Uniquely Radial Trees

C. M. Mynhardt*and J. Wodlinger†
Department of Mathematics and Statistics
University of Victoria, P.O. Box 3060 STN CSC
Victoria, BC, CANADA V8W 3R4

Abstract

A broadcast on a graph G is a function $f : V \to \{0, 1, \ldots, \operatorname{diam} G\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$. The broadcast number of G is the minimum value of $\sum_{v \in V} f(v)$ among all broadcasts f for which each vertex of G is within distance $f(v)$ from some vertex v with $f(v) \geq 1$. This number is bounded above by the radius of G. A graph is uniquely radial if its only minimum broadcasts are broadcasts f such that $f(v) = \operatorname{rad} G$ for some central vertex v, and $f(u) = 0$ if $u \neq v$. We characterize uniquely radial trees.

Keywords: broadcast; dominating broadcast; broadcast domination; radial tree

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1 Introduction

A *broadcast* on a graph G is a function $f : V(G) \to \{0, 1, \ldots, \operatorname{diam} G\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$. The *broadcast number* of G is the minimum value of $\sum_{v \in V} f(v)$ among all broadcasts f for which each vertex of G is within distance $f(v)$ from some vertex v with $f(v) \geq 1$, and is bounded above by the radius of G. A graph is *radial* if its broadcast number is equal to its radius, and *uniquely radial* if its only minimum broadcasts are broadcasts f such that $f(v) = \operatorname{rad} G$ for some central vertex v, while $f(u) = 0$ if $u \neq v$.

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We characterize uniquely radial trees, thus solving a problem posed in [5]. Before stating the characterization in Section 4, we give some definitions and background in Section 2, and some useful results and tools to aid the visualization and exposition of our results in Section 3. The proof of the main result is divided into three lemmas, which are also stated in Section 4. Their proofs are given in Section 6. Section 5 contains two corollaries, the first giving a simple characterization of uniquely radial caterpillars, and the second showing that there is no forbidden subtree characterization of uniquely radial trees.

2 Definitions and background

For undefined concepts see [1, 8]. A *broadcast vertex* is a vertex v for which $f(v) \geq 1$. The set of all broadcast vertices is denoted V_f^+. For $v \in V_f^+$, the *f-neighbourhood* $N_f[v]$ of v is the set $\{u : d(u,v) \leq f(v)\}$, while the *f-private neighbourhood* $\mathrm{PN}_f[v]$ of v consists of all vertices in $N_f[v]$ that are not also in $N_f[w]$ for any $w \in V_f^+ - \{v\}$. A vertex u *hears* a broadcast from $v \in V_f^+$, and v *broadcasts to* u, if $u \in N_f[v]$. A vertex v is *overdominated* if $f(u) - d(u,v) > 0$ for some $u \in V_f^+$.

A broadcast f is a *dominating broadcast* if every vertex hears at least one broadcast. The *cost* of a broadcast f is defined as $\mathrm{cost}(f) = \sum_{v \in V(G)} f(v)$. The broadcast number of G is denoted $\gamma_b(G)$, that is, $\gamma_b(G) = \min\{\mathrm{cost}(f) : f$ is a dominating broadcast of $G\}$. If f is a dominating broadcast such that $f(v) = 1$ for each $v \in V_f^+$, then V_f^+ is a *dominating set* of G, and the minimum cost of such a broadcast is the usual *domination number* $\gamma(G)$. A dominating broadcast f of a graph G for which $\mathrm{cost}(f) = \gamma_b(G)$ is called a *γ_b-broadcast*.

A *diametrical path* (abbreviated *d-path*) of a tree T is a path of length $\mathrm{diam}\, T$. A path is *even* or *odd*, corresponding to the parity of its length. A tree is either *central* or *bicentral*, depending on whether it has one or two (adjacent) central vertices; any d-path of a tree contains its centre. A broadcast f of a tree is called *central* if $V_f^+ = \{v\}$ and $f(v) = \mathrm{rad}\, T$ for some central vertex v of T, otherwise it is called *non-central*. A *stem* of a tree is a vertex adjacent to a leaf, and a *branch vertex* is a vertex of degree at least three. If f is a γ_b-broadcast of $T \neq K_2$ such that V_f^+ contains a leaf u, and v is the stem adjacent to u, then the broadcast g defined by $g(u) = 0$, $g(v) = f(u)$, and $g(w) = f(w)$ otherwise, is a γ_b-broadcast of T such that $|V_g^+| = |V_f^+|$. Therefore we consider only broadcasts without leaves as broadcast vertices.

Erwin [6, 7] was the first to consider the broadcast domination prob-

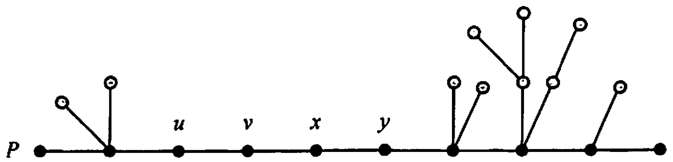

Figure 1: A tree with split-sets $\{uv\}$ and $\{xy\}$

lem, and to observe the trivial bound $\gamma_b(G) \leq \min\{\operatorname{rad} G, \gamma(G)\}$ for any graph G. The problem of characterizing radial trees was first addressed by Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi in [4] and also studied in [5, 12], and was solved by Herke and Mynhardt [11] (see Theorem 3.1). Minimum broadcast domination is solvable in polynomial time for any graph (Heggernes and Lokshtanov [9]) and in linear time for trees (Dabney, Dean and Hedetniemi [3]).

3 Radial trees, shadow trees and isosceles right triangles

A set M of edges of a d-path P of a tree T is a *split-P set* if, for each component T' of $T - M$, the path $P \cap T'$ is a d-path of T' of even, positive length. A *split-set* of T is a split-P set for some d-path P of T, and a *maximum split-set* of T is a split-set of maximum cardinality. For example, $\{uv\}$ and $\{xy\}$ are maximum split-P sets of the tree in Fig. 1. Radial trees are characterized as follows.

Theorem 3.1 [10, 11] *A tree is radial if and only if it has no nonempty split-set.*

The broadcast number of a tree can be expressed in terms of its radius and the cardinality of a maximum split-set.

Theorem 3.2 [10, 11] *For any tree T, let M be a maximum split-set of T of cardinality $m \geq 0$, and $T_1, \dots, T_{m+1}$ the components of $T - M$. Then*

$$\gamma_b(T) = \operatorname{rad} T - \left\lceil \frac{m}{2} \right\rceil = \sum_{i=1}^{m+1} \gamma_b(T_i).$$

We shall also need the following result from [5].

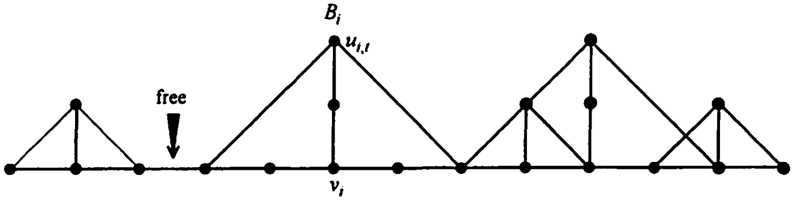

Figure 2: The triangles of a shadow tree

Proposition 3.3 [5] *If T' is a subtree of T, then $\gamma_b(T') \leq \gamma_b(T)$.*

Let $P = v_0, ..., v_d$ be a d-path of the tree T. For each i, let A_i be the set of all vertices of T that are connected to v_i by a (possibly trivial) path that is internally disjoint from P. Let B_i be a longest path in $T[A_i]$ that has initial vertex v_i. The *shadow tree of T with respect to P*, denoted $S_{T,P}$, is the subtree of T induced by $\bigcup_{i=0}^{d} V(B_i)$. If B_i has length at least one, it is called a *bough* of $S_{T,P}$ at v_i. If the d-path P is understood or irrelevant, we abbreviate $S_{T,P}$ to S_T. The relevance of shadow trees to the study of broadcast domination was demonstrated in [11].

Theorem 3.4 [11] *For any shadow tree S_T of T, $\gamma_b(S_T) = \gamma_b(T)$.*

Let S be a shadow tree with d-path $P = v_0, ..., v_d$. Draw S in the positive $X - Y$ plane with P on the X-axis, v_0 at the origin, each edge of unit length, and each edge not on P parallel to the Y-axis. We henceforth assume that all shadow trees are drawn as described above. We may thus describe a vertex v_i as being *to the left* of v_j, or v_j as being *to the right* of v_i, if $i < j$.

Let $H(t)$ be the tree obtained from $K_{1,3}$ by subdividing each edge $t-1$ times. If $H(t)$ is a subtree of S, then the leaves of $H(t)$ lie at the (geometric) vertices of an isosceles right triangle Δ whose hypotenuse lies on P and has length $2t$; we say that Δ has *radius t*. We use this observation below to better describe the positions of the boughs of S.

The vertices of the bough B of length t that begins at the vertex v_i are labelled $v_i = u_{i,0}, u_{i,1}, ..., u_{i,t}$. If $t \geq 1$, we place an isosceles right triangle $\Delta = \Delta_i$ of radius t with its hypotenuse on P, centred at v_i, with B_i on the median and $u_{i,t}$ at the apex of Δ (see Fig. 2). We say that the vertices $v_{i-t}, ..., v_{i+t}$, $u_{i,1}, ..., u_{i,t}$ are *vertices of Δ*, and that Δ is a *triangle of S*. An edge v_iv_{i+1} of P is *free* if it does not lie on a triangle of S. Note that all split-edges of S are free, but not all free edges are split-edges.

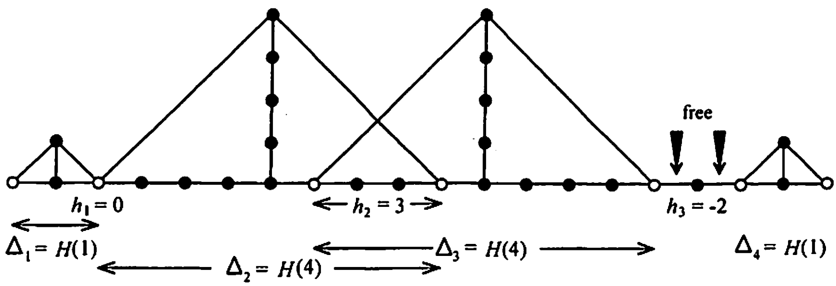

Figure 3: Enhanced shadow tree with overlaps $h_1 = 0$, $h_2 = 3$ and $h_3 = -2$

A triangle of S is a *nested* triangle if it is contained in another triangle. Suppose Δ is a nested triangle and let S' be the tree obtained by deleting the vertices on the bough of Δ. An edge is a split-edge of S if and only if it is a split-edge of S', hence $\gamma_b(S') = \gamma_b(S)$ by Theorem 3.2. Thus removing nested triangles from S does not change the radius or the broadcast number of S.

If $\deg v_1 = 2$ ($\deg v_{d-1} = 2$), join a new leaf to v_1 (v_{d-1} respectively) to form the tree S^*. The addition of these leaves does not change the radius or the broadcast number of S, but it simplifies the statement of the main theorem and other results. Note that the triangles Δ_1 and Δ_k (the last triangle) of S^* have radius one and may or may not be nested. Now remove all nested triangles of S^* **except** Δ_1 and Δ_k (i.e., delete the vertices on the boughs of the triangles). The resulting tree Z is called an *enhanced shadow tree.* While the shadow trees $S_{T,P}$ and $S_{T,P'}$ can be non-isomorphic if P, P' are distinct d-paths of T, the enhanced shadow trees $Z_{T,P}$ and $Z_{T,P'}$ are isomorphic, hence the choice of d-path is irrelevant when considering the enhanced shadow tree of T.

Let $v_{c_1}, ..., v_{c_k}$ be the branch vertices on P, let $B_i : v_{c_i} = u_{i,0}, u_{i,1}, ..., u_{i,t_i}$ be the bough of Z at v_{c_i} and let Δ_i be the triangle of Z with centre v_{c_i} and radius t_i associated with B_i. Let v_{ℓ_i} (v_{r_i}, respectively) be the vertex on P at distance t_i to the left (right) of v_{c_i}; that is, v_{ℓ_i} is the first and v_{r_i} is the last vertex of Δ_i on P. Define the *overlap* h_i of Δ_i and Δ_{i+1} by $h_i = r_i - \ell_{i+1}$ for $i = 1, ..., k-1$. See Fig. 3. Note that h_i may be positive, zero or negative. Thus Δ_i and Δ_{i+1} may have a negative overlap, which indicates that there are free edges on the $v_{r_i} - v_{\ell_{i+1}}$ path in Z. We use the notation defined here throughout the rest of the paper.

4 Uniquely Radial Trees

Let T be a tree with d-path $P = v_0, ..., v_d$ and enhanced shadow tree $Z = Z_{T,P}$. We state a number of conditions that will determine whether T is uniquely radial or not.

A1 The bough B_i of Z of length $t_i \geq 4$ occurs at v_{c_i}, where $t_i \equiv c_i \pmod 2$, and $h_{i-1}, h_i \leq 2$.

A2 The bough B_i of Z of length $t_i \geq 2$ occurs at v_{c_i}, where $t_i \not\equiv c_i \pmod 2$, and $h_{i-1}, h_i \leq 1$.

A3 The bough B_i of Z of length $t_i \geq 3$ occurs at v_{c_i}, where $t_i \equiv c_i \pmod 2$, and $h_{i-1} \leq 2$, $h_i \leq 2t_i - 3$.

A4 The bough B_i of Z of length $t_i \geq 3$ occurs at v_{c_i}, where $t_i \not\equiv c_i \pmod 2$, and $h_{i-1} \leq 2t_i - 3$, $h_i \leq 2$.

B Let d be even.

B1 Z has no free edges.

B2 Z does not have a zero overlap at a vertex labelled with an even subscript.

B3 If A1 holds, then T has a vertex w at distance $t_i - 2$ or $t_i - 1$ from v_{c_i} such that $w \notin V(Z)$.

B4 If A2 holds, then T has a vertex w at distance $t_i - 1$ or t_i from v_{c_i} such that $w \notin V(Z)$ and the $v_{c_i} - w$ path is internally disjoint from B_i.

C Let d be odd.

C1 Z has no free edges and no zero overlaps.

C2 Z has no overlap $h = 1$ of the form $v_{2j-1}v_{2j}$, $1 \leq j \leq \frac{d}{2}$.

C3 If A3 holds, then T has a vertex w at distance t_i or $t_i + 1$ from v_{c_i+2} such that $w \notin V(Z)$ and the $v_{c_i+2} - w$ path Q contains v_{c_i+1}; if v_{c_i+1} is the last vertex of P on Q, then $d(w, v_{c_i+2}) = t_i$.

C4 If A4 holds, then T has a vertex w at distance t_i or $t_i + 1$ from v_{c_i-2} such that $w \notin V(Z)$ and the $v_{c_i-2} - w$ path contains v_{c_i-1}; if v_{c_i-1} is the last vertex of P on Q, then $d(w, v_{c_i+2}) = t_i$.

Examples of uniquely radial trees and their enhanced shadow trees are displayed in Fig. 4 and Fig. 5, where the vertex w is indicated whenever conditions B3, B4, C3 or C4 hold. The statement of our main theorem follows.

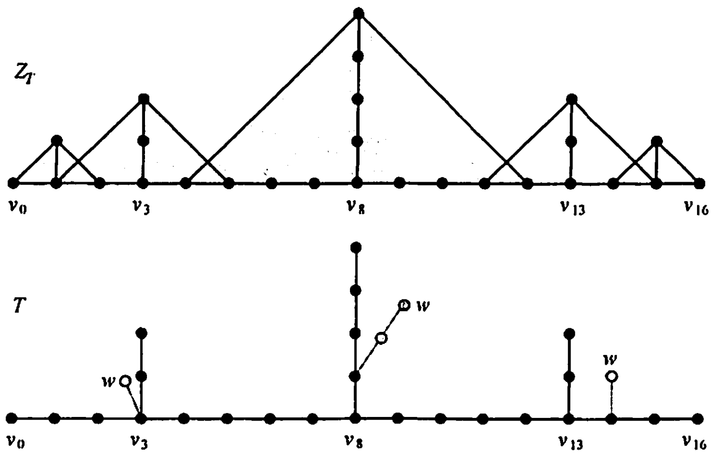

Figure 4: Uniquely radial tree of even diameter and enhanced shadow tree

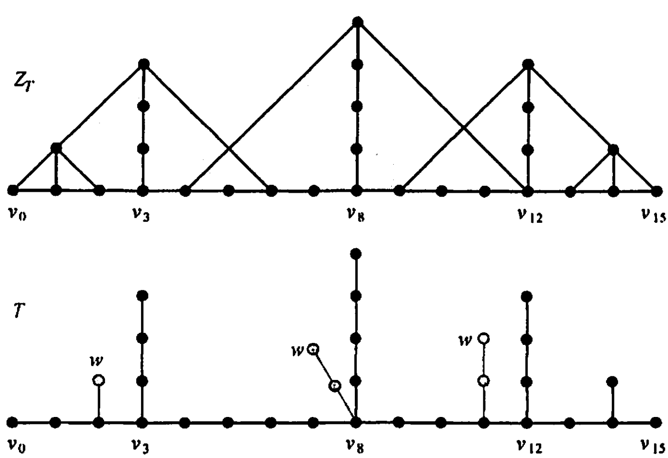

Figure 5: Uniquely radial tree of odd diameter and enhanced shadow tree

Theorem 4.1 *The tree T is uniquely radial if and only if* B1 - B4 *or* C1 - C4 *hold.*

The proof of Theorem 4.1 is divided into three lemmas, which we now state. We defer their proofs to Section 6. Lemmas 4.2 and 4.3 show that B1 - B4 or C1 - C4 are necessary conditions for T to be uniquely radial, while Lemma 4.4 shows that they are also sufficient. Theorem 4.1 follows.

Lemma 4.2 *If T is a tree such that one of* B1, B2, C1 *and* C2 *does not hold, then T is not uniquely radial.*

Lemma 4.3 *Let T be a tree with enhanced shadow tree Z such that* B1, B2, C1 *and* C2 *hold. If one of* B3, B4, C3 *or* C4 *does not hold, then T is not uniquely radial.*

Lemma 4.4 *If T is a tree with enhanced shadow tree Z such that* B1 - B4 *or* C1- C4 *hold, then T is uniquely radial.*

5 Corollaries

A caterpillar C consists of a d-path $P = v_0, ..., v_d$ together with any non-negative number of leaves attached to $v_1, ..., v_{d-1}$. The next result follows immediately from Theorem 4.1.

Corollary 5.1 *A caterpillar of order at least three is uniquely radial if and only if its diameter $v_0, ..., v_d$ is even and each v_i, where i is even and $i \notin \{0, d\}$, is a stem.*

A tree with diameter three is also called a *double star.* If the two stems of a double star have degrees a and b respectively, we denote it by $S(a, b)$. By Corollary 5.1, $S(a, b)$ is not uniquely radial. Moreover, the only supertrees of $S(a, b)$ with diameter three are double stars $S(a', b')$ with $a' \geq a$ and $b' \geq b$. Hence $S(a, b)$ is not contained in a uniquely radial tree of the same diameter. We show that double stars are the only trees with this property.

Corollary 5.2 *Every nontrivial tree T with* $\operatorname{diam} T = d \neq 3$ *is a subtree of a uniquely radial tree with diameter d.*

Proof. Let T be a tree with d-path $P = v_0, ..., v_d$. If d is even, add leaves to $v_1, ..., v_{d-1}$ until each v_i with i even is a leaf or a stem. By Proposition 6.1 and Corollary 5.1, the resulting tree is uniquely radial.

If d is odd, let $r = \operatorname{rad} T = \lceil \frac{d}{2} \rceil$. The result is obvious if $d = 1$, so assume $d \geq 5$. For $i \in \{r-1, r\}$, add the path $B_i = v_i, u_{i,1}, ..., u_{i,r-1}$ of length $r-1$ to T to form the tree S with $\operatorname{diam} S = d$. Theorem 4.1 implies that S is uniquely radial. ■

6 Proofs of lemmas

We begin with a proposition which shows that for trees with fixed diameter, being uniquely radial is a superhereditary property.

Proposition 6.1 *If the tree T has a uniquely radial subtree T' with the same diameter as T, then T is uniquely radial.*

Proof. Suppose to the contrary that T has a non-central γ_b-broadcast f. Since $|V_f^+| \geq 2$, $f(v) < \rho = \operatorname{rad} T = \operatorname{rad} T'$ for each $v \in V_f^+$, otherwise $\operatorname{cost}(f) > \operatorname{rad} T$, which is impossible.

For each $u \in V(T) - V(T')$, let u^* be the unique vertex of T' nearest to u. For each $v \in V(T')$, let $D_v = \{v\} \cup \{u \in V(T) - V(T') : v = u^*\}$ (note that $D_v \cap D_{v'} = \varnothing$ if $v \neq v'$) and $g_v = \max\{f(x) : x \in D_v\}$. Define the broadcast g on T' by $g(v) = g_v$ for each $v \in V(T')$. Then g is a dominating broadcast of T' such that $\operatorname{cost}(g) \leq \operatorname{cost}(f) = \rho = \gamma_b(T')$, so that g is a γ_b-broadcast of T'. But $g(v) < \rho$ for each $v \in V_g^+$, which implies that $|V_g^+| \geq 2$, contrary to T' being uniquely radial. ■

We need one more lemma before proving Lemma 4.2.

Lemma 6.2 *Let $B : v = u_0, u_1, ..., u_t$, $t \geq 2$, be a bough of a shadow tree S attached to the vertex v of the d-path of S. If S has a γ_b-broadcast f such that the leaf u_t of B hears a broadcast from u_j, $1 \leq j \leq t-1$, then S has a γ_b-broadcast g such that $g(u_{t-1}) = 1$.*

Proof. If $j = t-1$ and $f(u_j) > 1$, or if $j \leq t-2$, then $u_{j-1} \notin V_f^+$, otherwise there exists a broadcast f' with $\operatorname{cost}(f') < \operatorname{cost}(f)$. Define g by $g(u_{t-1}) = 1$, $g(u_{j-1}) = f(u_j) - 1$, and $g(x) = f(x)$ otherwise. Then g is a γ_b-broadcast of S. ■

Proof of Lemma 4.2. <u>B1 and part of C1</u> Let $P = v_0, ..., v_d$ be a d-path of T and suppose $v_i v_{i+1}$ is a free edge of Z_T. Then $2 \leq i \leq d-3$. First assume that $d = \operatorname{diam} T$ is odd, so that $\rho = \operatorname{rad} T = \frac{d+1}{2}$. Depending on the parity of i, the paths $v_0, ..., v_i$ and $v_{i+1}, ..., v_d$ are both even or both odd. Define the broadcast g of T as follows. If i is odd, say $i = 2m - 1$, $2 \le m \le \rho - 2$, let

$$g(v_m) = m, g(v_{m+\rho-1}) = \rho - m \text{ and } g(v) = 0 \text{ otherwise.} \tag{1}$$

If i is even, say $i = 2m$, $1 \le m \le \rho - 2$, let

$$g(v_m) = m, \ g(v_{m+\rho}) = \rho - m \text{ and } g(v) = 0 \text{ otherwise.} \tag{2}$$

Then g is a dominating broadcast of T such that $\text{cost}(g) = \rho$ and $|V_g^+| = 2$. Therefore T is not uniquely radial.

Now assume d is even, so $\rho = \frac{d}{2}$. Then one of $v_0, ..., v_i$ and $v_{i+1}, ..., v_d$ is even and one is odd; assume without loss of generality that the former path is even and say $i = 2m$, $1 \le m \le \rho - 2$. The result follows as above by defining g as in (2).

<u>B2</u> Let $i = 2m$, $1 \le m \le \rho - 1$ and define the broadcast g as in (2).

<u>Rest of C1</u> Consider d-paths $P = v_0, ..., v_d$ and $P^{-1} = u_0, ..., u_d$, where $u_i = v_{d-i}$, and apply B2.

<u>C2</u> In this case the paths $v_0, ..., v_{2m}$ and $v_{2m-1}, ..., v_d = v_{2\rho-1}$ are both even. Define g as in (1). ■

An easy consequence of Theorem 3.2 is that $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$, a result that was first proved by Erwin [6, 7]. Theorem 3.2 can also be used to determine the broadcast number of spiders. A *spider* $S(a_1, ..., a_s)$ is a tree with exactly one branch vertex v and $s \ge 3$ paths of lengths $a_1, ..., a_s$, where $a_1 \le ... \le a_s$, from v to a leaf. To simplify notation in the following formula, which is stated without proof, we write $S(a_1, ..., a_{s-3}, a, b, c)$ instead of $S(a_1, ..., a_s)$.

Proposition 6.3 *For any integers $a_1 \le ... \le a_{s-3} \le a \le b \le c$,*

$$\gamma_b(S(a_1, ..., a_{s-3}, a, b, c)) = \begin{cases} a + 1 + \frac{b+c-2a-2}{3} & \text{if } b - a \equiv c - a \\ & \equiv 1 \pmod 3 \\ a + \lceil \frac{b-a}{3} \rceil + \lceil \frac{c-a}{3} \rceil & \text{otherwise.} \end{cases}$$

Let $P = v_0, ..., v_d$ be a d-path of the enhanced shadow tree Z. For any $i \in \{0, ..., d\}$, let L_i (R_i, respectively) be the subtree of Z induced by $v_0, ..., v_i$ ($v_i, ..., v_d$, respectively) together with all boughs of Z attached to these vertices.

Proof of Lemma 4.3. Since Z has no free edges, Z is radial. Let Δ_i be the triangle that covers the bough $B_i = v_{c_i}, u_1, ..., u_{t_i}$ of Z. To simplify the notation we abbreviate Δ_i, B_i, t_i and c_i to Δ, B, t and c. Then v_{c-t} and v_{c+t} are the leftmost and rightmost vertices of Δ respectively.

A1 holds but B3 does not. Since $t_i \equiv c_i \pmod 2$, $c \pm t$ and $c \pm (t-2)$ are even. Since Z has no zero overlaps at $v_{c\pm t}$, $h_{i-1}, h_i \geq 1$. Since $h_{i-1}, h_i \leq 2$, the rightmost vertex of Δ_{i-1} is v_j, where $j \in \{c-t+1, c-t+2\}$, while the leftmost vertex of Δ_{i+1} is $v_{j'}$, where $j' \in \{c+t-1, c+t-2\}$. Consider the subtrees $L = L_{c-t+2}$ and $R = R_{c+t-2}$ of Z. Since $t \geq 4$ and $h_{i-1}, h_i \leq 2$, $\deg v_{c-t+2} = \deg v_{c+t-2} = 2$ and thus $v_0, ..., v_{c-t+2}$ is a d-path of L while $v_{c+t-2}, ..., v_d$ is a d-path of R. Since d is even, $\operatorname{diam} L$ and $\operatorname{diam} R$ are even, hence $\operatorname{rad} Z - \operatorname{rad} L - \operatorname{rad} R = \frac{1}{2}[(c+t-2) - (c-t+2)] = t-2$. Let $H = Z - L - R$ and note that $H = S(t-3, t-3, t)$.

The limits on j and j' and the radiality of Z imply that L and R are radial. Let f_L and f_R be central γ_b-broadcasts of L and R respectively. By Proposition 6.3, $\gamma_b(H) = t-2$. Define the γ_b-broadcast f_H of H by $f(u_{t-1}) = 1$, $f(v_c) = t-3$ and $f(v) = 0$ otherwise, and the broadcast g of Z by $g(v) = f_L(v)$ if $v \in V(L)$, $g(v) = f_R(v)$ if $u \in V(R)$ and $g(v) = f_H(v)$ if $u \in V(H)$. Then g is a dominating broadcast of Z and $\operatorname{cost}(g) = \operatorname{rad} L + \operatorname{rad} R + t - 2 = \operatorname{rad} Z$, hence g is a non-central γ_b-broadcast of Z.

Since B3 does not hold, each vertex w of $T - Z$ is either at distance at most $t-3$ or at distance at least t from v_c. If $d(w, v_c) \leq t-3$, then w hears the broadcast from v_c. If $d(w, v_c) > t$, then by definition of Z and Δ_i, the vertex of Z nearest to w belongs to Δ_j, $j \neq i$, so w hears a broadcast from the vertex in V_L^+ or V_R^+. If $d(w, v_c) = t$, then w is an isolated vertex of $T - Z$ (otherwise $T - Z$ has a vertex at distance $t-1$ from v_c), hence w is adjacent to v_{c-t+1}, v_{c+t-1} or u_{t-1} and hears a broadcast from a vertex in V_L^+, a vertex in V_R^+ or from u_{t-1}. Hence g is a non-central γ_b-broadcast of T, which shows that T is not uniquely radial.

A2 holds but B4 does not. Since $t_i \not\equiv c_i \pmod 2$, $c \pm (t-1)$ is even while $c \pm t$ is odd. Since $h_{i-1}, h_i \leq 1$, the rightmost vertex of Δ_{i-1} is v_j, where $j \in \{c-t, c-t+1\}$, while the leftmost vertex of Δ_{i+1} is $v_{j'}$, where $j' \in \{c+t, c+t-1\}$. Consider the subtrees $L = L_{c-t+1}$ and $R = R_{c+t-1}$ of Z and define $H = Z - L - R$. Similar to the above paragraph, $\operatorname{rad} Z - \operatorname{rad} L - \operatorname{rad} R = t-1$. In this case $H = S(t-2, t-2, t)$ and $\gamma_b(H) = t-1$ (Proposition 6.3). By defining f_H by $f_H(u_1) = t-1$ and $f_H(v) = 0$ otherwise, and f_L, f_R and g as above, we can show as above that Z is not uniquely radial.

Since B4 does not hold, each vertex w of $T - Z$ is either at distance at most $t-2$ or at distance at least $t+1$ from v_c. If $d(w, v_c) \leq t-2$, then w hears the broadcast from u_1, while if $d(w, v_c) \geq t+1$, then w hears a broadcast from the vertex in V_L^+ or V_R^+. As above g is a non-central γ_b-broadcast of T and the result follows.

<u>A3 holds but C3 does not.</u> Since $t_i \equiv c_i \pmod 2$, $c \pm t$ and $c - t + 2$ are even and $c - t + 3$ is odd. Since $h_{i-1} \leq 2$ and C1 holds, the rightmost vertex of Δ_{i-1} is v_j, where $j \in \{c-t+1, c-t+2\}$. Since $h_i \leq 2t-3$ and C1 and C2 hold, the leftmost vertex of Δ_{i+1} is $v_{j'}$, where $c - t + 3 \leq j' \leq c + t - 2$. Consider the subtrees $L = L_{c-t+2}$ and $R = R_{c-t+3}$ of $Z - \{u_{t-2}, u_{t-1}, u_t\}$. Since $t \geq 3$ and $h_{i-1} \leq 2$, $\deg v_{c-t+2} = 2$. If $t = 3$, then $v_{c-t+3} = v_c$; otherwise, since $h_i \leq 2t-3$, $\deg v_{c-t+3} = 2$. In either case $Z - \{u_{t-2}, u_{t-1}, u_t\} = L \cup R$.

Since d is odd, $\operatorname{diam} R$ is even, and clearly $\operatorname{diam} L$ is also even. Hence $\operatorname{rad} Z - \operatorname{rad} L - \operatorname{rad} R = \frac{1}{2}[(d+1) - (c-t+2) - (d-c+t-3)] = 1$. Also, any split-set of L or R has even cardinality. The limits on j' imply that the only possible free edge of R is $v_{c+t-3}v_{c+t-2}$, so R is radial, while L is clearly radial. Let f_L and f_R be the unique central γ_b-broadcasts of L and R. Define the broadcast g of Z by $g(v) = f_L(v)$ if $v \in V(L)$, $g(v) = f_R(v)$ if $v \in V(R)$, $g(u_{t-1}) = 1$ and $g(v) = 0$ otherwise. Then g is a dominating broadcast of Z and $\operatorname{cost}(g) = \operatorname{rad} L + \operatorname{rad} R + 1 = \operatorname{rad} Z$. Therefore g is a non-central γ_b-broadcast of Z, which is thus not uniquely radial.

Since C3 does not hold, each vertex w of $T - Z$ is either at distance at most $t-1$ or at least $t+2$ from v_{c+2}. Since $t \geq 3$ and $c + t \neq d$, the central vertex x of R lies to the right of v_{c+1}. If $d(w, v_{c+2}) \leq t-1$, then $d(w, v_{c+1}) \leq t-2$. But $d(v_{c-t+3}, v_{c+1}) = t-2$ and, by definition of f_R, v_{c-t+3} hears the broadcast from x. Hence w also hears the broadcast from x. If $d(w, v_{c+2}) \geq t+2$, it follows similar to the case where A1 holds that w hears at least one broadcast.

The result where A4 holds, but not C4, follows from the previous case by reversing the direction of the d-path P of Z. ■

Lemma 6.4 *If Z is an enhanced shadow tree such that* B1 *and* B2, *or* C1 *and* C2, *hold, and f is a non-central γ_b-broadcast of Z, then $V_f^+ \not\subseteq V(P)$.*

Proof. Since Z has no free edges, it is radial. Suppose to the contrary that f is a non-central γ_b-broadcast of Z such that $V_f^+ \subseteq V(P)$; say $V_f^+ = \{v_{a_1}, ..., v_{a_s}\}$, where $s \geq 2$ and $a_1 \leq \cdots \leq a_s$. For $i = 1, ..., s$, define $\lambda_i = a_i - f(v_{a_i})$ and $\rho_i = a_i + f(v_{a_i})$. Suppose $\rho_i < \lambda_{i+1}$ for some i. Since $V_f^+ \subseteq V(P)$ and each vertex on P hears at least one broadcast,

$\rho_i = \lambda_{i+1} - 1$. Since Z has no free edges, the edge $v_{\rho_i} v_{\lambda_{i+1}}$ is contained in a triangle Δ of Z. Let u be the vertex at the apex of Δ and let $x \in V(P)$ be the vertex that broadcasts to u. Then x broadcasts to v_{ρ_i} and to $v_{\lambda_{i+1}}$, so $x \in \{v_{a_i}, v_{a_{i+1}}\}$. But v_{a_i} does not broadcast to $v_{\lambda_{i+1}}$, and $v_{a_{i+1}}$ does not broadcast to v_{ρ_i}, a contradiction. Hence $\rho_i \geq \lambda_{i+1}$ for each i. Let $\sigma = \sum_{i=1}^{s-1}(\rho_i - \lambda_i)$. Since f is a dominating broadcast and Z is radial,

$$d = \operatorname{diam} Z \leq 2 \sum_{x \in V_f^+} f(x) - \sigma = 2 \operatorname{rad} T - \sigma = 2\lceil \tfrac{d}{2} \rceil - \sigma. \tag{3}$$

Therefore $\sigma = 0$ and $\rho_i = \lambda_{i+1}$ for each $i = 1, ..., s-1$ if d is even, while $\sigma \leq 1$ and $\rho_i = \lambda_{i+1}$ for at least $s-2$ values of i if d is odd.

Suppose $\rho_i = \lambda_{i+1}$ and Z does not have a zero overlap at v_{ρ_i}. Then some triangle Δ of Z contains both edges $v_{\rho_i - 1} v_{\rho_i}$ and $v_{\rho_i} v_{\rho_i + 1}$. Let u be the vertex at the apex of Δ and let $x \in V(P)$ be the vertex that broadcasts to u. Then x broadcasts to $v_{\rho_i - 1}$ and to $v_{\rho_i + 1}$, so $x \in \{v_{a_i}, v_{a_{i+1}}\}$ and a contradiction follows as above. Hence if $\rho_i = \lambda_{i+1}$, then Z has a zero overlap at v_{ρ_i}.

Assume d is even. By B2 each ρ_i and λ_{i+1} are odd; in particular, ρ_1 and λ_s are odd. Since $\rho_i - \lambda_i = 2f(v_i)$ is even and f is a dominating broadcast, $\lambda_1 < 0$ and $\rho_s > d$. Thus v_0 and v_d are overdominated. Hence $2\sum_{x \in V_f^+} f(x) \geq \operatorname{diam} Z + 2$, i.e. $\operatorname{cost}(f) \geq \operatorname{rad} Z + 1$, a contradiction.

Assume d is odd. Since Z has no zero overlaps, $\rho_i \neq \lambda_{i+1}$ for each i. By (3), $s = 2$ and $\rho_1 = \lambda_2 + 1$. By C2, λ_2 is even and ρ_1 is odd, thus the paths $v_0, ..., \rho_1$ and $\lambda_2, ..., v_d$ are odd. As above v_0 and v_d are overdominated, giving a contradiction as before. ■

The next lemma forms the first part of the proof of Lemma 4.4. We state it as a separate lemma because the proof of Lemma 4.4 is quite long.

Lemma 6.5 *Let Z be an enhanced shadow tree with* $\operatorname{diam} Z = d$ *such that* B1 *and* B2 *hold, but neither* A1 *nor* A2 *if d is even, and* C1 *and* C2 *hold, but neither* A3 *nor* A4 *if d is odd. Then Z is uniquely radial.*

Proof. Suppose Z has a non-central broadcast f. By Lemma 6.4, V_f^+ contains a vertex that does not lie on P. Let $B_i = v_c, u_1, ..., u_t$ be a bough that contains a vertex of $V_f^+ - V(P)$. By Lemma 6.2 we may assume that $t \geq 2$ and $f(u_{t-1}) = 1$. Let Δ_i be the triangle that covers B_i. Then Δ_i is neither the first nor the last triangle of Z, as these triangles have radius 1. Let Δ' be the triangle immediately preceding Δ_i, and $B' = v_{c'}, u_1, ..., u_{t'}$ the bough of Δ'. If $t = 2$, let $Z' = Z - \{u_1, u_2\}$, otherwise let $Z' =$

$Z - \{u_{t-2}, u_{t-1}, u_t\}$. Then $\operatorname{rad} Z' = \operatorname{rad} Z$. Let f' denote the restriction of f to Z' and note that $\operatorname{cost}(f') < \operatorname{cost}(f)$. We consider three cases, depending on the parities of c, d, and t.

Case 1 d is even and $t \equiv c \pmod 2$. Then any split-set of Z' has an even number of edges. Also, $c \pm t$ is even, and since Z does not have free edges or zero overlaps at $v_{c\pm t}$, $h_{i-1}, h_i \geq 1$. We consider the subcases $t = 2$ and $t \geq 3$ separately.

Subcase 1.1 $t = 2$. Since Δ_i is not a nested triangle, $h_{i-1}, h_i \leq 3$. Since $h_{i-1}, h_i \geq 1$, the only possible free edges of Z' are $v_{c-1}v_c$ and $v_c v_{c+1}$. Hence Z' is radial. Since $\operatorname{rad} Z' = \operatorname{rad} Z$, $\gamma_b(Z') = \gamma_b(Z)$. If $v_c \in N_f[w]$ for some $w \in V_f^+ - \{u_1\}$, then f' is a dominating broadcast of Z', which is a contradiction since $\operatorname{cost}(f') < \operatorname{cost}(f) = \gamma_b(Z')$. Hence $v_c \in \mathrm{PN}_f[u_1]$, so f' is a dominating broadcast of $Z' - v_c$.

Let L (R, respectively) be the component of $Z - v_c$ that contains v_0 (v_d, respectively). Then the $v_0, ..., v_{c'}$ subpath of P, followed by the bough B', is a d-path of L, the path $v_{c'}, v_{c'+1}, ..., v_{c-1}$ is a bough of L, and $\operatorname{diam} L = c - 2 + h_{i-1}$. Since Z has no free edges, the only possible split-sets of L are $\{v_{c'-t'}v_{c'-t'+1}\}$ or $\{v_{c'-t'+1}v_{c'-t'+2}\}$ if $h_{i-1} = 3$, or $\{v_{c'-t'}v_{c'-t'+1}\}$ if $h_{i-1} = 2$, while L has no free edges if $h_{i-1} = 1$. In each case Theorem 3.2 implies that $\gamma_b(L) \geq \lceil \frac{c-1}{2} \rceil = \frac{c}{2}$. Similarly, $\gamma_b(R) \geq \frac{d-c}{2}$. Since f' is a dominating broadcast of $Z' - v_c = L \cup R$, $\operatorname{cost}(f') = \frac{d}{2} = \operatorname{rad} Z = \operatorname{cost}(f) > \operatorname{cost}(f')$, a contradiction.

Subcase 1.2 $t \geq 3$. Then u_{t-1} does not broadcast to v_c and f' is a dominating broadcast of Z'. Since $h_{i-1}, h_i \geq 1$, Z' has at most four free edges if $t = 3$, namely the edges of the subpath $v_{c-2}, v_{c-1}, v_c, v_{c+1}, v_{c+2}$. But as $c \pm 2$ is odd, Z' has no nonempty split-set in this case, so Z' is radial. If $t \geq 4$, then $\max\{h_{i-1}, h_i\} \geq 3$ since A1 does not hold. Assume without loss of generality that $h_i \geq 3$. Then Z' has at most two free edges, namely $v_{c-t+1}v_{c-t+2}$ and $v_{c-t+2}v_{c-t+3}$. Since these edges are adjacent, Z' has no nonempty split-set and is radial. In either case $\operatorname{cost}(f') = \operatorname{rad} Z' = \operatorname{rad} Z = \operatorname{cost}(f) > \operatorname{cost}(f')$, a contradiction.

Case 2 d is even and $t \not\equiv c \pmod 2$. Then $c \pm t$ is odd. Since A2 does not hold, $\max\{h_{i-1}, h_i\} \geq 2$. Assume without loss of generality that $h_i \geq 2$. If $t = 2$, we proceed as in Subcase 1.1 to show that $\gamma_b(L) \geq \frac{c-1}{2}$ and $\gamma_b(R) \geq \frac{d-c+1}{2}$, and obtain a similar contradiction, so we assume $t \geq 3$. Now Z' has at most four free edges, and if Z' has exactly four free edges, then $h_{i-1} = 0$ and $h_i = 2$, so that the free edges are the edges of the path $v_{c-t}, v_{c-t+1}, v_{c-t+2}, v_{c-t+3}$ and the edge $v_{c+t-3}v_{c+t-2}$. The parity of the subscripts implies that no pair of these edges forms a split-set. Hence Z'

is radial and we obtain a contradiction as in Subcase 1.2.

Case 3 d is odd and $t \equiv c \pmod 2$. As d is odd, a free edge $v_j v_{j+1}$ is a split-edge of Z' if and only if j is even. Since Z has no zero overlaps, $h_{i-1}, h_i \geq 1$. If $h_i = 1$, then the edge $v_{c+t-1}v_{c+t}$ is an overlap. Since $c+t-1$ is odd, this contradicts C2. Thus $h_i \geq 2$. Again we consider the cases $t = 2$ and $t \geq 3$ separately.

Subcase 3.1 $t = 2$. Since Z has no nested triangles, $h_{i-1}, h_i \leq 3$. Since $h_i \geq 2$, $v_{c-1}v_c$ is the only possible free edge of Z'. Since $c-1$ is odd, $\{v_{c-1}v_c\}$ is not a split-set of Z'. Hence Z' is radial. Proceeding as in Subcase 1.1 we show that $\gamma_b(L) \geq \frac{c}{2}$ and $\gamma_b(R) \geq \frac{d-c+1}{2}$. Thus $\operatorname{cost}(f') \geq \frac{d+1}{2} = \operatorname{rad} Z$ and this leads to a contradiction as before.

Subcase 3.2 $t \geq 3$. As shown above, $h_i \geq 2$. Since C3 does not hold, $h_{i-1} \geq 3$ or $h_i \geq 2t-2$. If $h_{i-1} \geq 3$, then $v_{c+t-3}v_{c+t-2}$ is the only possible free edge of Z', and since $c+t-3$ is odd, Z' is radial. If $h_i \geq 2t-2$, then $v_{c-t+1}v_{c-t+2}$ is the only possible free edge of Z', and since $c-t+1$ is odd, Z' is radial. A contradiction follows as in the previous cases.

The case where d is odd and $t \not\equiv c \pmod 2$ follows from Case 3 by reversing the direction of the path P. ■

Proof of Lemma 4.4. Let $I = \{i :$ the bough B_i of Z satisfies one of A1 - A4$\}$. By Lemma 6.5 we may assume that $I \neq \varnothing$. For each $i \in I$, let w_i be a vertex of T whose existence is guaranteed by condition B3, B4, C3 or C4 respectively. Let $W = \bigcup_{i \in I}\{w_i\}$ and let H be a minimal subtree of T that contains $V(Z) \cup W$. Then each w_i is a leaf of H. Also, $\operatorname{diam} H = \operatorname{diam} T = \operatorname{diam} Z$ and $\gamma_b(H) = \gamma_b(T) = \gamma_b(Z)$. We show that H is uniquely radial; the result will then follow from Proposition 6.1.

Suppose to the contrary that H is not uniquely radial. For any broadcast ϕ, let $A_\phi = \{u \in V_\phi^+ - V(P) : u$ broadcasts to the leaf u_{t_i} of B_i for some $i\}$. Let f be a non-central γ_b-broadcast of H such that $\sum_{u \in A_f} f(u)$ is minimum. Let $U = V_f^+ - V(Z)$. For $y \in V(H-Z)$, let x_y be the vertex of Z nearest to y. We first show that

(I) if $u \in U$, then $f(u) = 1$, u is adjacent to x_u, and $x_u \in \mathrm{PN}_f[u]$.

For $u \in U$, let $D_u = \{y \in U : x_y = x_u\}$ and $g_u = \max\{f(x_u), \sum_{y \in D_u} f(y) - 1\}$. Then $g_u = 0$ if and only if $D_u = \{u\}$, $f(u) = 1$ and $f(x_u) = 0$. Let H' be the subtree of H obtained by deleting all $y \in V(H-Z)$ such that $x_y = x_u$. If $g_u \neq 0$, define the broadcast g on H' by $g(x_u) = g_u$ and $g(v) = f(v)$ otherwise. Then g is a dominating broadcast of H' such that

$\text{cost}(g) < \text{cost}(f)$. Hence $\gamma_b(H') < \gamma_b(Z)$, contradicting Proposition 3.3. Therefore $g_u = 0$, so that $f(u) = 1$ and u is adjacent to x_u. If $x_u \notin \text{PN}_f[u]$, then x_u hears a broadcast from some vertex of Z, so that the restriction g' of f to H' is a dominating broadcast of H' such that $\text{cost}(g') < \text{cost}(f)$, which is impossible. ♦

By (I) and our convention that leaves are not broadcast vertices, each vertex in U is adjacent to a vertex in W. We show that

(II) V_f^+ contains a vertex of $B_i - P$ for some i.

Consider the broadcast f' of Z defined by $f'(x_u) = 1$ for each $u \in U$ and $f'(v) = f(v)$ otherwise. By (I), $\text{PN}_{f'}[x_u] = \{x_u\}$ for each $u \in U$ and f' is a dominating broadcast of Z with $\text{cost}(f') = \text{cost}(f)$, hence a γ_b-broadcast of Z. Also, $|V_{f'}^+| = |V_f^+|$, so f' is non-central. By Lemma 6.4, $V_{f'}^+$ contains a vertex y' of $B_i - P$ for some i. If $y' \in V_f^+$ we are done, hence suppose $y' = x_u$ for some $u \in U$ and let $w \in W$ be adjacent to u. Since $t_i \geq d(w, v_c) = d(w, x_u) + d(x_u, v_c) = 2 + d(x_u, v_c)$, $d(x_u, v_c) \leq t_i - 2$ and so x_u is not the stem of B_i. Since $f(x_u) = 1$, some vertex $y \in V_f^+$ broadcasts to the leaf u_{t_i} of B_i. Since y does not broadcast to x_u (because $\text{PN}_{f'}[x_u] = \{x_u\}$), $y \in V(B_i - P)$. ♦

Let z be the vertex on $B_i - P$ that broadcasts to $u_t = u_{t_i}$ and say $f(z) = a$. We may assume that $d(z, u_t) = a$, i.e., $z = u_{t-a}$, otherwise we may redefine f so that this is the case. Then $a \leq t - 1$.

(III) If $w = w_i \in N_f[z]$, then the $w - z$ path π is contained in $\text{PN}_f[z]$:

Suppose some vertex $y \in V_f^+$ broadcasts to a vertex of π. By (I), $y \notin U$. Also, y does not lie on the $z - u_t$ path, nor on the $z - w$ path, otherwise there exists a dominating broadcast of H with cost less than that of f, which is impossible. Therefore either $y = u_s$, where $d(u_s, v_c) < \min\{d(z, v_c), d(x_w, v_c)\}$, or $y \in V(Z - B_i)$.

Suppose x_w lies on the $z - u_t$ path. If $d(w, z) = a$, then $d(w, v_c) = a + (t - a) = t$. But $d(w, v_c) < t$ unless B4 holds, in which case the $v_c - w$ path is internally disjoint from B_i, a contradiction; hence $d(w, z) < a$. Now y broadcasts to z but not to w, otherwise there exists a dominating broadcast of H with cost less than that of f. Thus, if $\delta = d(y, w) - f(y)$, then $\delta > 0$. Since $d(w, z) < a = d(z, u_t)$ and y broadcasts to z, $\delta < d(y, u_t) - f(y) \leq a$. Define g by $g(y) = f(y) + \delta$, $g(u_{t-a+\delta}) = a - \delta$, $g(z) = 0$ and $g(v) = f(v)$ otherwise; note that $u_t \in N_g[u_{t-a+\delta}]$. Then g is a non-central γ_b-broadcast of H such that $\sum_{u \in A_g} g(u) < \sum_{u \in A_f} f(u)$, contrary to the choice of f.

Suppose x_w lies on the $v_c - z$ path and $x_w \neq z$. Then y broadcasts to x_w. As above, $\delta = d(y, w) - f(y) > 0$, and since $x_w \neq z$, $\delta < a$. Defining g as above we contradict the choice of f.

Hence assume x_w does not lie on B_i. Since $d(v_c, w) \leq t$ in all cases, $x_w \in \{v_{c-t+1}, ..., v_{c-1}\} \cup \{v_{c+1}, ..., v_{c+t-1}\}$. Assume without loss of generality that $x_w \in \{v_{c-t+1}, ..., v_{c-1}\}$. Since y does not lie on the $z - u_t$ or the $z - w$ path, y is to the right of v_c and broadcasts to v_c, or y is to the left of x_w and broadcasts to x_w. In either case we obtain a contradiction similar to the above two cases. ♦

(IV) $a = 1$ if and only if $w_i \notin N_f[z]$:

If $w = w_i \notin N_f[z]$, then either w does not exist (i.e., $i \notin I$) or w hears a broadcast from $z' \neq z$. If $a > 1$, define the broadcast g as in the proof of Lemma 6.2. Then $g(u_{t-1}) < g(z)$, hence g is a γ_b-broadcast of H that violates the choice of f. Conversely, if $a = 1$ and $w \in N_f[z]$, then w is adjacent to z and $d(w, v_c) = t$. But $d(w, v_c) = t$ only if B4 holds, and we get a contradiction as in the proof of (III). ♦

Let $F = H - \mathrm{PN}_f[z]$ and let f' be the restriction of f to F. Then f' is a dominating broadcast of F with $\mathrm{cost}(f') < \mathrm{cost}(f)$. Hence F is either disconnected, or connected and non-radial, because $\operatorname{diam} F = \operatorname{diam} H$ if F is connected. We prove next that

(V) F is connected.

Suppose F is disconnected. If K is a component of F that consists of vertices of $H - Z$, then $w \in V(K)$. Hence $w \notin \mathrm{PN}_f[z]$, so by (III), $w \notin N_f[z]$. Since no vertex of K is adjacent in F to a vertex of Z, some vertex y of the $w - x_w$ path is contained in $\mathrm{PN}_f[z]$. Hence $x_w \in N_f[z]$. Thus, if u is any vertex that broadcasts to w, then (I) implies that $u \notin U$. So $u \in V(Z)$. Since $y \in \mathrm{PN}_f[z]$, $u = z$, a contradiction. Hence no component of F consists of vertices of $H - Z$ only. Since F is disconnected, $v_c \in \mathrm{PN}_f[z]$ and $a \geq \lceil \frac{t}{2} \rceil$. By (IV) there are two cases: $a = 1$ and $w_i \notin N_f[z]$, and $a \geq 2$ and $w_i \in N_f[z]$. The first case was considered in Subcase 1.1 of Lemma 6.5, hence we assume henceforth that $a \geq 2$ and $w_i \in N_f[z]$.

Since z broadcasts to precisely the vertices $v_{c-2a+t}, ..., v_{c+2a-t}$ of P, $v_{c\pm(2a-t+1)} \notin N_f[z]$ and F contains the subtree $L = L_{c-2a+t-1}$ of H consisting of the path $v_0, ..., v_{c-2a+t-1}$ and all boughs and vertices in W connected to this path, and the subtree $R = R_{c+2a-t+1}$ of H defined similarly. Since $\mathrm{PN}_f[z] \subseteq N_f[z]$ and this inclusion may be strict, it is possible that some of the vertices $v_{c-2a+t}, ..., v_{c+2a-t}$ belong to F, and that some of them are broadcast vertices that broadcast to L or R. Suppose $v \in V_f^+$ for some $v \in \{v_{c-2a+t}, ..., v_{c+2a-t}\}$. Since $v_c \in \mathrm{PN}_f[z]$, v does not broadcast to v_c, and similarly v does not broadcast to w. Assume without loss of generality that v is to the left of v_c. If $f(v) = 1$, then $v = v_{c-2a+t}$, else $N_f[v] \subset N_f[z]$.

Then f' defined by $f'(v_{c-2a+t-1}) = 1$, $f'(v) = 0$, and $f'(x) = f(x)$ otherwise, is a dominating broadcast of H such that $\text{cost}(f') = \text{cost}(f)$, and we consider f' instead of f. If $f(v) > 1$ and v' is the vertex of P immediately preceding v, then f'' defined by $f''(v') = f(v) - 1$, $f''(v) = 0$, and $f''(x) = f(x)$ otherwise, is a dominating broadcast of H such that $\text{cost}(f'') < \text{cost}(f)$, which is impossible. Therefore we may assume that the restrictions f_L and f_R of f to L and R, respectively, are dominating broadcasts of L and R. Hence

$$\operatorname{rad} H = \gamma_b(H) = \gamma_b(F) + a \geq \gamma_b(L) + \gamma_b(R) + a. \tag{4}$$

Say L (R, respectively) has a maximum split-set of cardinality m (m', respectively). By Theorem 3.2, $2\gamma_b(L) = \operatorname{diam} L - m$ and $2\gamma_b(R) = \operatorname{diam} R - m'$. From (4) we have

$$2 \operatorname{rad} H \geq \operatorname{diam} L + \operatorname{diam} R + 2a - m - m'. \tag{5}$$

Suppose d is even and $t \equiv c \pmod 2$. Then A1 holds, $c \pm t$ is even and Z has no zero overlaps at $v_{c\pm t}$. Hence $1 \leq h_{i-1}, h_i \leq 2$. Let Δ' be the triangle immediately preceding Δ_i, and $B' = v_{c'}, u'_1, ..., u'_{t'}$ the bough of Δ'. If $a = t-1$ and $h_{i-1} = 2$, then the leftmost and rightmost vertices of Δ' are $v_{c'-t'}$ and $v_{c'+t'} = v_{c-t+2}$, respectively, the $v_0, ..., v_{c'}$ subpath of P followed by the bough B' is a d-path of L, the path $P_L : v_{c'}, v_{c'+1}, ..., v_{c-t+1}$ is a bough of L, $\operatorname{diam} L = c - t + 2$ (which is even), and the only free edges of L on P_L are $u'_{t'-1}u'_{t'}$ and $v_{c'-t'}v_{c'-t'+1}$. Since $\operatorname{diam} L$ is even, m is even. But $u'_{t'-1}u'_{t'}$ is not a split-edge (being the last edge of P_L), thus $m = 0$. Similarly, if $a = t-1$ and $h_i = 2$, then $\operatorname{diam} R = d-(c+t-2)$ and $m' = 0$.

For all other values of a and h_{i-1}, h_i, $\operatorname{diam} L = c - 2a + t - 1$ and $\operatorname{diam} R = d - (c + 2a - t + 1)$. The possible free edges of L are the edges of the path $Q = v_{c-t+1}, ..., v_{c-2a+t-1}$ of length $2(t - a - 1)$. Similarly, R has $2(t - a - 1)$ possible free edges. But since $c \pm (t-1)$ is odd and d is even, neither $v_{c-t+1}v_{c-t+2}$ nor $v_{c+t-2}v_{c+t-1}$ is a split-edge. Thus by symmetry, and regardless of the values of h_{i-1} and h_i, $m = m'$, and (5) becomes

$$\operatorname{diam} H \geq \operatorname{diam} L + \operatorname{diam} R + 2a - 2m. \tag{6}$$

If L has a split-set $M \subseteq E(Q)$ with $|M| = m$, then $Q - M$ has $m+1$ components. It is possible that one component consists of $v_{c-t+1}v_{c-t+2}$ only, because v_{c-t+1} is adjacent to a vertex of $L - Q$, but all other components have at least two edges, giving a total of at least $2m+1$ edges on Q. Hence $2m + 1 \leq 2(t - a - 1)$, i.e., $2m < 2(t - a - 1)$. Now, if $a = t - 1$ and $h_{i-1} = h_i = 2$, then (5) becomes

$$d = \operatorname{diam} H \geq (c - t + 2) + d - (c + t - 2) + 2(t - 1) = d + 2,$$

which is impossible. Similarly, if exactly one of h_{i-1} and h_i is equal to 2, then we still have $m = m' = 0$, and (5) gives $\operatorname{diam} H \geq d+1$, a contradiction. In all other cases (6) gives

$$\begin{aligned} d = \operatorname{diam} H &\geq (c-2a+t-1) + d - (c+2a-t+1) + 2a - 2m \\ &= d + 2(t-a-1) - 2m > d, \end{aligned}$$

a contradiction.

Suppose d is even and $t \not\equiv c \pmod 2$. Then by B4, $d(v_c, w) \in \{t, t-1\}$ and $x_w \in V(P)$, so $d(z,w) > t-1$ and z does not broadcast to w, a contradiction.

Suppose d is odd and $t \equiv c \pmod 2$. Since Z has no zero overlaps and no overlap $v_{2j-1}v_{2j}$, $h_{i-1} \geq 1$ and $h_i \geq 2$. The possible free edges of L are the edges of the path $Q = v_{c-t+1}, \ldots, v_{c-2a+t-1}$ of length $2(t-a-1)$, and since $c-t+1$ is odd, $v_{c-t+1}v_{c-t+2}$ is not a split-edge. The possible free edges of R are the edges of the path $Q' = v_{c+2a-t+1}, \ldots, v_{c+t-2}$. Hence again $m = m'$ and we obtain a contradiction as above. The case d is odd and $t \not\equiv c \pmod 2$ follows by symmetry. Hence F is connected. ♦

Since $\gamma_b(F) < \gamma_b(H)$ and $\operatorname{rad} F = \operatorname{rad} H$, F is nonradial and has a non-empty split-set. Also, $v_c \notin \mathrm{PN}_f[z]$ (otherwise F would be disconnected). Now we prove that

(VI) $a \neq 1$ and thus $w \in \mathrm{PN}_f[z]$.

Suppose $a = 1$, i.e. $f(u_{t-1}) = 1$. Then the path $B_F = B_i - \mathrm{PN}_f[z]$ has length at least $t-3$. First suppose d is even and $t \equiv c \pmod 2$. Then w satisfies B3. Since d is even, any split-set of F has an even number of edges. Since Z has no zero overlaps at $v_{c\pm t}$, $h_{i-1}, h_i \geq 1$. Therefore the only possible split-set of F is $M = \{v_{c-t+2}v_{c-t+3}, v_{c+t-3}v_{c+t-2}\}$. If B_F has length at least $t-2$, then these edges are not free, and M is not a split-set, a contradiction. Hence assume B_F has length $t-3 \geq 1$. Now, if $x_w \in V(B_F)$ (including the possibility $x_w = v_c$), then by B3, $d(w, v_c) \geq t-2$ and w is the leaf of the bough B_i' of the enhanced shadow tree $Z_{F,P}$. Again M is not a split-set. Therefore $x_w \in V(P) - \{v_c\}$. Assume without loss of generality that x_w is to the left of v_c, say $x_w = v_j$, $j < c$. By B3, $c-t+2 \leq j \leq c-1$. But then w is the leaf of the bough B_j of $Z_{F,P}$ attached to v_j, and in all cases $v_{c-t+2}v_{c-t+3}$ is contained in the triangle that covers B_j. Thus M is not a split-set, once again a contradiction.

Suppose d is even and $t \not\equiv c \pmod 2$. Then w satisfies B4. If $t = 2$, the only possible free edges are the edges of the path $v_{c-2}, v_{c-1}, v_c, v_{c+1}, v_{c+2}$, and since $c \pm 2$ is odd, F does not have a nonempty split-set. Hence assume $t \geq 3$. Then the only possible free edges are the edges of the paths

$v_{c-t}, v_{c-t+1}, v_{c-t+2}, v_{c-t+3}$ and $v_{c+t-3}, v_{c+t-2}, v_{c+t-1}, v_{c+t}$. Since $c \pm t$ is odd, $\{v_{c-t+1}v_{c-t+2}, v_{c+t-2}v_{c+t-1}\}$ is the only possible split-set. By B4 we may assume that $x_w = v_j$ where $c - t + 1 \leq j \leq c$ and we obtain a contradiction as above.

Finally, assume d is odd and $t \equiv c \pmod 2$, so that C3 holds. By C1 and C2, $h_{i-1} \geq 1$ and $h_i \geq 2$. The only possible split-set is $\{v_{c-t+2}v_{c-t+3}\}$. If B_F has length at least $t-2$, or if $x_w \in V(B_F)$, or x_w lies on P to the left of v_c, we obtain a contradiction as in the case where d is even. If x_w lies on $V(P)$ to the right of v_c, then $x_w = v_{c+1}$ and $d(x_w, w) = t - 1$. Again it is easy to see that $v_{c-t+2}v_{c-t+3}$ is not a free edge of $Z_{F,P}$. The case where d is odd and $t \not\equiv c \pmod 2$ follows by symmetry, hence we have proved (VI). ♦

Thus $a \geq 2$ and B_F has length at least $t-2a-1$. Also, since $w \in \mathrm{PN}_f[z]$ but $v_c \notin \mathrm{PN}_f[z]$, (III) implies that $x_w \in V(B_i) - \{v_c\}$. Hence B4 and thus A2 does not hold. Suppose A1 and B3 hold. As before, $h_{i-1}, h_i \geq 1$. It is easy to see that $\{v_{c-t+2}v_{c-t+3}, v_{c+t-3}v_{c+t-2}\}$ is a split-set of F, and, moreover, is contained in a maximum split-set of F. Consider $L = L_{c-t+2}$ and $R = R_{c+t-2}$. Then L and R are radial and have even diameters, and by Theorem 3.2,

$$\gamma_b(F) = \gamma_b(L) + \gamma_b(R) + \gamma_b(F - L - R). \tag{7}$$

If $u_1 \notin \mathrm{PN}_f[z]$, then $F - L - R$ contains the spider $S(t - 2a - 1, t - 3, t - 3)$ as subgraph, and if $u_1 \in \mathrm{PN}_f[z]$, i.e. if $a \geq \lceil \frac{t-1}{2} \rceil$, then $F - L - R$ is the path P_{2t-5}. Since $\operatorname{diam} H - \operatorname{diam} L - \operatorname{diam} R = 2t - 4$,

$$\gamma_b(H) = \gamma_b(L) + \gamma_b(R) + t - 2. \tag{8}$$

But by Proposition 6.3,

$$\begin{aligned}\gamma_b(S((t-2a-1, t-3, t-3)) &= \begin{cases} t - 2a + \frac{4a-6}{3} & \text{if } a \equiv 0 \pmod 3 \\ t - 2a - 1 + 2\lceil \frac{2a-2}{3} \rceil & \text{otherwise} \end{cases} \\ &\geq \begin{cases} t - a + \frac{a-6}{3} & \text{if } a \equiv 0 \pmod 3 \\ t - a - 1 + \lceil \frac{a-4}{3} \rceil & \text{otherwise} \end{cases} \\ &\geq t - a - 1,\end{aligned}$$

and, as shown in [7], $\gamma_b(P_{2t-5}) = \lceil \frac{2t-5}{3} \rceil$. Therefore if $u_1 \notin \mathrm{PN}_f[z]$, then

$$\gamma_b(H) = \gamma_b(L) + \gamma_b(R) + \gamma_b(F - L - R) + a \geq \gamma_b(L) + \gamma_b(R) + t - 1,$$

contrary to (8). If $u_1 \in \mathrm{PN}_f[z]$, then $a \geq \frac{t-1}{2}$ and

$$a + \gamma_b(F - L - R) \geq \left\lceil \frac{t-1}{2} \right\rceil + \left\lceil \frac{2t-5}{3} \right\rceil \geq t - 1 + \left\lceil \frac{t-7}{6} \right\rceil = t - 1$$

since $t \geq 2$. This again contradicts (8).

Suppose A3 and C3 hold. By C1 and C2, $h_{i-1} \geq 1$ and $h_i \geq 2$. Since d is odd, any split-set of F has an odd number of edges. Let $e_1 = v_{c-t+2}v_{c-t+3}$ and $e_2 = v_{c+t-4}v_{c+t-3}$. Then $\{e_1\}$ and $\{e_2\}$ are split-sets, each of which is contained in a maximum split-set of F. If such a set consists of one edge, then by Theorem 3.2, $\gamma_b(F) = \frac{d-1}{2} = \operatorname{rad} F - 1 = \operatorname{rad} H - 1$, hence $\gamma_b(H) = a + \operatorname{rad} H - 1 > \operatorname{rad} H$, which is impossible. Therefore any maximum split-set contains at least three edges. But if $t = 3$, then $e_1 = e_2$ is the only possible split-edge of F, so we assume that $t \geq 4$.

Since $L = L_{c-t+2}$ and $R = R_{c+t-3}$ are radial and have even diameters, $\{e_1, e_2\}$ can be extended to a maximum split-set of F by adding edges of P between e_1 and e_2. Again (7) holds. If $u_1 \notin \mathrm{PN}_f[z]$, then $F - L - R$ contains the spider $S(t-2a-1, t-4, t-3)$ as subgraph, and if $u_1 \in \mathrm{PN}_f[z]$, then $F - L - R$ is the path P_{2t-6}. Thus $\operatorname{diam} H - \operatorname{diam} L - \operatorname{diam} R = 2t - 5$ and

$$\gamma_b(H) = \frac{d+1}{2} = \gamma_b(L) + \gamma_b(R) + t - 2. \tag{9}$$

By Proposition 6.3,

$$\begin{aligned}
&a + \gamma_b(S((t-2a-1, t-4, t-3)) \\
&= a + t - 2a - 1 + \left\lceil \frac{2a-3}{3} \right\rceil + \left\lceil \frac{2a-2}{3} \right\rceil \\
&= t - a - 1 + \begin{cases} \frac{4a-3}{3} & \text{if } a \equiv 0 \pmod 3,\ a \geq 3 \\ 2(\frac{2a-2}{3}) & \text{if } a \equiv 1 \pmod 3,\ a \geq 4 \\ 2(\frac{2a-1}{3}) & \text{if } a \equiv 2 \pmod 3,\ a \geq 2 \end{cases} \\
&\geq t - 1
\end{aligned}$$

and

$$a + \gamma_b(P_{2t-6}) \geq \left\lceil \frac{t-1}{2} \right\rceil + \left\lceil \frac{2t-6}{3} \right\rceil \geq t - 1 - \left\lceil \frac{t-9}{6} \right\rceil = t - 1$$

since $t \geq 4$. Following the method above we obtain a contradiction of (9) in each case. The case where A4 and C4 hold follows by symmetry. This completes the proof of Lemma 4.4. ■

References

[1] G. Chartrand, L. Lesniak, *Graphs and Digraphs.* Fourth Edition, Chapman & Hall, Boca Raton, 2005.

[2] E. J. Cockayne, S. Herke, C. M. Mynhardt, Broadcasts and domination in trees, *Discrete Math.* **311** (2011), 1235–1246.

[3] J. Dabney, B. C. Dean, S. T. Hedetniemi, A linear-time algorithm for broadcast domination in a tree, *Networks* **53** (2009) 160–169.

[4] J. Dunbar, D. Erwin, T. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, Broadcasts in graphs, *Discrete Applied Math.* **154** (2006), 59–75.

[5] J. Dunbar, S. M. Hedetniemi, S. T. Hedetniemi, Broadcasts in trees, Manuscript, 2003.

[6] D. Erwin, Cost domination in graphs, *Dissertation, Western Michigan University*, 2001.

[7] D. Erwin, Dominating broadcasts in graphs, *Bulletin of the ICA* **42** (2004), 89–105.

[8] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[9] P. Heggernes, D. Lokshtanov, Optimal broadcast domination in polynomial time, *Discrete Math.* **36** (2006), 3267-3280.

[10] S. Herke, Dominating broadcasts in graphs, *Master's thesis, University of Victoria*, 2009.

[11] S. Herke, C. M. Mynhardt, Radial Trees, *Discrete Math.* **309** (2009), 5950–5962.

[12] S. M. Seager, Dominating broadcasts of caterpillars, *Ars Combin.* **88** (2008), 307–319.