X' and X'' with $2x - y$ and $y - x$ elements, respectively, and the second partite set Y into Y' and Y'' with $2x - y$ and $2(y - x)$ elements, respectively. Let Z' be a set of six new vertices to be added to the third partite set Z. Use Lemma 2.1 twice with the sets X', Y' and Z' to produce $4(2x - y)$ gregarious 3-paths using precisely the edges from X' to Z' and from Y' to Z'. Use Lemma 2.2 three times with the sets X'', Y'' and Z' to produce $6(y - x)$ gregarious 3-paths using precisely the edges from X'' to Z' and from Y'' to Z'. These new 3-paths together with the $(xy + xz + yz)/3$ gregarious 3-paths in the decomposition of $K(x, y, z)$ give the $(xy + x(z + 6) + y(z + 6))/3$ gregarious 3-paths of the decomposition of $K(x, y, z + 6)$. □

Henceforth, let $K(r, s, t)$ be the complete tripartite graph on sets R, S, T of size r, s, t, respectively. If $K(r, s, t)$ can be decomposed into gregarious 3-paths, then the total number of edges, $rs + rt + st$, must be divisible by 3, so one of the following must hold:

(i) $r \equiv s \equiv t \equiv 1 \pmod 3$;

(ii) $r \equiv s \equiv t \equiv 2 \pmod 3$;

(iii) at least two of r, s, t must be congruent to $0 \pmod 3$.

Also, if $r \leqslant s \leqslant t$, then every gregarious 3-path must contain at least one edge having a vertex in R, so

$$(rs + rt + st)/3 \leqslant r(s + t) \leqslant 2rt,$$

which implies that $s \leqslant 4r$ and $t(s - 2r) \leqslant 2rs$. The latter is true for all t whenever $s \leqslant 2r$, but limits the size of t otherwise. Any triple, (r, s, t), that satisfies this condition plus one of (i), (ii), or (iii) is said to be *admissible.*

We now have the tools to decompose all graphs $K(r, s, t)$, $r \leqslant s \leqslant t$, whenever $s \leqslant 2r$ and (r, s, t) is admissible. The methods are basically iterative and so require a number of base decompositions. Unfortunately these methods cannot be applied when $s > 2r$, since they are based on Lemma 2.3 which requires $2r - s \geqslant 0$. In order to produce the base cases, we introduce latin representations and trades.

3 Latin representations and trades

For a detailed description of a latin representation, we refer the reader to Section 3 of [5]. However, for convenience, we briefly describe this here as well.

Let the vertices of $K(r,s,t)$ be $R \cup S \cup T$ where $R = \{i_a \mid 1 \leqslant i \leqslant r\}$, $S = \{i_b \mid 1 \leqslant i \leqslant s\}$ and $T = \{i_c \mid 1 \leqslant i \leqslant t\}$. A latin rectangle, $\mathcal{L}$, of order $r \times s$, with rows indexed by R, columns indexed by S, and entries from T, corresponds to rs disjoint triangles in the graph $K(r,s,t)$. Indeed, entry k_c in cell (i_a, j_b) corresponds to the triangle or 3-cycle (i_a, j_b, k_c).

Now not all edges of the tripartite graph $K(r,s,t)$ are used in $\mathcal{L}$, but we can add extra entries to the ends of the r rows so that these r rows use all t symbols in T, and we can add extra entries to the bottom of each of the s columns so that these s columns use all t symbols in T. This then forms what we term a *latin representation*; see Figure 2. This is "latin" in the first r rows, and "latin" in the first s columns. Each entry within the latin *rectangle* corresponds to a triangle, while the entries at the "*side*" (of the first r rows) and the entries at the "*bottom*" (of the first s columns) correspond to single edges, respectively between R and T, and between S and T. The entry k_c corresponds to the 3-cycle (i_a, j_b, k_c), the entry ℓ_c corresponds to the single edge $i_a\ell_c$, and the entry m_c corresponds to the single edge $j_b m_c$. Since all the entries are elements of the set T, we can omit the subscripts on the entries without fear of ambiguity.

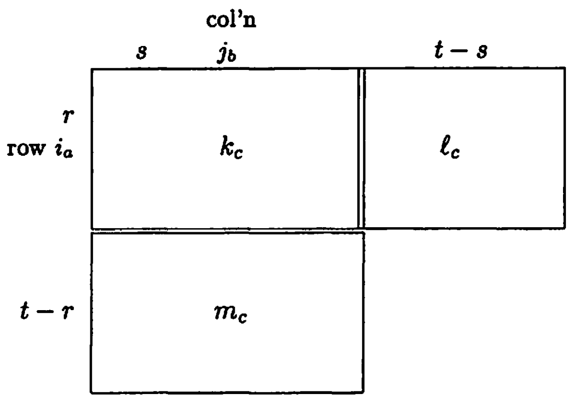


Figure 2: Illustrating a latin representation

Throughout this paper, we make use of the following *standard latin representation.* Let A^* be a $u \times u$ cyclic latin square, where

$$u = \begin{cases} 0 & \text{if } s \equiv 0 \pmod 3 \\ r & \text{if } s \equiv r \equiv t \not\equiv 0 \pmod 3 \\ r+1 & \text{if } s \equiv 1 \pmod 3,\ r \equiv 0 \pmod 3 \\ r+2 & \text{if } s \equiv 2 \pmod 3,\ r \equiv 0 \pmod 3). \end{cases}$$

With this choice, $s - u$ is always congruent to $0 \pmod 3$. So let B^* be an $(s-u) \times (s-u)$ array made up of 3×3 blocks, B_i, where $i = 1, \ldots, \frac{s-u}{3}$, arranged as a cyclic latin square. If $i \leqslant \lceil r/3 \rceil$, then B_i lies (at least partly) within the latin rectangle, and so must be a latin square (left below); otherwise, B_i is of the second type below.

$$B_i = \begin{array}{|ccc|} \hline 3i-2 & 3i-1 & 3i \\ 3i-1 & 3i & 3i-2 \\ 3i & 3i-2 & 3i-1 \\ \hline \end{array} \qquad B_i = \begin{array}{|ccc|} \hline 3i-2 & 3i-2 & 3i-2 \\ 3i-1 & 3i-1 & 3i-1 \\ 3i & 3i & 3i \\ \hline \end{array}$$

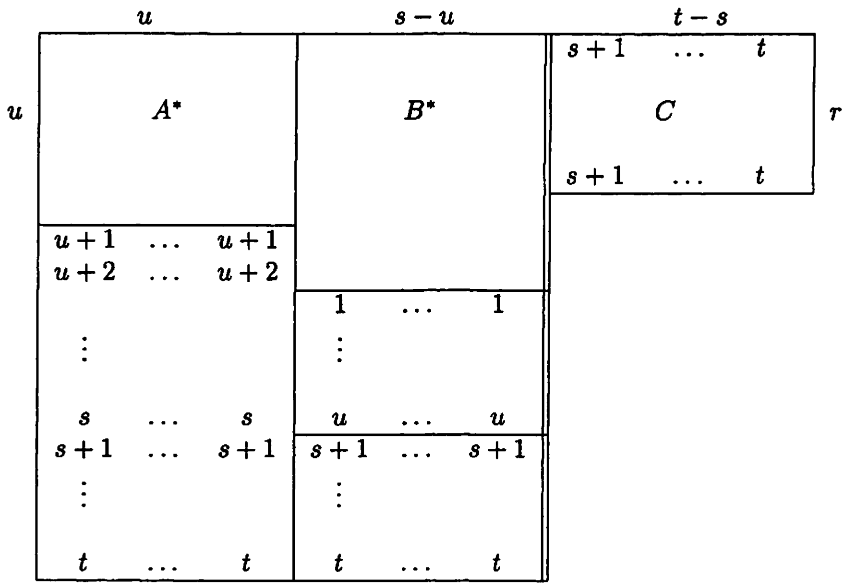


Figure 3: Standard latin representation with $u > 0$

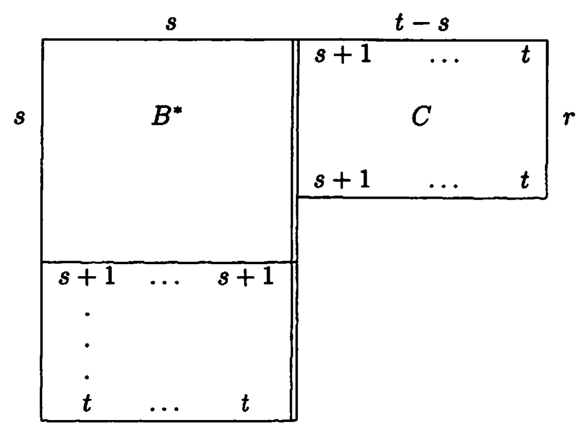
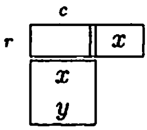


Figure 4: Standard latin representation with $u = 0$

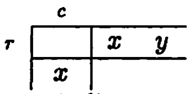
Row and column numbers are given with respect to the whole latin representation. The latin rectangle, $\mathcal{L}$, consists of the entries in rows 1 to r and columns 1 to s. Let $A = A^* \cap \mathcal{L}$ and $B = B^* \cap \mathcal{L}$. Rows $r+1$ to t are the bottom of the latin representation, with D and E consisting of columns 1 to u and $u+1$ to s, respectively. (Note that when $u = r+1$ or $r+2$, D and E overlap A^* and B^*.)

A *trade* gives us a way to transform a particular pattern of entries in the latin representation into a set of gregarious 3-paths; we "trade" triangles and edges for 3-paths. For example,

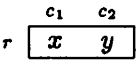
T_1 is: 

and this is the single path $[r, x, c, y]$.

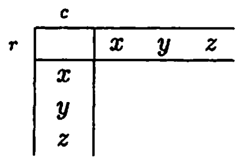
Each trade listed here may also be used in transpose format; for example,

$\overline{T_1}$ is 

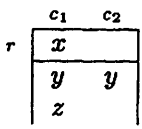
and this is also a single path, $[c, x, r, y]$. (We use an overline to indicate transpose of a trade here.) The following trades are all useful in our subsequent decompositions.

T_2^a : 

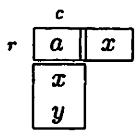
2 paths:
$[x, c_1, r, y]$, $[x, r, c_2, y]$

T_2^b : 

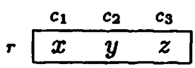
2 paths:
$[r, x, c, y]$, $[y, r, z, c]$

T_2^c : 

2paths:
$[r, x, c_1, z]$, $[r, c_1, y, c_2]$

T_2^d : 

2 paths:
$[y, c, r, a]$, $[r, x, c, a]$

T_3^a : 

3 paths:
$[x, c_1, r, c_2]$, $[x, r, c_3, z]$, $[c_2, y, r, z]$

T_4^a :

	c_1	c_2
r	a	b
	x	y
	z	z
	w	w

4 paths: $[r, a, c_1, x]$, $[r, b, c_2, y]$, $[r, c_1, w, c_2]$, $[r, c_2, z, c_1]$
The paths are still distinct if $x = y$ or if one or both of $x = b$ or $y = a$ holds.

T_4^b :

	c_1	c_2	c_3		
r_1				x	y
r_2				x	y
r_3				x	y
	x	x	x		
	y	y	y		

4 paths: $[x, r_1, y, c_1]$, $[c_1, x, r_2, y]$, $[r_3, x, c_2, y]$, $[r_3, y, c_3, x]$

T_4^c :

	c_1	c_2	c_3
r	a	b	w
	x	y	z

4 paths: $[x, c_1, a, r]$, $[c_1, r, c_2, y]$ $[c_2, b, r, c_3]$, $[r, w, c_3, z]$
The paths are still distinct if two or more of x, y, z are equal or if $a = z$.

T_4^d :

	c_1	c_2	c_3
r	w	x	y
	x	y	z

4 paths: $[x, c_1, w, r]$, $[c_1, r, c_2, y]$, $[c_2, x, r, c_3]$, $[r, y, c_3, z]$
The paths are still distinct if $z = w$.

T_5^a :

	c_1	c_2	c_3
r	x	y	z
	y	z	x
	z	x	y

5 paths: $[r, c_1, x, c_2]$, $[r, x, c_3, z]$, $[y, r, c_2, z]$, $[r, z, c_1, y]$, $[r, c_3, y, c_2]$

T_5^b :

	c_1	c_2	c_3
r	a	b	d
	x	y	z
	w	w	w

5 paths: $[r, c_1, w, c_2]$, $[w, c_3, r, c_2]$, $[r, b, c_2, y]$, $[r, a, c_1, x]$, $[r, d, c_3, z]$.
The paths are still distinct if $x = y = z$ or if one or more of $x = b, y = d, z = a$.

T_6^a :

	c_1	c_2	c_3
r	a	b	w
	x	x	x
	y	y	y
	z	z	z

6 paths: $[r, c_1, x, c_2]$, $[r, a, c_1, y]$, $[r, c_2, y, c_3]$, $[r, b, c_2, z]$, $[r, w, c_3, x]$, $[r, c_3, z, c_1]$

T_6^b :

	c_1	c_2	c_3
r	a	b	d
	b	d	w
	d	w	b
	x	y	z

6 paths $[r, a, c_1, x]$, $[r, b, c_2, y]$, $[r, d, c_3, z]$, $[r, c_1, b, c_3]$, $[r, c_2, d, c_1]$, $[r, c_3, w, c_2]$
The paths are still distinct if $w = a$ and/or if two or more of x, y, z are equal.

Note that distinct letters denote distinct entries, unless otherwise noted. Also entries shown to be in the same row of the bottom of a particular trade need not in fact be in the same row, because the bottom is only *column* latin. The same holds for entries in the same column of the side, because the side is only *row* latin. For example, trade T_4^a may appear as:

	c_1	c_2
r	a	b
	x	z
	z	y
	w	w

Whenever latin representations and trades are used in the following sections, we suggest that the reader crosses off the cells in the latin representations given, as each trade is specified, in order to check that all entries in the representations are used.

4 Small s: $r \leqslant s \leqslant 2r$

The following lemma holds for all $r > 1$.

Lemma 4.1 *The graph $K(r,r,r)$, where $r > 1$, can be decomposed into gregarious 3-paths.*

PROOF. Use any latin square of order r as the latin representation. We have a great deal of choice. One possible decomposition is as follows: for even r, take $r^2/2$ trades T_2^a, and for odd r, take r trades T_3^a (in the first three columns, say) and then $r(r-3)/2$ trades T_2^a. □

4.1 $r \equiv s \equiv t \pmod 3$

Lemma 4.2 *Let $1 < r < s \leqslant 2r$, such that $r \equiv s \pmod 3$. Then $K(r,s,s)$ can be decomposed into gregarious 3-paths.*

PROOF. Suppose first that $s - r \equiv 0 \pmod 6$. Then there exists an integer $k \geqslant 1$ such that $s = r + 6k$. Starting with the decomposition of $K(r,r,r)$ from Lemma 4.1, (since $r > 1$), apply Lemma 2.1 precisely $2k$ times to obtain a decomposition of $K(r,r,r+6k) = K(r,r,s)$. Since $s \leqslant 2r$, apply Lemma 2.3 exactly k times to obtain a decomposition of $K(r,r+6k,s) = K(r,s,s)$.

Now suppose that $s - r \equiv 3 \pmod 6$ and $r \geqslant 6$. Then there exists an integer $k \geqslant 1$ such that $s = r - 3 + 6k$. Starting with the decomposition of $K(r-3, r-3, r-3)$, apply Lemma 2.1 to obtain a decomposition of $K(r-3, r-3, r)$. Since $r \leqslant 2(r-3)$, apply Lemma 2.3 precisely k times to give a decomposition of $K(r-3, r, r-3+6k) = K(r-3, r, s)$. Since $s \leqslant 2r$, apply Lemma 2.3 another k times to give a decomposition of $K(r, s, r-3+6k) = K(r, s, s)$.

The only remaining cases (with $s \leqslant 2r$) are: $K(3,6,6)$, $K(4,7,7)$ and $K(5,8,8)$; these can be found in the Appendix. □

Theorem 4.3 *Let* $1 \leqslant r \leqslant s \leqslant t$, $(r,s,t) \neq (1,1,1)$, $r \equiv s \equiv t \pmod 3$ *with* $s \leqslant 2r$. *Then* $K(r,s,t)$ *has a decomposition into gregarious* 3*-paths.*

PROOF. If $r = 1$, then $s = 1$. The graph $K(1,1,4)$, with vertex set $\{a\} \cup \{b\} \cup \{w,x,y,z\}$, can be decomposed into the paths $[a,x,b,w]$, $[y,a,z,b]$, $[w,a,b,y]$. Use Lemma 2.1 to get decompositions for the remaining graphs $K(1,1,t)$ with $t \equiv 1 \pmod 3$. (Recall that t is unbounded if $s \leqslant 2r$.)

Now consider $K(r,s,t)$, where $1 < r \leqslant s \leqslant t$, $r \equiv s \equiv t \pmod 3$ and $s \leqslant 2r$.

Suppose first that $t - s \equiv 0 \pmod 6$. Then there exists an integer k with $k \geqslant 0$ such that $t = s + 6k$. By Lemmas 4.2 or 4.1, $K(r,s,s)$ can be decomposed into gregarious 3-paths. Since $s \leqslant 2r$, we can apply Lemma 2.3, k times, to obtain a decomposition of $K(r,s,s+6k) = K(r,s,t)$.

Next suppose that $t - s \equiv 3 \pmod 6$ and $s - r \equiv 0 \pmod 6$. If $r = s$, then the decomposition is given by Lemmas 4.1 and 2.1. Otherwise, there exist integers $u, v > 0$ such that $t = s + 6u - 3$ and $s = r + 6v$. Starting with a decomposition of $K(r,r,r)$, $(r > 1)$, apply Lemma 2.1, $2v - 1$ times, to obtain a decomposition of $K(r,r,r+6v-3) = K(r,r,s-3)$. Since $s - 3 < s \leqslant 2r$, apply Lemma 2.3 v times, giving a decomposition of $K(r, r+6v, s-3) = K(r,s,s-3)$. Then apply Lemma 2.3 u times to obtain a decomposition of $K(r,s,s-3+6u) = K(r,s,t)$.

Finally, suppose that $t - s \equiv s - r \equiv 3 \pmod 6$. Then $t - r \equiv 0 \pmod 6$ and there exists an integer w with $w > 0$ such that $t = r + 6w$. Starting with the decomposition of $K(r,r,r)$, with $r > 1$, apply Lemma 2.1, $(s-r)/3$ times, to get a decomposition of $K(r,r,s)$. Next apply Lemma 2.3 w times to give a decomposition of $K(r,s,r+6w) = K(r,s,t)$, as required. □

4.2 Exactly two of $r, s, t \equiv 0 \pmod 3$

The inductive decompositions are given in Theorems 4.9, 4.10 and 4.11. The necessary base cases are given first. Note that when r is small, s can be larger than $2r$ in these lemmas. In addition, Lemma 4.4 holds for $s \leqslant 4r$.

Lemma 4.4 *If $s \equiv 0 \pmod 3$, $0 < r \leqslant s \leqslant 4r$, then $K(r, s, s)$ has a decomposition into gregarious 3-paths.*

PROOF. If $s = r$, we can use Lemma 4.1; otherwise, we take the standard latin representation with $u = 0$.

Clearly, $s - r = 3j + k$, where $k \in \{0, 1, 2\}$ and j is a non-negative integer. If $k = 1$, use $s/3$ lots of trade T_4^d on the elements of rows r and $r+1$; if $k = 2$, use $s/3$ lots of trade T_5^a on the elements of rows r, $r+1$ and $r+2$; if $k = 0$, this step is not needed. Next, use $\frac{sj}{3}$ lots of trade T_6^a on the remaining entries in the bottom and on j rows of B. This is possible since $j = \frac{s-r-k}{3} \leqslant r - \frac{k}{3}$, so there are enough rows remaining in B. All remaining entries in B can be used in T_3^a trades. □

Corollary 4.5 *For $r \equiv 1 \pmod 3$, $K(r, r+2, r+2)$ has a decomposition into gregarious 3-paths; in addition, $K(r, r+5, r+5)$ has a decomposition if $r > 1$. For $r \equiv 2 \pmod 3$, $K(r, r+1, r+1)$ and $K(r, r+4, r+4)$ have decompositions into gregarious 3-paths.*

Lemma 4.6 *If $r \equiv 1 \pmod 3$, then $K(r, r+2, r+5)$ has a decompostion into gregarious 3-paths. If $r \equiv 2 \pmod 3$, then $K(r, r+1, r+4)$ has a decomposition into gregarious 3-paths.*

PROOF. In each of these cases, $s \equiv 0 \pmod 3$, so use the standard latin representation with $u = 0$.

A decomposition of $K(1, 3, 6)$ is given in the Appendix, so assume that $r \equiv 1 \pmod 3$ and $r \geqslant 4$. First, use $s/3$ lots of trade T_5^a on the entries in rows $r, r+1, r+2$. Then use r lots of trade T_2^b on all the entries of C and the entries in columns 1 to r and rows $r+3, r+4, r+5$ of E. Use one lot of T_4^a on the entries in columns $r+1, r+2$ and rows $1, r+3, r+4, r+5$. The remaining entries in the latin rectangle can be used in two lots of trade T_2^a and $(s-6)/3 + s(r-2)/3$ lots of trade T_3^a.

A decomposition of $K(2, 3, 6)$ is given in the Appendix, so assume that $r \equiv 2 \pmod 3$ and $r > 2$. Use $r-2$ lots of trade T_2^b on the entries in rows 1 to $r-2$ of C and the entries in columns 1 to $r-2$ and rows $r+1, r+2, r+3$ of E. Use $(r-2)/3$ lots of T_4^d on the entries in columns 1 to $r-2$ and rows $r, r+1$. This leaves the entries in three columns and four rows of the bottom and two rows of the side. Use these side entries in six T_1 trades. Finish the decomposition with $r + (r-1)(r-2)/3$ lots of trade T_3^a. □

Lemma 4.7 *If $r \equiv 0 \pmod 3$, $r > 0$, then $K(r,r,r+1)$, $K(r,r,r+2)$, $K(r,r+3,r+4)$, and $K(r,r+3,r+5)$ all have decompositions into gregarious 3-paths.*

PROOF. Suppose that $r \equiv 0 \pmod 3$, $r > 0$, so $u = 0$ in the standard latin representation.

First, consider $K(r,r,r+1)$, with $r > 3$. In columns 1,2,3, use the entries 1,2,3 along with the 3 entries in row $r+1$ in a T_4^c trade. Repeat this with triples of columns $(4,5,6), \ldots, (r-2, r-1, r)$, using up all copies of the entries 1,2,3 in B. In rows 1,2,3, use the entries 4,5,6 along with the 3 entries in column $r+1$ in a $\overline{T_4^c}$ trade. Repeat this with the other triples of rows using up all copies of the entries 4,5,6 in B. The remaining entries in the latin rectangle can be used in T_2^a and T_3^a trades. To decompose $K(3,3,4)$, first obtain a decomposition for $K(1,3,3)$ by applying trade T_5^a to its standard latin representation. Then apply Lemma 2.1.

For $K(r,r,r+2)$, use $r/3$ lots of T_4^b on the entries in C and E and $r^2/3$ lots of trade T_3^a.

For $K(r,r+3,r+4)$, use r lots of trade T_1 on the entries in C and the entries in rows $r+3$ and $r+4$ and columns 1 to r of E. Then $(r+6)/3$ lots of T_5^b use up the remaining entries in E; the remaining entries of B are used in T_3^a trades.

For $K(r,r+3,r+5)$, use $r/3$ lots of trade T_4^b to use up all entries in C plus the entries in columns 1 to r and rows $r+4, r+5$ of E. One lot of T_5^b takes care of the entries in columns $r+1, r+2, r+3$ and rows $r+4, r+5$, while $(r+3)/3$ lots of T_6^a finish off all the remaining entries in E. The remaining entries of B are used in T_3^a trades. □

Lemma 4.8 *If $r \equiv 0 \pmod 3$, $r > 0$, then $K(r,r+1,r+3)$ and $K(r,r+2,r+3)$ can be decomposed into gregarious 3-paths.*

PROOF. The decomposition of $K(3,4,6)$ is given in the Appendix. For $K(r,r+1,r+3)$, with $r > 3$, use the standard latin representation with $u = r+1$, so both B and E are omitted. Use all the entries in rows 1 to $r-3$ of C in $r-3$ lots of $\overline{T}_1$ with the entries in row $r+3$ and columns 1 to $r-3$ of D. Use the entries in rows $r-2, r-1, r$ of C in $\overline{T}_1$ trades with the entries in row $r+2$ and columns 1 to 3 of D. Use one lot of T_4^d and $(r-6)/3$ lots of T_5^b to finish using up all the entries in columns 1 to $r-3$

of D plus the entries in columns 1 to $r-3$ of row r. Two lots of T_4^a take care of the entries in rows r to $r+3$ and columns $r-2$ to $r+1$.

For $K(r,r+2,r+3)$, use the standard latin representation with $u=r+2$. Use up all the entries in C in r lots of trade T_1 with the entries in rows $r+2, r+3$ and columns 1 to r of the bottom. Use one lot of T_4^a on the entries in columns $r+1, r+2$ and rows $r, r+1, r+2, r+3$ and $r/3$ lots of T_4^d on the entries in rows $r, r+1$ and columns 1 to r.

In both cases, the remaining entries in the latin rectangle are used in T_2^a and T_3^a trades. □

The next three theorems, together with Theorem 4.3 above, complete the problem whenever $s \leqslant 2r$.

Theorem 4.9 *If $s,t \equiv 0 \pmod 3$ and $r \equiv 1,2 \pmod 3$ with $s \leqslant 2r$, then $K(r,s,t)$ can be decomposed into gregarious 3-paths.*

PROOF. Suppose first that $s,t \equiv 0 \pmod 6$ and $r \equiv 2 \pmod 6$, where $s \leqslant 2r$. If $r=2$, then $s \leqslant 4 < 6 \leqslant s$, so we need only consider $r \geqslant 8$. Then there exist integers q,v,w with $r=6q+2$, $s=6v$ and $t=6w$. Now $K(r,r+4,r+4)$ can be decomposed into gregarious 3-paths by Lemma 4.4. Apply Lemma 2.3, $(v-q-1)$ times, to get a decomposition for $K(r,r+4+6(v-q-1),r+4) = K(r,s,r+4)$, and an additional $(w-q-1)$ times to get a decomposition for $K(r,s,r+4+6(w-q-1)) = K(r,s,t)$.

For the remaining cases, we tabulate below what may be used, similar to this case, to obtain a decomposition of $K(r,s,t)$.

(r,s,t) mod 6	start with	(r,s,t) mod 6	start with
(2,0,3)	$K(r,r+4,r+1)$	(1,0,0)	$K(r,r+5,r+5)$
(2,3,0)	$K(r,r+1,r+4)$	(1,0,3)	$K(r,r+5,r+2)$
(2,3,3)	$K(r,r+1,r+1)$	(1,3,0)	$K(r,r+2,r+5)$
		(1,3,3)	$K(r,r+2,r+2)$
(5,0,0)	$K(r,r+1,r+1)$	(4,0,0)	$K(r,r+2,r+2)$
(5,0,3)	$K(r,r+1,r+4)$	(4,0,3)	$K(r,r+2,r+5)$
(5,3,0)	$K(r,r+4,r+1)$	(4,3,0)	$K(r,r+5,r+2)$
(5,3,3)	$K(r,r+4,r+4)$	(4,3,3)	$K(r,r+5,r+5)$

□

Theorem 4.10 *If $r \equiv t \equiv 0 \pmod 3$ and $s \equiv 1,2 \pmod 3$ with $s \leqslant 2r$, then $K(r,s,t)$ can be decomposed into gregarious 3-paths.*

PROOF. Suppose first that $r \equiv t \equiv 0$ or $3 \pmod 6$ and $s \equiv 2 \pmod 3$, where $s \leqslant 2r$. Then there exist integers v and w such that $t - r = 6v$ and $s - r = 2 + 3w$. By Lemma 4.7, $K(r, r, r+2)$ can be decomposed into gregarious 3-paths. Apply Lemma 2.1, w times, to obtain a decomposition of $K(r, r, r+2+3w) = K(r, r, s)$. Since $s \leqslant 2r$, apply Lemma 2.3, v times, to obtain a decomposition of $K(r, s, r+6v) = K(r, s, t)$. If $s \equiv 1 \pmod 3$, use the same method starting with a decomposition of $K(r, r, r+1)$ from Lemma 4.8.

Now suppose that $r, t \equiv 0 \pmod 3$ and $t - r \equiv 3 \pmod 6$. There exists an integer v such that $t - r = 3 + 6v$, and if $s \equiv 2 \pmod 6$, there exists an integer w such that $s = r + 2 + 6w$. By Lemma 4.7, $K(r, r+2, r+3)$ can be decomposed into gregarious 3-paths. Apply Lemma 2.3 precisely w times to obtain a decomposition for $K(r, r+2+6w, r+3) = K(r, s, r+3)$. Since $s \leqslant 2r$, apply the same lemma v times to give a decomposition for $K(r, s, r+3+6v) = K(r, s, t)$. If $s \equiv 5 \pmod 6$, then there exists an integer q such that $s = 6q + r + 5$. By Lemma 4.7, $K(r, r+3, r+5)$ can be decomposed into gregarious 3-paths. Apply Lemma 2.3 just q times to obtain a decomposition of $K(r, r+5+6q, r+3) = K(r, s, r+3)$. Apply the same lemma v more times to get a decomposition for $K(r, s, r+3+6v) = K(r, s, t)$. If $s \equiv 1 \pmod 3$, use the same methods starting with decompositions of $K(r, r+1, r+3)$ and $K(r, r+3, r+4)$ from Lemma 4.8.

□

Theorem 4.11 *If $r \equiv s \equiv 0 \pmod 3$ and $t \equiv 1, 2 \pmod 3$ with $s \leqslant 2r$, then $K(r, s, t)$ can be decomposed into gregarious 3-paths.*

PROOF. Suppose that $r \equiv s \equiv 0 \pmod 3$, $r > 3$ and $t \equiv 1, 2 \pmod 3$ with $s \leqslant 2r$. We use the following initial decompositions.

s mod 6	$t - r$ mod 6	start with	s mod 6	$t - r$ mod 6	start with
r	1	$K(r, r, r+1)$	$r+3$	1	$K(r, r+1, r+3)$
	4	$K(r, r, r+4)$		4	$K(r, r+3, r+4)$
	2	$K(r, r, r+2)$		2	$K(r, r+3, r+2)$
	5	$K(r, r, r+5)$		5	$K(r, r+3, r+5)$

If $r = 3$, then $s = 3$ or 6. The graphs $K(2, 3, 3)$ and $K(3, 3, 4)$ can be decomposed by Lemmas 4.4 and 4.7, respectively, so, by Lemma 2.1, there are decompositions of $K(3, 3, 5)$, $K(3, 3, 8)$ and $K(3, 3, 7)$. Use $K(3, 3, 4)$, $K(3, 3, 5)$, $K(3, 3, 8)$ and $K(3, 3, 7)$ as initial decompositions.

Starting with the initial decomposition in each case, use Lemma 2.3 $\lfloor \frac{s-r}{6} \rfloor$ times to get a decompositions for $K(r, s, r+i)$, where $i = 1, 2, 4$ or 5

as appropriate. Then use Lemma 2.3 v times to get a decomposition for $K(r,s,r+i+6v) = K(r,s,t)$. □

5 Large s: $2r < s \leqslant 4r$

5.1 $t \leqslant 4r$

Since $s \leqslant t \leqslant \dfrac{2rs}{s-2r} = 4r$, we note that if $s = 4r$, then $t = 4r$ as well.

Theorem 5.1 *If $2r < s \leqslant t \leqslant 4r$, with (r,s,t) admissible, then $K(r,s,t)$ has a decomposition into gregarious 3-paths.*

Proof. Case 1: $r \equiv s \equiv t \pmod 3$. Start with the standard latin representation and use up all of C and all entries in columns 1 to r and rows $s+1$ to t in $r(t-s)/3$ lots of trade T_2^b. Then $(s-r)(t-s)/9$ lots of T_6^a will use up the remaining entries in rows $s+1$ to t plus the entries in rows 1 to $(t-s)/3$ of columns $r+1$ to s in the latin rectangle. Note that $C = \emptyset$ if and only if $t = s$, in which case no entries have been used.

(i) If $r \equiv s \equiv t \equiv 0 \pmod 3$, $\frac{s(s-r)}{9}$ lots of T_6^a will use up the remaining bottom entries plus the entries in $\frac{s-r}{3}$ rows of the latin rectangle. (This is fine since $\frac{t-s}{3} + \frac{s-r}{3} \leqslant r$.) The remaining entries in the latin rectangle are used in $\frac{(4r-t)(s-r)}{9} + \frac{r(4r-s)}{9}$ lots of trade T_3^a.

(ii) Let $r \equiv s \equiv t \equiv 1 \pmod 3$. If $4 \leqslant r$, use up the remaining bottom entries in: $\frac{2(s-r)}{3}$ lots of T_4^a on the entries in columns 1 to 4; $\frac{s-r}{3}$ lots of T_6^b and $\frac{(s-r-3)(s-r)}{9}$ lots of T_6^a on the entries in columns 5 to s. The rest of the entries in the latin rectangle are used in $\frac{2(4r-s)}{3}$ lots of T_2^a and $\frac{(r-4)(4r-s)}{9} + \frac{(s-r)(4r-t)}{9}$ lots of T_3^a. For $(r,s,t) = (1,4,4)$, just use two lots of trade T_4^a.

(iii) Let $r \equiv s \equiv t \equiv 2 \pmod 3$. If $6 \leqslant s-r$, use up the bottom entries in columns $s - 6\lfloor\frac{s-r}{6}\rfloor + 1$ to s in $(t-r)\lfloor\frac{s-r}{6}\rfloor$ lots of T_4^a. If $s-r$ is even, all entries in E have been used, so let $q = r$. If $s-r$ is odd, the entries in exactly 3 columns of E are unused, so use $\frac{t-s}{3}$ lots of T_6^a to take care of the entries in rows $s+1$ to t of columns $r+1, r+2, r+3$ and let $q = r+3$. The unused bottom entries are precisely those in columns 1 to q and rows $r+1$ to s. If q is even, use $\frac{q(s-r)}{6}$ lots of T_4^a; if q is odd, use $\frac{s-r}{3}$ lots of T_6^a on the entries in columns 1 to 3 and $\frac{(q-3)(s-r)}{6}$ lots of T_4^a on the rest. The remaining entries in the latin rectangle can be used in T_2^a and T_3^a trades. The case $(r,s,t) = (2,5,8)$ is covered in the Appendix.

Case 2: $s \equiv t \equiv 0 \not\equiv r \pmod 3$. Here $t < 4r$. We use the standard latin representation with $u = 0$. Use up all of C and all entries in columns 1 to r and rows $s+1$ to t in $r(t-s)/3$ lots of trade T_2^b. Since $s-r$ can be written as $3a+2b$, where $b = 0, 1$ or 2, use $a(t-s)/3$ lots of T_6^a and $b(t-s)/3$ lots of T_4^a to use up the remaining entries in rows $s+1$ to t in the bottom and the entries in rows 1 to $(t-s)/3$ of columns $r+1$ to s in the latin rectangle.

If $r \equiv 1 \pmod 3$, we use $s/3$ lots of T_5^a on rows $r, r+1, r+2$ and $s(s-r-2)/9$ lots of T_6^a; if $r \equiv 2 \pmod 3$, we use $s/3$ lots of T_4^d on rows $r, r+1$ and $s(s-r-1)/9$ lots of T_6^a. Finish the decomposition using T_2^a and T_3^a trades.

Case 3: $r \equiv s \equiv 0 \pmod 3$, $t \equiv 1 \pmod 3$. So $t \leqslant 4r-2$. Use the latin representation with $u = 0$. Use columns $s+2$ to t of C and the entries in rows $s+2$ to t and columns 1 to r of the bottom in $r(t-s-1)/3$ lots of trade T_2^b. Use the entries in column $s+1$ of C in T_1 trades with the entries in rows s and $s+1$ of columns 1 to r of the bottom. For the other entries in the bottom of columns 1 to r, use $r/3$ lots of T_5^b and $r(s-r-3)/9$ lots of T_6^a.

To finish up the rest of the bottom, use $(s-r)/3$ lots of T_4^c and $(s-r)(t-r-1)/9$ lots of T_6^a. The remaining entries in the latin rectangle can be used in T_3^a trades.

Case 4: $r \equiv s \equiv 0 \pmod 3$, $t \equiv 2 \pmod 3$. Then $t < 4r$ and $u = 0$ in the standard latin representation. Use columns $s+3$ to t of C and the entries in rows $s+3$ to t and columns 1 to r of the bottom in $r(t-s-2)/3$ lots of trade T_2^b. Use the entries in columns $s+1$ and $s+2$ of C in T_1 trades with the entries in rows $s-1$ to $s+2$ of columns 1 to r of the bottom. Use $r/3$ lots of T_4^c and $r(s-r-3)/9$ lots of T_6^a to finish off the entries in columns 1 to r of the bottom.

To finish columns $r+1$ to s of the bottom, use $(s-r)/3$ lots of T_5^b and $(s-r)(t-r-2)/9$ lots of T_6^a. The remaining entries in the latin rectangle can be used in T_3^a trades.

Case 5: $r \equiv t \equiv 0 \pmod 3$, $s \equiv 1 \pmod 3$. Then $u = r+1$ in the latin representation. Use $r(t-s-2)/3$ lots of trade T_2^b on the entries in rows 1 to r and columns $s+3$ to t of C and the entries in columns 5 to $r+4$ and rows $s+3$ to t of D and E. Use $r/3$ lots of trade T_4^b on the entries in rows 1 to r and columns $s+1, s+2$ of C and the entries in columns 5 to $r+4$ and rows $s+1, s+2$ of D and E. There are $s-r \equiv 1 \pmod 3$ unused bottom entries in each of the columns 5 to $r+4$, so use $(r-3)/3$ lots of trade T_4^d, one T_4^c trade, and $r(s-r-1)/9$ lots of trade T_6^a.

There are still $t-r$ unused bottom entries in each of the columns 1 to 4. Use $2(t-r)/3$ lots of trade T_4^a to finish these off.

Use up the $t-r$ unused bottom entries in each of the columns $r+5$ to s in $(s-r-4)(t-r)/9$ lots of trade T_6^a. Any unused entries in the latin rectangle can be used in T_2^a and T_3^a trades.

Case 6: $r \equiv t \equiv 0 \pmod 3$, $s \equiv 2 \pmod 3$. Use the latin representation with $u = r+2$.

Use $r(t-s-1)/3$ lots of trade T_2^b on all the entries in columns $s+2$ to t of C and the entries in columns 1 to r and rows $s+1$ to t of D. Use r lots of trade T_1 on the entries in column $s+1$ of C and the entries in columns 1 to r and rows $r+2$ and $s+1$ of D. All entries of C have been used. On columns 1 to r of D, use $r/3$ lots of trade T_4^d and $r(s-r-2)/9$ lots of trade T_6^a. For columns $r+1$ and $r+2$, use $(t-r)/3$ lots of trade T_4^a. Use $(t-r)(s-r-2)/9$ lots of trade T_6^a to finish the bottom entries. Any remaining entries in the latin rectangle can be used in T_2^a or T_3^a trades. □

5.2 $t > 4r$, t even

Starting with the standard latin representation, the basic steps in each of the theorems in this section are:

1. use A to take care of $3r$ rows of D and use the minimum number of elements of C to finish off the elements in D with T_1 trades;
2. spread the remaining elements of C as evenly as possible over the $s-u$ columns of E in a cyclic fashion: pairing the r entries $s+1$ from C with the r entries $s+1$ in columns $u+1$ to $u+r$ in E, the r entries $s+2$ in C with the entries $s+2$ in columns $u+r+1$ to $u+2r$ in E, wrapping around to column 1 again when necessary to create the matching pairs needed for T_1 trades;
3. allocate the additional points from E needed for the T_1 trades;
4. use the rest of the entries in E in trades with elements of B (relatively easy because the number of columns in B and E is divisible by 3);
5. use T_3^a and T_2^a trades to use up remaining entries in A and B.

NOTE: In the proof of Theorem 5.3, we provide specific details needed to complete each step. In the proofs of the subsequent theorems, we provide details only where the steps differ from those in Theorem 5.3.

Lemma 5.2 *If (r, s, t) is admissible and t is even, then*

$$t + r - 2\lfloor \tfrac{tr}{s} \rfloor \leqslant 3r, \qquad t + r - 2\lfloor \tfrac{r(t+2r)}{2(s-r)} \rfloor \leqslant 3r,$$

$$t + r - 2\lfloor \tfrac{rt+2r^2+4r-2t}{2(s-r-2)} \rfloor \leqslant 3r, \qquad t + r - 2\lfloor \tfrac{rt+2r^2+2r-t}{2(s-r-1)} \rfloor \leqslant 3r.$$

PROOF. Since (r, s, t) is admissible, $t(s-2r) \leqslant 2rs$, so $\frac{t-2r}{2} \leqslant \frac{tr}{s}$. However, t is even, so $\frac{t-2r}{2}$ is an integer and thus $\frac{t-2r}{2} \leqslant \lfloor \frac{tr}{s} \rfloor$ also. Therefore, $t + r - 2\lfloor \frac{tr}{s} \rfloor \leqslant 3r$.

The other inequalities are proved in a similar fashion. □

Theorem 5.3 *If* (r, s, t) *is admissible with* $2r < s < 4r < t$, $s \equiv 0 \pmod 3$, $r \not\equiv 2 \pmod 3$, *and* t *even, then* $K(r, s, t)$ *has a decomposition into gregarious 3-paths.*

PROOF. Use the standard latin representation with $u = 0$. Then Step 1 is omitted and the columns of the latin representation have been partitioned into triples.

Step 2: Since $s \equiv 0 \pmod 3$, at least one of r or $t \equiv 0 \pmod 3$, so $r(t-s) \equiv 0 \pmod 3$. The cyclic distribution of elements of C will end on the boundary between 2 column-triples. Each column gets either

$$\lfloor \tfrac{r(t-s)}{s} \rfloor = \lfloor \tfrac{rt}{s} \rfloor - r \qquad \text{or} \qquad \lceil \tfrac{rt}{s} \rceil - r \qquad \text{elements of } C.$$

Step 3: If $r \equiv 0 \pmod 3$, then all columns in a given column-triple have been assigned exactly the same elements of C. To complete the T_1 trades, continue to respect the column-triples when assigning the last entry (i.e., pick entries in the same row for all 3 entries in a single row of a column-triple), but the elements in any row may be used.

If $r \not\equiv 0 \pmod 3$, then the columns in some triples of E may have been assigned different elements of C (the indicated entries are the ones that have been matched):

i	i	
		$i+1$

or

i		
	$i+1$	$i+1$

To complete each of these T_1 trades, just use whichever of i or $i+1$ is still available in the column. As in the $r \equiv 0 \pmod 3$ case, any row can be used to complete the T_1 trades when all 3 columns in a column-triple have been matched with the same element of C.

Step 4: There are either

$$t - r - 2\lfloor \tfrac{rt}{s} \rfloor - 2r = t + r - 2\lfloor \tfrac{rt}{s} \rfloor \qquad \text{or} \qquad t + r - 2\lceil \tfrac{rt}{s} \rceil$$

unused bottom entries in each column and all columns in a triple have the same number of unused bottom entries.

By Lemma 5.2, each column contains at most $3r$ unused elements. Each column-triple contains only complete rows of unused entries. If $r \equiv 1 \pmod 3$, use a T_6^b trade on the entries in rows $r, r+1, r+2, r+3$ of each column triple. Then, in both cases, complete the decomposition by using as many T_6^a trades on each column triple as possible, ending with a T_5^b or a T_4^c, if necessary. □

Theorem 5.4 *If (r, s, t) is admissible with $2r < s < 4r < t$, $s \equiv 0 \pmod 3$, $r \equiv 2 \pmod 3$, and t even, then $K(r, s, t)$ has a decompostion into gregarious 3-paths.*

PROOF. First, we note that under the conditions of the theorem, $s \leqslant 4r-2$, $u = 0$ in the standard latin representation and the columns of B and E have been partitioned into triples.

If $r = 2$, the only possibility is $(r, s, t) = (2, 6, 12)$, which is given in the Appendix. If $r \geqslant 5$ and $s = 4r - 2$, then $t \leqslant \lfloor \frac{2r(4r-2)}{2r-2} \rfloor = 4r + 2 \equiv 10 \pmod{12}$ and so, in fact, $t \leqslant 4r-2$. This case is covered by Theorem 5.1. Henceforth, assume that $r \geqslant 5$ and $s \leqslant 4r - 5$.

Step 4: If $t + r - 2\lfloor \frac{rt}{s} \rfloor \leqslant 3r - 1$, use a T_5^b trade on the entries in rows $r, r+1, r+2$ of each column triple. Then complete the decomposition by using as many T_6^a trades on each column triple as possible, ending with a T_5^b or a T_4^c, if necessary.

Now suppose that $t + r - 2\lfloor \frac{rt}{s} \rfloor = 3r$. Then $t + r - 2\lceil \frac{rt}{s} \rceil = 3r$ or $3r - 2$ depending on whether $\frac{rt}{s} \in \mathbb{Z}$ or not. In any event, $\frac{t-2r}{2} \leqslant \frac{rt}{s} < \frac{t-2r+2}{2}$, so $2s(r-1) < t(s-2r) \leqslant 2sr$. We have to revisit the T_1 trades above and make sure that we use up the entries in row $r+1$ in these trades. If each column-triple has at least one row where all 3 columns are matched with the same entry, then we can use the entries in row $r+1$ to complete these T_1 trades. If there are column-triples which do not have rows of this type, we can shift things around provided there are enough of these rows in some column-triple.

There are $(r-2)/3$ column-triples where all entries in the same row are matched with the same entry; there are $t-s$ different types of entries in C, so we need $(t-s)(r-2)/3 \geqslant s/3$ (the number of column-triples). However, $s \leqslant 4r - 5$ implies that $s - 2r < 2(r-2)$ and so

$$s = s(r-1) - s(r-2) < \frac{t(s-2r)}{2} - s(r-2) < t(r-2) - s(r-2) = (t-s)(r-2).$$

Once we have adjusted this assignment of elements of C, we can use up all the entries in row $r+1$ in the T_1 trades. Finish using the entries in E in T_6^a trades. □

Theorem 5.5 *If (r,s,t) is admissible with $2r < s < 4r < t$, $r \equiv s \not\equiv 0 \pmod 3$, and t even, then $K(r,s,t)$ has a decomposition into gregarious 3-paths.*

PROOF. Note that $t \equiv s \pmod 3$ also and that the columns of B and E have been partitioned into triples.

Step 1: Use the last $\frac{t-4r}{2}$ entries in each column of D with all the entries in the last $\frac{t-4r}{2}$ columns of C as the matching items in $\frac{t-4r}{2}$ lots of trade T_1. For the third elements in these trades, use the entries in rows $r+1,\ldots,r+\frac{t-4r}{2}$ of D. There are $3r$ unused entries in each column of D. Since $r \geqslant 2$, $r = 2a+3b$, where $a = 1$ or 2, so the columns of A and D can be partitioned into triples together with either a pair or a quadruple. Then ar lots of trade T_4^a and br T_6^a trades use up all the remaining entries in A and D.

Step 2: There are $t-s-\frac{t-4r}{2}$ columns of unused elements in C. Since both $t-s-\frac{t-4r}{2}$ and $s-r \equiv 0 \pmod 3$, the cyclic distribution of elements of C ends on the boundary between two column-triples in E. Each column gets either

$$\left\lfloor \frac{r(t/2-s+2r)}{s-r} \right\rfloor = \left\lfloor \frac{r(t+2r)}{2(s-r)} \right\rfloor - r \qquad \text{or} \qquad \left\lceil \frac{r(t+2r)}{2(s-r)} \right\rceil - r \qquad \text{elements of } C.$$

Step 4: After completing the T_1 trades, the number of unused elements in each column of E is either

$$t+r-2\left\lfloor \frac{r(t+2r)}{2(s-r)} \right\rfloor \qquad \text{or} \qquad t+r-2\left\lceil \frac{r(t+2r)}{2(s-r)} \right\rceil.$$

By Lemma 5.2, each column contains $3r$ or fewer unused elements, so we can finish in the same way as in Theorem 5.3 if $s \not\equiv 2 \pmod 3$. If $r \equiv s \equiv 2 \pmod 3$, then $t \equiv 2 \pmod 6$ and we can finish as in Theorem 5.4 once we make the following observations for the case where $t+r-2\lfloor \frac{r(t+2r)}{2(s-r)} \rfloor = 3r$.

If $r = 2$, then the only possibility is $(r,s,t) = (2,5,20)$ which is given in the Appendix.

If $r \geqslant 5$, then $s \leqslant 4r-3$. However, if $s = 4r-3$, then $t \leqslant \lfloor \frac{2r(4r-3)}{2r-3} \rfloor = 4r+3+\lfloor \frac{9}{2r-3} \rfloor \leqslant 4r+4$, which actually implies that $t \leqslant 4r$. So we may assume that $s \leqslant 4r-6$.

When the last $r(t-s-\frac{t-4r}{2})$ entries of C are distributed over the $s-r$ columns of E, there are $\frac{(r-2)}{3}(t-s-\frac{t-4r}{2})$ instances where all three columns of a column-triple are assigned the same entry from C. However, $t > 4r$ and $4r-6 \geqslant s$ imply that $t-2s+4r > 8r-2s > 6$ and $6(r-2) \geqslant 2(s-r)$, so $\frac{(r-2)}{3}(t-s-\frac{t-4r}{2}) \geqslant \frac{s-r}{3}$ (the number of column-triples). □

Theorem 5.6 *If (r,s,t) is admissible with $2r < s < 4r < t$, $s \equiv 2 \pmod 3$, $r \equiv 0 \pmod 3$, and t even, then $K(r,s,t)$ has a decompostion into gregarious 3-paths.*

PROOF. Note that $t \equiv 0 \pmod 6$ and in the standard latin representation, $u = r+2$. First suppose that $r > 3$.

Step 1: Use the r entries in each of the last $\frac{t-4r}{2}$ columns of C in T_1 trades with the entries in columns 1 to r and rows $\frac{t+4r+2}{2}$ to t of D plus the entries in rows $r+1$ to $\frac{t-2r}{2}$ of the same columns in D. Use the entries in rows 1 and 2 of columns $4r+1$ to $\frac{t+4r}{2}$ of C in T_1 trades with the entries in columns $r+1$ and $r+2$ and rows $4r+1$ to $\frac{t+4r}{2}$ of D and the entries in rows $r+1$ to $\frac{t-2r}{2}$ of the same columns in D. Each column, 1 to $r+2$, has $t-r-(t-4r) = 3r$ unused bottom entries which can be used in T_4^a (for columns $r+1$ and $r+2$) and T_6^a trades.

Step 2: In C, we have used $(r+2)\frac{t-4r}{2}$ entries, leaving $r(t-s)-(r+2)\frac{t-4r}{2}$ entries to distribute among the $s-r-2$ columns of E to use in T_1 trades. Since $r(t-s)-(r+2)\frac{t-4r}{2}$ and $(s-r-2) \equiv 0 \pmod 3$, the cyclic distribution of these entries ends on the boundary between two column-triples.

Step 4: So there are $t-r-2\lceil\frac{rt-2rs+4r^2-2t+8r}{2(s-r-2)}\rceil = t+r-2\lceil\frac{rt+2r^2+4r-2t}{2(s-r-2)}\rceil$ or $t+r-2\lfloor\frac{rt+2r^2+4r-2t}{2(s-r-2)}\rfloor$ unused bottom elements in each column of E. By Lemma 5.2, $t+r-2\lfloor\frac{rt+2r^2+4r-2t}{2(s-r-2)}\rfloor \leqslant 3r$, so all columns have at most $3r$ unused bottom elements. Since $r \equiv 0 \pmod 3$, there are no problematic rows in E, so we can finish as in Theorem 5.3.

Now suppose that $r = 3$. Then $(r,s,t) = (3,8,18), (3,8,24)$.

Cyclically distribute all the entries of C across all columns of D and E, starting with the entry 9 in rows 1 to 3. The columns are partitioned into 2 column-triples (columns 1 to 3 and columns 4 to 6) and one column-pair. Select the third entry for each T_1 trade in the usual way ensuring that the entries in rows 4 and 5 in each column are used. Finish using the entries in columns 1 to 6 using T_6^a trades plus two T_4^c trades if $t = 18$. There are 9 unused entries in each of columns 7 and 8, so use three T_4^a trades to finish. □

Theorem 5.7 *If* (r, s, t) *is admissible with* $2r < s < 4r < t$, $s \equiv 1 \pmod 3$, $r \equiv 0 \pmod 3$, *and* t *even, then* $K(r, s, t)$ *has a decomposition into gregarious 3-paths.*

PROOF. Note that $t \equiv 0 \pmod 6$ and that $u = r + 1$ in the standard latin representation.

Step 1: Perform T_1 trades on the following sets of entries: rows 1 to $r - 1$ and columns $\frac{t+4r+2}{2}$ to t of C with columns 1 to $r - 1$ and rows $r + 1$ and $4r + 2$ to t of D; row r and columns $\frac{t+4r+2}{2}$ of C to t with column r and rows $4r + 1$ to t of D; and row 1 and columns $4r + 1$ to $\frac{t+4r}{2}$ of C with column $r + 1$ and rows $4r + 1$ to t of D.

Finish using the entries in columns 1 to $r+1$ by using $r(r-3)/3$ lots of trade T_6^a on columns 1 to $r-3$; $2r$ lots of trade T_4^a on columns $r-2, r-1, r, r+1$.

Step 2: There are $r(t - s) - (r + 1)\frac{t-4r}{2}$ (which is congruent to 0 (mod 3)) unused entries in C, which can be distributed evenly over the $s - r - 1$ other columns to be used in T_1 trades in the same manner as before.

Step 4: By Lemma 5.2, $t - r - 2\lfloor\frac{rt-2rs+4r^2-t+4r}{2(s-r-1)}\rfloor \leqslant 3r$, so each column has at most $3r$ unused bottom entries. Use trades T_6^a plus T_5^b or T_4^c, if necessary, to use up these entries. □

5.3 $t > 4r$, t odd

In this section, we use the methods of the previous section after using two new combinations, T_{15} and T_{10}, of T_1 and T_2^c trades to use up 4 entries in each column of D and E. For ease of reference, label the rows of D and E with $u + 1, u + 2, \ldots, t$, and the columns of C with $s + 1, \ldots, t$. In the standard latin representation, all entries in row k of D and E, and column k of C, are equal to k, for all $s + 1 \leqslant k \leqslant t$.

The letter a denotes the *cells* whose entries are used in a single T_2^c trade (rather than the entries themselves). Similarly, the letters $b, \ldots, f$ indicate the cells used in other T_2^c trades. The T_1 trades in T_{15} use 3 entries in column k of C with entries in columns c_1, c_3, c_4 of rows k and $j + 2$; the T_1 trades in T_{10} use the entries in columns c_1, c_4 of rows k and l plus 2 entries from column k or from column l of C or one from each.

T_{15} :

	c_1	c_2	c_3	c_4	c_5	c_6
1	a	f	b	c	e	d
j	a	a	b	c	c	d
$j+1$	a	b	b	c	d	d
$j+2$		f			e	f
k		f			e	e

T_{10} :

	c_1	c_2	c_3	c_4
1	a	d	c	b
j	a	a	b	b
$j+1$	a	c	c	b
k		d	c	
l		d	d	

Lemma 5.8 *If (r,s,t) is admissible and t is odd, then*

(i) $t+r-4-2\lfloor\frac{2rt-s}{2s}\rfloor \leqslant 3r-3$; and

(ii) $t+r-4-2\lfloor\frac{2rt-s-3}{2s}\rfloor \leqslant 3r-3$, if $s \equiv 3 \pmod 6$.

PROOF. We prove (ii); part (i) is similar. Suppose that $t+r-4-2\lfloor\frac{2rt-s-3}{2s}\rfloor > 3r-3$, so that $t+r-4-2\lfloor\frac{2rt-s-3}{2s}\rfloor \geqslant 3r-2$. However, t is odd which means that $t+r-4-2\lfloor\frac{2rt-s-3}{2s}\rfloor \geqslant 3r-1$.

Then $\lfloor\frac{2rt-s-3}{2s}\rfloor \leqslant \frac{t-2r-3}{2} \in \mathbb{Z}$ and so $\frac{2rt-s-3}{2s} < \frac{t-2r-3}{2}+1$. Therefore, $2rs-3 < t(s-2r) \leqslant 2rs$. (The latter inequality follows since (r,s,t) is admissible.)

Here $s \equiv 3 \pmod 6$, so at least one of r, $t \equiv 0 \pmod 3$, and both $2rs$, $t(s-2r) \equiv 0 \pmod 3$, so from above, $t(s-2r) = 2rs$. However, s and t are both odd, so we have a contradiction. Hence $t+r-4-2\lfloor\frac{2rt-s-3}{2s}\rfloor \leqslant 3r-3$, as required. □

Theorem 5.9 *If (r,s,t) is admissible with $2r < s < 4r < t$, $s \equiv 0 \pmod 3$ and t odd, then $K(r,s,t)$ has a decomposition into gregarious 3-paths.*

PROOF. When $s \equiv 0 \pmod 3$ and t is odd, $t+r-2\lfloor\frac{tr}{s}\rfloor \leqslant 3r+1$. By parity, $t+r-2\lfloor\frac{tr}{s}\rfloor \neq 3r$, so we can use the methods of Theorems 5.3 and 5.4 unless $t+r-2\lfloor\frac{tr}{s}\rfloor = 3r+1$. This would imply that some columns, say exactly β of them $(0 < \beta < s)$, would contain $3r+1$ unused elements in Step 4 and we could not complete this decomposition.

Suppose that $t+r-2\lfloor\frac{tr}{s}\rfloor = 3r+1$. This means that $\lfloor\frac{tr}{s}\rfloor = \frac{t-2r-1}{2}$, $\frac{tr}{s} \notin \mathbb{Z}$, t must be odd and $2rs-s < t(s-2r) \leqslant 2rs$. No case with $r=2$ meets these conditions. The only ones with $r < 5$ are $(r, s, t) = (3, 9, 17), (4, 9, 69)$. Use the method below for $(3, 9, 17)$; the case $(4, 9, 69)$ is in the Appendix.

If $r \geqslant 5$ or $(r, s, t) = (3, 9, 17)$, then

$$\beta(3r + 1) + (s - \beta)(3r - 1) = (t - r)s - 2r(t - s),$$

which implies that $2\beta = t(s - 2r) - 2rs + s \leqslant s < 4r$, and since at least one of t or $r \equiv 0 \pmod 3$, $2\beta \equiv 0 \pmod 6$. We start again setting up the trades on the standard latin representation.

First make a change to the latin representation to make it easier to specify which trades are used. Since either $(r, s, t) = (3, 9, 17)$ or $r \geqslant 5$, $2r < s$ and $s \equiv 0 \pmod 3$, we have $s - r \geqslant 6$. The last six rows of B^* (rows $s-5, \ldots, s$) consist of the blocks below.

$B_{s/3-1}$	$B_{s/3}$	B_1	$\cdots$	$B_{s/3-2}$
$B_{s/3}$	B_1	B_2	$\cdots$	$B_{s/3-1}$

Since these are outside the latin rectangle, the entries need only be column-latin, so rearrange these by switching the blocks in every second column as below.

$B_{s/3-1}$	B_1	B_1	B_3	$\cdots$
$B_{s/3}$	$B_{s/3}$	B_2	B_2	$\cdots$

Let r_1 be such that $r \equiv r_1 \pmod 3$ and $0 \leqslant r_1 < 3$. Use $(r - r_1)/3$ lots of T_{15}, with $j = s - 2$ and $k = t$, on columns 1 to $2(r - r_1)$. Use $\lfloor \frac{s}{6} \rfloor - \frac{r-r_1}{3}$ lots of T_{15}, with $j = s - 2$ and $k = t - 1$, on columns $2(r - r_1) + 1$ to $6\lfloor \frac{s}{6} \rfloor$. Note that we have used T_{15} on at least 2β columns of E and that we have used the same rows in all columns of each column-triple. If $s \equiv 0 \pmod 6$, then we have used all entries in row 1 of B and $s/2$ elements of C. If $s \equiv 3 \pmod 6$, we have used the entries in columns 1 to $s - 3$ of row 1 in B and $(s - 3)/2$ entries of C.

If $s \equiv 3 \pmod 6$, to deal with the last 3 columns, use two T_2^c trades on the entries in rows $s - 1$ and s, plus row 1 columns $s - 2$ and s. Use three T_1 trades on 2 rows (these can be rows $t - 1$ and t unless $s = 4r - 1$) and 3 entries of C. So we have used a total of $(s + 3)/2$ entries in C.

Use Steps 2 and 3.

<u>Step 4:</u> The number of unused bottom entries per column is:

$$\begin{array}{lll} t + r - 4 - 2\lceil \frac{2rt-s}{2s} \rceil & \text{or} \quad t + r - 4 - 2\lfloor \frac{2rt-s}{2s} \rfloor & \text{if } s \equiv 0 \pmod 6 \\ t + r - 4 - 2\lceil \frac{2rt-s-3}{2s} \rceil & \text{or} \quad t + r - 4 - 2\lfloor \frac{2rt-s-3}{2s} \rfloor & \text{if } s \equiv 3 \pmod 6. \end{array}$$

Each of these, by Lemma 5.8, is less than or equal to $3r - 3$. Finish the decomposition as in Theorems 5.3 and 5.4. □

Theorem 5.10 *If (r, s, t) is admissible with $2r < s < 4r < t$, $s \not\equiv 0 \pmod 3$ and t odd, then $K(r, s, t)$ has a decomposition into gregarious 3-paths.*

PROOF. All cases are done in a similar manner. For all T_{15} trades, except one in Case 3, use $j = s - 2$ and $k = t - 1$ or t; all T_{10} trades use the four rows $s - 1, s, t - 1$ and t. We give complete details in Case 1 and then indicate any changes for the others. Any unused entries left in the latin rectange at the end can be used in T_3^a trades.

Case 1: Suppose that $s \equiv 4 \pmod 6$. Then $u \equiv 1 \pmod 3$, $u \geqslant 4$ and $t \geqslant 4r + 3$. In addition, either $r \equiv t \equiv 1 \pmod 3$ and $2r + 2 \leqslant s \leqslant 4r - 6$ or $r \equiv t \equiv 0 \pmod 3$ and $2r + 4 \leqslant s \leqslant 4r - 2$.

On columns 5 to s, use $\frac{s-4}{6}$ lots of T_{15}. For $\lfloor \frac{r}{3} \rfloor$ of these use $k = t$ and for the remaining $\frac{s-4}{6} - \lfloor \frac{r}{3} \rfloor$, use $k = t - 1$. If $r \equiv 0 \pmod 3$, there will be no t entries and at least three $t - 1$ entries left in C; if $r \equiv 1 \pmod 3$, there will be one t entry and at least four $t - 1$ entries left in C.

On columns 1 to 4, use a T_{10} trade with two $t - 1$ entries from C if $r \equiv 0 \pmod 3$ and with one t entry and one $t - 1$ entry from C if $r \equiv 1 \pmod 3$. All the t entries in C have been used and there are $r(t - s) - \frac{s}{2}$ unused entries in C.

Distribute $\frac{s(t-4r-1)}{2}$ of the remaining C elements over the s columns, in the usual cyclic fashion, to create the matching pairs for T_1 trades. This is possible since $2rs \geqslant t(s - 2r)$. Note that if $r \equiv 0 \pmod 3$, $r \geqslant 6$, the first r columns of D are matched with $s + 1$, while the last column of D is matched with $s + 2$; so in this case, switch this around so that the last 3 columns of D are matched with $s + 2$ and the first two columns of E with $s + 1$; use the entries in row $r + 1$ of D to complete the T_1 trades using these $s + 1$ and $s + 2$ in D. Now select the third entry of the bottom for the remaining T_1 trades, respecting column-pairs in columns 1 to 4 and respecting column triples elsewhere. If $r \equiv 1 \pmod 3$, avoid using entries from rows $r + 1, r + 2$. Each column in D and E now has $3r - 3$ unused bottom entries. For columns 1 to 4, use the remaining bottom entries in $2(r - 1)$ lots of trade T_4^a.

There are $\frac{2rs-t(s-2r)}{2} \equiv 0 \pmod 3$ unused entries in C to be used in more T_1 trades. Distribute them over columns 5 to s, so all colums in a column triple get the same number of matches and each column is matched with either

$$\left\lfloor \frac{2rs-t(s-2r)}{2(s-4)} \right\rfloor \quad \text{or} \quad \left\lceil \frac{2rs-t(s-2r)}{2(s-4)} \right\rceil$$

elements of C. In all cases, $2\lceil\frac{2rs-t(s-2r)}{2(s-4)}\rceil \leqslant 3r-3$. Use the remaining bottom elements in as many T_6^a or T_6^b trades as possible, finishing with T_4^c, T_4^d, T_5^a or T_5^b if needed.

Case 2: Suppose that $s \equiv 2 \pmod 6$. Then $u \equiv 2 \pmod 3$ and $t \geqslant 4r+3$. Also, $r \equiv t \equiv 2 \pmod 3$ and $2r+4 \leqslant s \leqslant 4r-6$ or $r \equiv t \equiv 0 \pmod 3$ and $2r+2 \leqslant s \leqslant 4r-4$. If $r=3$, then $(r,s,t) = (3,8,15),(3,8,21)$, which are given in the Appendix, so assume that $r > 3$.

Use $\frac{s-8}{6}$ T_{15} trades in columns 9 to s and two T_{10} trades on columns 1 to 8 so that r copies of entry t and $\frac{s-2r}{2}$ copies of entry $t-1$ from C are used. If $r \equiv 5 \pmod 6$, interchange the entries in columns $r+1, r+2, r+3$ of rows $s-r-2, s-r-1, s-r$ with those in rows $s-2, s-1, s$ to make this possible.

Distribute $\frac{s(t-4r-1)}{2}$ of the remaining C elements over the s columns, in the usual cyclic fashion, to create the matching pairs for T_1 trades and complete the trades respecting the column-pair, $(1,2)$, and column-triples elsewhere. Each column now has $3r-3$ unused bottom entries. Use r-1 T_4^a trades to use up the bottom entries in columns 1 and 2.

Distribute the remaining $\frac{2rs-t(s-2r)}{2} \equiv 0 \pmod 3$ entries in C over the columns 3 to s and complete the T_1 trades, respecting the column-triples. If $r \equiv 0 \pmod 3$, then use two lots of T_6^b on columns 3 to 8, followed by as many T_6^a trades as possible on the entries in columns 2 to s and T_4^c or T_5^d if needed. If $r \equiv 0 \pmod 3$ and $\lfloor\frac{2rs-t(s-2r)}{2(s-2)}\rfloor \neq 0$, then each column is matched with at least one more entry of C and so there are at most $3r-5$ unused bottom entries in each of the columns 2 to s. Use T_5^a or T_4^d and T_6^a trades to use up the bottom entries.

If $r \equiv 2 \pmod 6$ and $\lfloor\frac{2rs-t(s-2r)}{2(s-2)}\rfloor = 0$, use $\frac{s-r}{2}$ lots of T_4^a on columns $r+1$ to s and rows $2, r+1, r+2, r+3$. If $r \equiv 5 \pmod 6$, $r > 5$, and $\lfloor\frac{2rs-t(s-2r)}{2(s-2)}\rfloor = 0$, use $\frac{s-r-3}{2}$ lots of T_4^a on columns $r+4$ to s and rows $2, r+1, r+2, r+3$ and three lots of T_4^a on the entries in columns $r+1$ to $r+6$ of rows $3, r+5, r+6$ plus row $r+1$ for columns $r+1$ to $r+3$ and row $r+4$ for columns $r+4$ to $r+6$. Finish both cases as above. For $(r,s,t) = (5,14,35)$, see the Appendix.

Case 3: Suppose that $s \equiv 1 \pmod 6$. Then $u \equiv 1 \pmod 3$ and $4r+3 \leqslant t$. Also either $r \equiv t \equiv 1 \pmod 3$ and $2r+5 \leqslant s \leqslant 4r-3$ or $r \equiv t \equiv 0 \pmod 3$ and $2r+1 \leqslant s \leqslant 4r-5$, so $t(s-2r) \leqslant 2rs-3$ and there are no admissible triples in this case with $r < 6$.

On column 1, use a T_1 trade on entries $s-1, t-1$ and a $t-1$ from C plus a T_2^d trade on entries in rows $1, s, t$ and a t from C. Use $\frac{s-7}{6}$ lots of T_{15} on columns 8 to s, $\lfloor\frac{r-1}{3}\rfloor$ of them using t and the rest $t-1$; use T_{15} on columns

2 to 7 using rows $1, s-1, s, t-1, t$ plus three $t-1$ entries from C, if $r \equiv 1 \pmod 3$, and two t entries and one $t-1$ entry from C, if $r \equiv 0 \pmod 3$.

Distribute $\frac{s(t-4r-1)}{2}$ of the remaining C elements over the s columns, in the usual cyclic fashion, to create the matching pairs for T_1 trades. Columns 1 to 3 have each been matched with $s+1$, so complete these T_1 trade using the entries from row $r+1$. Complete the rest of the T_1 trades respecting the column-pairs $(4,5)$ and $(6,7)$ and the column-triples elsewhere. Each column has $3r-3$ unused bottom entries. Use up the rest of the bottom entries in columns 4 to 7 in $2(r-1)$ lots of trade T_4^a.

Distribute the remaining $\frac{2rs-t(s-2r)-3}{2}$ C entries among the $s-4$ other columns (note that $0 \leqslant \frac{2rs-t(s-2r)-3}{2} \leqslant \frac{(3r-3)(s-4)}{2}$ for all admissible (r,s,t) in this case) and complete the T_1 trades respecting the column-triples. Finish using the bottom entries in as many T_6^a trades as possible, ending with T_4^c or T_5^b trades if needed.

Case 4: Suppose that $s \equiv 5 \pmod 6$. Then $u \equiv 2 \pmod 3$ and $4r+3 \leqslant t$. Also either $r \equiv t \equiv 2 \pmod 3$ and $2r+1 \leqslant s \leqslant 4r-3$ or $r \equiv t \equiv 0 \pmod 3$ and $2r+5 \leqslant s \leqslant 4r-7$, so $t(s-2r) \leqslant 2rs-3$ and there are no admissible triples in this case with $r < 5$.

On column 1, use a T_1 trade on entries $s-1, t-1$ and a $t-1$ from C plus a T_2^d trade on entries in rows $1, s, t$ and a t from C. Use $\frac{s-5}{6}$ lots of T_{15} on columns 6 to s, $\lfloor \frac{r-2}{3} \rfloor$ of them using t and the rest $t-1$; a T_{10} trade on columns 2 to 5 using a $t-1$ and a t if $r \equiv 2 \pmod 3$ and two t entries from C if $r \equiv 0 \pmod 3$.

Distribute $\frac{s(t-4r-1)}{2}$ of the remaining C elements over the s columns, in the usual cyclic fashion, to create the matching pairs for T_1 trades. Complete the trades using entries from the rows $s+1$ to t (possible since $t-s-1-(t-4r-1) = 4r-s \geqslant 3$), respecting the column-pair $(1,2)$ and the column-triples elsewhere. Each column now has $3r-3$ unused bottom entries. Finish columns 1 and 2 by using $r-1$ T_4^a trades.

If $r \equiv 2 \pmod 3$ and $\lfloor \frac{2rs-t(s-2r)-3}{2(s-2)} \rfloor = 0$, use the entries in rows $2, r+1, r+2, r+3$ and columns 6 to s in $\frac{s-5}{2}$ lots of T_4^a (possible since $r+3 < s-2$). Each of these columns now has $3r-6 \geqslant 9$ unused bottom entries. Distribute the remaining $\frac{2rs-t(s-2r)-3}{2} \equiv 0 \pmod 3$ unused entries of C across the $s-2$ columns 3 to s (note that some columns will get one, others none) and complete the T_1 trades respecting column-triples. Otherwise, omit the T_4^a trades and just distribute the $\frac{2rs-t(s-2r)-3}{2} \geqslant 3$ unused entries in C over the columns 3 to s and use them in T_1 trades, selecting the third entry for each respecting the column-triples. This is possible since $3r-3 \geqslant \lceil \frac{2rs-t(s-2r)-3}{2(s-2)} \rceil$ for all admissible (r,s,t) satisfying the conditions for this case. We can use up all the remaining bottom entries in T_6^a, T_6^b, T_5^a, T_5^b, T_4^c or T_4^d trades. □

This completes the proof of our main theorem.

Main Theorem: *There exists a gregarious 6-cycle decomposition of the complete tripartite graph $K(\rho,\sigma,\tau)$ if and only if the following two conditions hold:*

(a) *ρ,σ,τ are all even, say $\rho = 2r$, $\sigma = 2s$ and $\tau = 2t$, and*

  (i) $r \equiv s \equiv t \equiv 1 \pmod 3$; *or*

  (ii) $r \equiv s \equiv t \equiv 2 \pmod 3$; *or*

  (iii) *at least two of r, s, t must be* $0 \pmod 3$;

(b) *if $r \leqslant s \leqslant t$, then $s \leqslant 4r$ and $t(s-2r) \leqslant 2rs$.*

Appendix

$K(1,3,6)$ (below left): The subscripts in the latin representation indicate the trades used: entries subscripted 1 form a trade T_6^b, and entries subscripted 2, 3, 4 form three trades T_1 .

$K(2,3,6)$ (below right): T_1 trades are subscripted 1 through 6; remaining entries form two T_3^a trades.

1_1	2_1	3_1	4_2	5_3	6_4
2_1	3_1	1_1			
3_1	1_1	2_1			
4_1	4_1	4_2			
5_4	5_3	5_2			
6_4	6_3	6_1			

1	2	3	4_1	5_2	6_3
2	3	1	4_4	5_5	6_6
3_6	1_4	2_5			
4_1	4_4	4_3			
5_1	5_2	5_5			
6_6	6_2	6_3			

1_7	2_7	3	4	5	6_1	7_2	8_3
2_8	1_8	4	5	3	6_4	7_5	8_6
3_7	3_7	5_1	3_3	4_5			
4_7	4_7	1	1	1			
5_7	5_7	2	2	2			
6_8	6_8	6_1	6_4	6_6			
7_8	7_8	7_2	7_4	7_5			
8_8	8_8	8_2	8_3	8_6			

$K(2,5,8)$ (above): T_1 trades are subscripted 1 through 6; T_4^a trades 7 and 8. The remaining entries form T_5^b and T_3^a trades.

$K(2,5,20)$: Use the same 8 trades as for $K(2,5,8)$ plus 12 lots of T_1 on the remaining 24 unused entries in columns 1 and 2 with the entries $9,\dots,14$ in the side. The remaining entries (below) are used in 2 lots of T_6^a, subscripted 1 and 2, and in 12 lots of T_1.

3_1	4_1	5_1	15_3	16_5	17_7	18_9	19_{11}	20_{13}
4_2	5_2	3_2	15_4	16_6	17_8	18_{10}	19_{12}	20_{14}
1_1	1_1	1_1						
2_1	2_1	2_1						
9_1	9_1	9_1						
10_2	10_2	10_2						
11_2	11_2	11_2						
12_2	12_2	12_2						
13_3	13_4	13_6						
14_5	14_7	14_8						
15_3	15_4	15_{12}						
16_5	16_{10}	16_6						
17_9	17_7	17_8						
18_9	18_{10}	18_{14}						
19_{11}	19_{13}	19_{12}						
20_{11}	20_{13}	20_{14}						

$K(2,6,12)$ (below): All side and starred entries are used in 12 T_1 trades; the rest of the entries form six T_4^a trades.

1	2	3	4	5	6	7	8	9	10	11	12
2	3	1	5	6	4	7	8	9	10	11	12
3_*	1_*	2	6	4_*	5_*						
4_*	4_*	4	2	1_*	1_*						
5	5	5_*	1_*	2	2						
6	6	6_*	3_*	3	3						
7_*	7_*	7	7	7	7						
8_*	8_*	8	8	8	8						
9	9	9_*	9_*	9	9						
10	10	10_*	10_*	10	10						
11	11	11	11	11_*	11_*						
12	12	12	12	12_*	12_*						

$K(3,4,6)$ (below left): $\overline{T_2^c}$ trades are subscripted 1, 2; T_4^a trades are subscripted 3, 4; remaining entries form two T_3^a trades.

$K(3,6,6)$ (below right): T_6^a trades are subscripted 1, 2; remaining entries form four T_3^a trades.

1	2	3	4_1	5_1	6_1
2	3	4	1_2	5_2	6_2
3_3	4_3	1_4	2_4	5_1	6_2
4_3	1_3	2_4	3_4		
5_3	5_3	5_4	5_4		
6_3	6_3	6_4	6_4		

1	2	3	4	5	6
2	3	1	5	6	4
3_1	1_1	2_1	6_2	4_2	5_2
4_1	4_1	4_1	1_2	1_2	1_2
5_1	5_1	5_1	2_2	2_2	2_2
6_1	6_1	6_1	3_2	3_2	3_2

$K(3,8,15)$ (below): In column 5 of the standard latin representation, interchange the entries 3 and 4 with the entries 6 and 7. T_{10} trades are subscripted 0,1; subcripts 2 through 6 each indicate three T_1 trades and subscript 7, two T_1 trades (set up in a cyclic way); use three T_5^b trades and one T_3^a on columns 1 to 6 and two T_4^a on columns 7 and 8.

1_0	2_0	3_0	4_0	5_1	6_1	7_1	8_1		14_1	15_0
2	3	4	5	1	7	8	6	…	14_4	15_0
3	4	5	1	2	8	6	7		14_4	15_1
4	5	1	2	6	1	1	1			
5_2	1_2	2_2	3	7	2	2	2			
6_0	6_0	6_0	6_0	3_1	3_1	3_1	3_1			
7_0	7_0	7_0	7_0	4_1	4_1	4_1	4_1			
8_0	8_0	8_0	8_0	8_3	5_3	5	5			
9_2	9_2	9_2	9	9	9	9	9			
10	10	10	10_3	10_3	10_3	10	10			
11_4	11_5	11_5	11	11	11	11_4	11_4			
12_4	12_5	12_5	12_5	12_6	12_6	12_4	12_4			
13_6	13_7	13_7	13_5	13_6	13_6	13	13			
14_6	14_7	14_7	14_3	14_1	14_1	14_1	14_1			
15_0	15_0	15_0	15_0	15_1	15_1	15_1	15_1			

$K(3,8,21)$: In column 5 of the standard latin representation, interchange the entries 3 and 4 with the entries 6 and 7. Then use two T_{10} trades on rows 1, 6, 7, 19 (columns 1 to 4), 17 (columns 5 to 8) and 21, using three 21 entries and one 19 entry from C. Distribute the entries 9 through 18 and the two remaining 19 entries from C in the usual cyclic manner and select the remaining entry for T_1 trades respecting the column-pair (1,2) and the 2 column-triples, except that 18 in column 5 should be matched with 19 rather than 17 to accommodate one T_{10} trade. Use the three 20 entries in T_1 trades with columns 6,7, 8. This leaves 6 unused entries in each of columns 1 through 5 and four unused entries in columns 6,7, 8. Use two T_4^a, one T_6^b, two T_6^a and one T_4^c.

$K(4,7,7)$ (below left): T_4^a trades are subscripted 1, 2; T_3^a trades 3, 4; a T_5^a and a T_4^c trade are marked 5 and 6, respectively; rest of entries form six T_2^a trades.

1_3	2_3	3_3	4	5	6	7
2_4	3_4	4_4	5	6	7	1
3_6	4_6	5_6	6	7	1	2
4_5	5_5	6_5	7_1	1_1	2_2	3_2
5_5	6_5	1_5	1_1	4_1	3_2	6_2
6_5	1_5	2_5	2_1	2_1	4_2	4_2
7_6	7_6	7_6	3_1	3_1	5_2	5_2

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5_1	6_1	7_2	8_2	1_3	2_3	3_4	4_4
6_1	5_1	8_2	7_2	2_3	1_3	4_4	3_4
7_1	7_1	5_2	5_2	3_3	3_3	1_4	1_4
8_1	8_1	6_2	6_2	4_3	4_3	2_4	2_4

$K(5,8,8)$ (above right): T_4^a trades are subscripted 1 to 4; remaining entries form 16 T_2^a trades.

$K(4,9,69)$: First, 240 lots of trade T_1 are used; the first bottom element for each can be selected in a cyclic manner as in Step 2 of Section 5.2; the freedom of choice is such that after selecting the second entry for each T_1 trade we can have the following (below right) unused entries remaining in the latin representation. Three T_6^b trades are subscripted 1, 2, 3; one T_5^b marked 4; three T_2^c trades, 5, 6, 7; the remaining entries are used in seven T_6^a trades.

1_5	2_6	3_7	4_4	5_4	6_4	7_6	8_6	9_6
2	3	1	5	6	4	8	9	7
3	1	2	6	4	5	9	7	8
4_1	5_1	6_1	7_2	8_2	9_2	1_3	2_3	3_3
5_1	6_1	4_1	8_2	9_2	7_2	2_3	3_3	1_3
6_1	4_1	5_1	9_2	7_2	8_2	3_3	1_3	2_3
7_1	7_1	7_1	1_2	1_2	1_2	4_3	4_3	4_3
8	8	8	2	2	2	5	5	5
9	9	9	3	3	3	6	6	6
11	11	11	12	12	12	10	10	10
13	13	13	14	14	14	15	15	15
18	18	18	16	16	16	17	17	17
20	20	20	21	21	21	19	19	19
67_5	67_6	67_7	23_4	23_4	23_4	24	24	24
69_5	69_6	68_7	25_4	25_4	25_4	26	26	26
						28	28	28
						68_7	69_5	69_6

$K(5,14,35)$: To make the decomposition easier to see, rearrange the entries in rows 6 through 15 of column 5 of the standard latin representation as below. Use 7 lots of T_4^a on rows 2,6,7,10; 7 lots of T_4^a on rows 3,8,9,10; T_{10} on rows 1 and 12–15 of columns 1–4 and 2 copies of the entry 15 in the side; T_{15} on rows 1 and 12–15 of columns 9–14 and the remaining 3 copies of the entry 15 in the side; plus 4 lots of T_2^c, subscripted 1 to 4, on the indicated entries and the entries in the same columns of row 1. Distribute the remaining 100 side entries in the usual cyclic manner over the 14 columns (each gets 7) with the 2 extra entries going to columns 5 and 6. Complete T_1 trades in the usual way, respecting pairs of columns for columns 1 to 8 and triples for columns 9 through 14. Finish up with 8 lots of T_4^a on columns 1 to 8 and 4 lots of T_6^a on columns 9 to 14.

						...							
6	6	6	6	11	11	9	10	14	12	13	8	6	7
7	7	7	7	12	12	12	12	6	6	6	9	9	9
8	8	8	8	13	13	13	13	7	7	7	10	10	10
9	9	9	9	14	14	14	14	8	8	8	11	11	11
10	10	10	10	10	1	1	1	1	1	1	1	1	1
11	11	11	11	6	2	2	2	2	2	2	2	2	2
12	12	12	12	7	3	3_1	3_1	3	3	3	3	3	3
13	13	13	13	8	4	4_2	4_1	4	4	4	4	4	4
14	14	14	14	9_3	5_4	5_2	5_2	5	5	5	5	5	5
15	15	15	15	15_3	15_4	15_4	15_3	15	15	15	15	15	15
						...							

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