Some minimal $(r, 2, k)$-regular graphs containing a given graph

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Abstract

A graph G is said to be $(2, k)$-regular graph if each vertex of G is at a distance two away from k vertices of G. A graph G is called $(r, 2, k)$-regular graph if each vertex of G is at a distance 1 away from r vertices of G and each vertex of G is at a distance 2 away from k vertices of G [9]. This paper suggests a method to construct a $((m+n-2), 2, (m-1)(n-1))$-regular graph of smallest order mn containing a given graph G of order $n \geq 2$ as an induced subgraph for any $m > 1$.

Keywords. induced subgraph; clique number; independent number; distance degree; regular graph; (d, k)-regular graphs; $(2, k)$-regular graphs; semiregular.
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1 Introduction

In this paper, we consider only finite, simple, connected graphs. For basic definitions and terminologies we refer Harary [7] and J.A Bondy and U.S.R. Murty [4]. We denote the vertex set and edge set of a graph G by $V(G)$

and $E(G)$ respectively. The degree of a vertex v is the number of edges incident at v. A graph G is regular if all its vertices have the same degree.

For a connected graph G, the distance $d(u, v)$ between two vertices u and v is the length of a shortest (u, v) path. Therefore, the degree of a vertex v is the number of vertices at a distance 1 from v, and it is denoted by $d(v)$. This observation suggests a generalization of degree. That is, $d_d(v)$ is defined as the number of vertices at a distance d from v. Hence $d_1(v) = d(v)$ and $N_d(v)$ denote the set of all vertices that are at a distance d away from v in a graph G. That is, $N_1(v) = N(v)$ and $N_2(v)$ denotes the set of all vertices that are at a distance 2 away from v in a graph G and the closed neighbourhood of v is defined as $N[v] = N(v) \cup \{v\}$.

The concept of distance d-regular graph was introduced and studied by G.S. Bloom, J.K. Kennedy and L.V.Quintas [3]. A graph G is said to be distance d-regular if every vertex of G has the same number of vertices at a distance d from it. If each vertex of G has exactly k number of vertices at a distance d from it, then we denote this graph by (d, k)-regular graph. That is, a graph G is said to be (d, k)-regular if $d_d(v) = k$, for all v in G. The concept of (d, k) regular graphs is a natural extension of the idea of regular graphs. The $(1, k)$-regular graphs are nothing but our usual k-regular graphs.

A graph G is $(2, k)$ regular if $d_2(v) = k$, for all v in G. The concept of the semiregular graph was introduced and studied by Alison Northup [2]. A graph is said to be k-semiregular graph if each vertex of G is at distance two away from exactly k vertices of G. That is, if $d_2(v) = k$, for all v in G. We observe that $(2, k)$ - regular and k - semiregular graphs are the same. A graph G is said to be $(r, 2, k)$-regular if $d(v) = r$ and $d_2(v) = k$, for all $v \in V(G)$.

An induced subgraph of G is a subgraph H of G such that $E(H)$ consists of all edges of G whose end points belong to $V(H)$. In 1936, *König* [8] proved that if G is any graph, whose largest degree is r, then there is an r-regular graph H containing G as an induced subgraph. In 1963, Paul Erdos and Paul Kelly [6] determined the smallest number of new vertices which must be added to a given graph G to obtain such a graph.

The above results motivate us to suggest a method to construct a $(m + n - 2), 2, (m - 1)(n - 1))$-regular graph S of order mn containing a given graph G of order $n \geq 2$ as an induced subgraph, for any $m > 1$.

A clique of a simple graph G is a subset S of V such that the subgraph of G induced by S, is denoted by $G[S]$, is complete. The clique number of

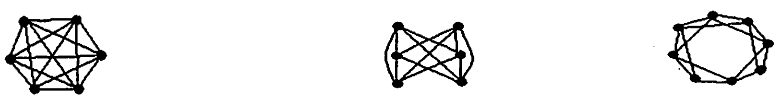
G is the number of vertices in a maximum clique in G. A subset S of V is called an independent set of G if no two vertices of S are adjacent in G. An independent set is maximum if G has no independent S' with $|S'| > |S|$. The number of vertices in a maximum independent set of G is called the independent number of G.

2 $(2, k)$-regular graphs

Definition 2.1. *A graph G is said to be $(2, k)$-regular graph if each vertex of G is at a distance two away from exactly k vertices. That is, $d_2(v) = k$, for all vertices in G.*

Remark 2.2. *There are two types of $(2, k)$-regular graphs exist. They are non-regular graphs which are $(2, k)$-regular and regular graphs which are $(2, k)$-regular.*

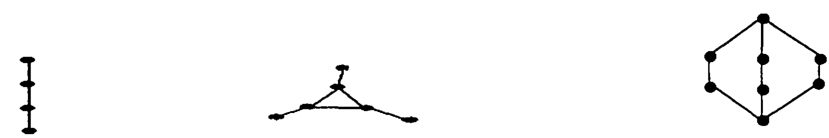
Example 2.3. *(i) Regular graphs which are $(2, k)$-regular.*



(2 , 0) regular . (2 , 1) regular . (2 , 3) regular.

Figure 1

(ii) Non-regular graphs which are $(2, k)$-regular.

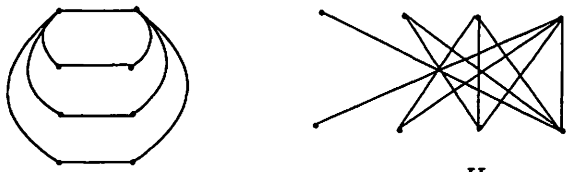


(2, 1)–regular. (2, 2)–regular . (2 , 3) – regular.

Figure 2

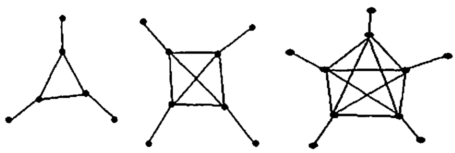
(iii) Book graph $B_n = S_n \times P_2 (n \geq 2)$, (where S_n is the star graph of order n and P_2 is the path graph of order 2) is $(2, (n-1))$-regular graph.

(iv) Let $H_{n,n}$ [1, 5] denote the bipartite graph having two partite sets $V_1 = \{v_1, v_2, v_3, v_4, \ldots, v_n\}$ and $V_2 = \{u_1, u_2, u_3, u_4, \ldots, u_n\}$ and edge set $E(H_{n,n}) = \bigcup_{i=1}^{n} E_i$, where $E_i = \{v_i u_j : n - i + 1 \leq j \leq n \text{ and } (1 \leq i \leq n)\}$. This graph $H_{n,n}$ is $(2, (n-1))$-regular graph.



B_4 *Figure 3* $H_{4,4}$

(v) Consider, the graph which contains K_n and a pendant vertex attached to each vertex of K_n is of order $2n$. This graph is a $(2,(n-1))$-regular graph with clique number n and independent number n having the smallest order $2n$.



G_1 G_2 G_3

Figure 4

From (v), we observe the following result.

Result 2.4. *For any $n \geq 2$, the smallest order of $(2,(n-1))$-regular graph with clique number n and independent number n is $2n$.*

3 $(r,2,k)$-regular graph

Definition 3.1. *A graph G is called $(r,2,k)$-regular if each vertex in graph G is at a distance one from exactly r-vertices and at a distance two from exactly k vertices. That is, $d(v)=r$ and $d_2(v)=k$, for all v in G.*

The following facts are known from literature:

Fact 3.2. *[7] If G is $(r,2,k)$-regular graph, then $0 \leq k \leq r(r-1)$.*

Fact 3.3. *[9] For any $r>1$, a graph G is $(r,2,r(r-1))$-regular if G is r-regular with girth at least five.*

Fact 3.4. *[10] For any odd $r \geq 3$, there is no $(r,2,1)$-regular graph.*

Fact 3.5. *[10] Any $(r,2,k)$-regular graph has at least $k+r+1$ vertices.*

Fact 3.6. *[10] If r and k are odd, then $(r,2,k)$-regular graph has at least $k+r+2$ vertices.*

Fact 3.7. *[10] For any $r \geq 2$ and $k \geq 1$, G is a $(r,2,k)$-regular graph of order $r+k+1$ if and only if $diam(G)=2$.*

Fact 3.8. *[10] For any $r \geq 2$, there is a $(r, 2, (r-1)(r-1))$-regular graph on $4 \times 2^{r-2}$ vertices.*

Fact 3.9. *[10] For $r > 1$, if G is a $(r, 2, (r-1)(r-1))$-regular graph, then G has girth four.*

Fact 3.10. *[9] For any $n \geq 5$, $(n \neq 6, 8)$ and any $r > 1$, there exists a $(r, 2, r(r-1))$-regular graph on $n \times 2^{r-2}$ vertices with girth five.*

Fact 3.11. *[11] For any $r \geq 2$, there is a $(r, 2, (r-2)(r-1))$-regular graph on $3 \times 2^{r-2}$ vertices.*

4 Minimal $(r, 2, k)$-regular graphs containing a given graph as an induced subgraph

König [8] proved that if G is any graph, whose largest degree is r, then it is possible to add new points and to draw new lines joining either two new points or a new point to an existing point, so that the resulting graph H is a regular graph containing G as an induced subgraph. Paul Erdös and Paul Kelly [6] determined the smallest number of new vertices which must be added to a given graph G to obtain such a graph. We now suggest a method that may be considered an analogue to König's theorem for $(r, 2, k)$-regular graph.

Theorem 4.1. *For any $m > 1$, every graph G of order $n \geq 2$ is an induced subgraph of a $(n+m-2, 2, (m-1)(n-1))$-regular graph H of order mn.*

Proof. Let G be the given graph of order $n \geq 2$ with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let G_t denote a copy of G with $V(G_t) = \{v_1^t, v_2^t, v_3^t \cdots, v_n^t\}$, $t = 1, 2, \cdots, m$. Let H be the graph with vertex set

$$V(H) = \{v_i^t | i = 1, 2, 3, \cdots, n, t = 1, 2, 3, \cdots, m\} = \bigcup_{t=1}^{m} V(G_t)$$

and the edge set

$$E(H) = \bigcup_{t=1}^{m} E(G_t) \cup \bigcup_{t=1}^{m-1} \{v_j^t v_i^{t+1}, v_j^m v_i^1 | v_j^1 v_i^1 \notin E(G_1), 1 \leq j \leq n, j+1 \leq i \leq n\}$$

$$\cup \bigcup_{k=1}^{n} \{v_k^i v_k^{i+j} | 1 \leq i \leq m-1, 1 \leq j \leq m-i\}$$

The resulting graph H contains G as an induced subgraph. Moreover in H, $d(v_i^t) = m+n-2$, for $1 \leq i \leq n, 1 \leq t \leq m$. That is, H is $(m+n-2)$-regular graph with mn vertices.

To find the d_2-degree of each vertex in H. We examine the following cases:

Case 1. When $t = 1$.

If $v \in V(G_1)$, then $v = v_j^1$, for some j. Let $v_j^1 \in V(H) - N[v_i^1]$. Then v_j^1 and v_i^1 are non-adjacent vertices in H. By our construction, v_j^1 is adjacent to v_i^2 and v_i^2 is adjacent to v_i^1. That is, $d(v_j^1, v_i^1) = 2$. Therefore, $v_j^1 \in N_2(v_i^1)$. This implies that $V(H) - N[v_i^1] \subseteq N_2(v_i^1)$. Suppose $v_j^1 \in N_2(v_i^1)$, then, v_j^1 is non-adjacent with v_i^1. Therefore $v_j^1 \in V(H) - N[v_i^1]$. Therefore, $N_2(v_i^1) = V(H) - N[v_i^1]$, for $1 \leq i \leq n$ and $d_2(v_i^1) = (m-1)(n-1)$, for $1 \leq i \leq n$

Case 2 When $2 \leq t \leq m-1$.

If $v \in V(G_t)$, then $v = v_j^t$, for some j. Let $v_j^t \in V(H) - N[v_i^1]$, Then v_j^t and v_i^1 are non-adjacent vertices in H. By our construction, v_j^t is adjacent to v_i^t and v_i^t is adjacent to v_i^1. That is, $d(v_j^t, v_i^1) = 2$. Therefore, $v_j^t \in N_2(v_i^1)$, This implies that $V(H) - N[v_i^1] \subseteq N_2(v_i^1)$. Suppose $v_j^t \in N_2(v_i^1)$, then v_j^t is non-adjacent with v_i^1. Therefore $v_j^t \in V(H) - N[v_i^1]$. Therefore, $N_2(v_i^1) = V(H) - N[v_i^1]$, for $1 \leq i \leq n$ and $d_2(v_i^1) = (m-1)(n-1)$, for $1 \leq i \leq n$.

Case 3 When $t = m$.

If $v \in V(G_m)$, then $v = v_j^m$, for some j. Let $v_j^m \in V(H) - N[v_i^1]$, Then v_j^m and v_i^1 are non-adjacent vertices in H. By our construction, v_j^m is adjacent to v_i^m and v_i^m is adjacent to v_i^1. That is, $d(v_j^m, v_i^1) = 2$. Therefore, $v_j^m \in N_2(v_i^1)$, .This implies that $V(H) - N[v_i^1] \subseteq N_2(v_i^1)$. Suppose $v_j^m \in N_2(v_i^1)$, then v_j^m is non-adjacent with v_i^1. Therefore $v_i^m \in V(H) - N[v_i^1]$. Therefore, $N_2(v_i^1) = V(H) - N[v_i^1]$ for $1 \leq i \leq n$ and $d_2(v_i^1) = (m-1)(n-1)$, for $1 \leq i \leq n$.

Similarly, for $2 \leq t \leq m$, $N_2(v_i^t) = V(H) - N[v_i^t]$ and $d_2(v_i^t) = (m-1)(n-1)$, for $1 \leq i \leq n$. Therefore, H is a $(m+n-2, 2, (m-1)(n-1))$-regular graph of order mn containing the given G as an induced subgraph.

For any graph of order $n \geq 2$, there exists a $(m+n-2, 2, (m-1)(n-1))$-regular graph H of order mn containing a given graph as an induced subgraph.

Every graph G of order $n \geq 2$ is an induced subgraph of a $(n+m-2, 2, (m-1)(n-1))$-regular graph H of order mn. □

Corollary 4.2. *For any $m > 1$, the smallest order of a $(n+m-2, 2, (m-1)(n-1))$-regular graph H containing a given graph G of order $n \geq 2$ as an induced subgraph is mn.*

Proof. The graph H constructed in Theorem 4.1, is $(n+m-2)$-regular with $(2, (m-1)(n-1))$-regular graph of smallest order mn. Suppose H is $n+m-2$-regular with $(2, (m-1)(n-1))$-regular graph of order $mn-1$. That is, for each $v_i \in H$, $d_2(v_i) = (m-1)(n-1)$ and $d(v_i) = n+m-2, 1 \leq i \leq 2n$.

Therefore, H has at least $(m-1)(n-1)+n+m-2+1 = mn$ vertices, which is a contradiction. □

Corollary 4.3. *Every graph G of order $n \geq 2$ is an induced subgraph of $(n, 2, (n-1))$-regular graph of smallest order $2n$.*

Example 4.4. *Figure.5. illustrates the Corollary 4.3 for $n = 3$ and $m = 2$.*

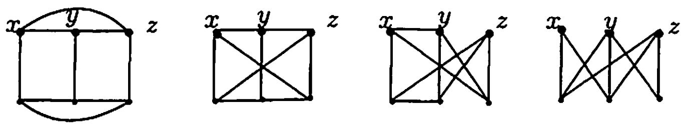


Figure 5

In the above graphs, the graph G is induced by the vertices x, y, z.

Corollary 4.5. *Every graph G of order $n \geq 2$ is an induced subgraph of $(n+1, 2, 2(n-1))$-regular graph of smallest order $3n$.*

Example 4.6. *Figure 6 illustrates the Corollary 4.5 for $n = 3$ and $m = 3$.*

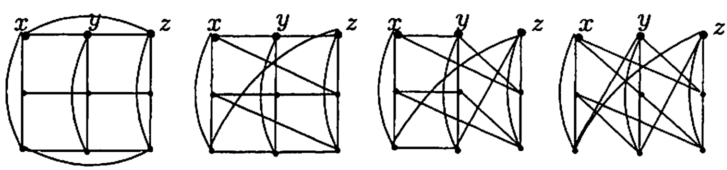


Figure 6

In the above graphs, the graph G is induced by the vertices x, y, z.

Corollary 4.7. *Every graph G of order $n \geq 2$ is an induced subgraph of $(n+2, 2, 3(n-1))$-regular graph of smallest order $4n$.*

Remark 4.8. *There are at least as many $((n+m-2), 2, (m-1)(n-1))$-regular graphs of order mn as there are graphs G of order $n \geq 2$. If we put $m = 2, 3, 4, 5, \ldots n, \ldots$, then there are $(n, 2, (n-1)), (n+1, 2, 2(n-1)), (n+2, 2, 3(n-1)), (n+3, 2, 4(n-1)), \ldots, (2n-2, 2, (n-1)^2) \ldots$ regular graphs of smallest order $2n, 3n, 4n, 5n, \ldots, n^2, \ldots$ respectively containing any graph G of order $n \geq 2$ as an induced subgraph.*

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