Decompositions of λK_n into $S(4,3)$'s

Derek W. Hein and Dinesh G. Sarvate

ABSTRACT. A *Stanton–type graph* $S(n,m)$ is a connected multigraph on n vertices such that for a fixed integer m with $n-1 \le m \le \binom{n}{2}$, there is exactly one edge of multiplicity i (and no others) for each $i = 1,2,\ldots,m$. In a recent paper, the authors decomposed λK_n (for the appropriate minimal values of λ) into two of the four possible types of $S(4,3)$'s. In this note, decompositions of λK_n (for the appropriate minimal values of λ) into the remaining two types of $S(4,3)$'s are given.

1. Introduction

A *simple graph* G is an ordered pair (V,E) where V is an n–set (of *points*), and E is a subset of the set of the $\binom{n}{2}$ pairs of distinct elements of V (the *edges*). This definition can be generalized to that of a *multigraph* (without loops) by allowing E to be a multiset, where edges can occur with *frequencies* greater than 1.

DEFINITION 1. *A complete multigraph λK_n (for $\lambda \ge 1$) is a graph on $n \ge 2$ points with λ edges between every pair of distinct points.*

A graph can be decomposed into a collection of subgraphs such that every edge of the graph is contained in one of the subgraphs. In particular:

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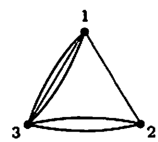
DEFINITION 2. *A G–*decomposition *of λK_n is a collection of subgraphs (each isomorphic to G) such that the multiunion of their edge sets contains λ copies of all edges of K_n.*

Decomposing λK_n into simple graphs has been well–studied in the literature. For a well–written survey on the decomposition of a complete graph into simple graphs with small numbers of points and edges, see [**1**]. The decomposition of copies of a complete graph into proper multigraphs has not received much attention (see [**2, 5, 6, 7**]).

DEFINITION 3. [**2**] *A* Stanton graph *S_n is a multigraph on $n \geq 2$ vertices such that for each $i = 1, 2, \ldots, \binom{n}{2}$, there is exactly one edge of multiplicity i (and no others).*

EXAMPLE 1. *The unique (up to isomorphism) Stanton graph S_3 on $V = \{1, 2, 3\}$ with edge set $E = \big\{\{1,2\}, \{1,3\}, \{1,3\}, \{1,3\}, \{2,3\}, \{2,3\}\big\}$ can be drawn as*



Chan and Sarvate [**2**] decomposed λK_n into Stanton graphs S_3 for the appropriate minimal λ. El–Zanati, Lapchinda, Tangsupphathawat and Wannasit [**3**] have proved that the necessary conditions are sufficient for a decomposition of λK_n into S_3's.

DEFINITION 4. [**4**] *Given a positive integer $n \geq 2$ and an integer m such that $n - 1 \leq m \leq \binom{n}{2}$, a* Stanton–type graph *$S(n, m)$ is a connected multigraph on n vertices such that for $i = 1, 2, \ldots, m$, there is exactly one edge of multiplicity i (and no others).*

NOTE 1. *It should be noted that an $S(n, n-1)$ is a cycle–free connected multigraph.*

NOTE 2. *It is also true that an $S(n, \binom{n}{2})$ is the same as an S_n.*

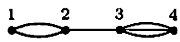
There are exactly 4 nonisomorphic $S(4, 3)$ (each having 6 edges): the LOE–type and OLE–type graphs (described in [**4**]), and the so–called LEO–type and ELO–type graphs (described in the sequel).

DEFINITION 5. [**4**] *Let $V = \{a, b, c, d\}$. An* LOE–type graph *$\langle a, b, c, d\rangle$ on V is a graph with 6 edges where the frequencies of edges $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$ are 1, 2 and 3 respectively.*

EXAMPLE 2. *Consider* $G_1 = (V, E)$ *where* $V = \{1, 2, 3, 4\}$ *and* $E = \{\{1, 2\}, \{2, 3\}, \{2, 3\}, \{3, 4\}, \{3, 4\}, \{3, 4\}\}$. *Then* G_1 *is an LOE–type graph, denoted* $\langle 1, 2, 3, 4 \rangle$.

DEFINITION 6. [4] *Let* $V = \{a, b, c, d\}$. *An* OLE–type graph $[a, b, c, d]$ *on* V *is a graph with 6 edges where the frequencies of edges* $\{a, b\}$, $\{b, c\}$ *and* $\{c, d\}$ *are 2, 1 and 3 respectively.*

EXAMPLE 3. *Consider* $G_2 = (V, E)$ *where* $V = \{1, 2, 3, 4\}$ *and* $E = \{\{1, 2\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 4\}, \{3, 4\}\}$. *Then* G_2 *is an OLE–type graph, denoted* $[1, 2, 3, 4]$.

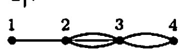


Hein and Sarvate [4] showed how to decompose λK_n into LOE–type and OLE–type graphs. In the sequel, we show how to decompose λK_n into LEO–type and ELO–type graphs (described in the next section), thereby completing the result for all nonisomorphic $S(4, 3)$. Though the main technique used is to construct appropriate base graphs and to develop them cyclically, an additional lemma is used in each type of decomposition. Another difference is that when $n = 4$, the minimum λ for LEO–decompositions is 6 (not 3, as it is for LOE–, OLE– and ELO–decompositions).

2. Preliminaries

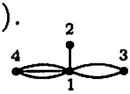
DEFINITION 7. *Let* $V = \{a, b, c, d\}$. *An* LEO–type graph $|a, b, c, d|$ *on* V *is a graph with 6 edges where the frequencies of edges* $\{a, b\}$, $\{b, c\}$ *and* $\{c, d\}$ *are 1, 3 and 2 respectively.*

EXAMPLE 4. *Consider* $G_3 = (V, E)$ *where* $V = \{1, 2, 3, 4\}$ *and* $E = \{\{1, 2\}, \{2, 3\}, \{2, 3\}, \{2, 3\}, \{3, 4\}, \{3, 4\}\}$. *Then* G_3 *is an LEO–type graph, denoted* $|1, 2, 3, 4|$.



DEFINITION 8. *Let* $V = \{a, b, c, d\}$. *An* ELO–type graph (a, b, c, d) *on* V *is a graph with 6 edges where the frequencies of edges* $\{a, b\}$, $\{a, c\}$ *and* $\{a, d\}$ *are 1, 2 and 3 respectively.*

EXAMPLE 5. *Consider* $G_4 = (V, E)$ *where* $V = \{1, 2, 3, 4\}$ *and* $E = \{\{1, 2\}, \{1, 3\}, \{1, 3\}, \{1, 4\}, \{1, 4\}, \{1, 4\}\}$. *Then* G_4 *is an ELO–type graph, denoted* $(1, 2, 3, 4)$.



LEMMA 1. *The multigraph λK_n can be decomposed by LEO–type or ELO–type graphs only if 6 divides $\lambda\binom{n}{2}$.*

THEOREM 1. *The minimum number of copies λ of the complete graph K_n that could be decomposed into LEO–type or ELO–type graphs is*

a) $\lambda = 3$, *when* $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$,

b) $\lambda = 4$, *when* $n \equiv 3, 6, 7, 10 \pmod{12}$ *and*

c) $\lambda = 6$, *when* $n \equiv 2, 11 \pmod{12}$

with the exception of an LEO–decomposition for $n = 4$, which has a minimum $\lambda = 6$.

PROOF. The results can be obtained by two–way counting, divisibility requirements and Lemma 1. We now address the exception. We let $V = \{v_1, \ldots, v_4\}$, $e_1 = \{v_1, v_2\}$, $e_2 = \{v_3, v_4\}$, and note that we must have λ LEO–type graphs in each decomposition.

Suppose that an LEO–decomposition of $3K_4$ exists. Without loss of generality, if the edge e_1 has frequency 1 in a graph (say, G_1) in this decomposition, then there are two cases for the remaining frequencies on edge e_1: either e_1 has frequency 1 in each of two other graphs (say, G_2 and G_3) in the decomposition, or else e_1 has frequency 2 in another graph (say, G_2) in the decomposition. In the former case, the edge e_2 occurs in G_1, G_2 and G_3 with total frequency 6, which cannot occur. In the latter case, the edge e_2 has frequency 2 in G_1 and frequency 1 in G_2. Hence, graphs in this decomposition occurs in pairs; that is, there must be an even number of graphs in this decomposition. This contradicts the fact that the number of graphs in the decomposition is odd. Hence, the minimal $\lambda \neq 3$. ♦

Suppose that an LEO–decomposition of $4K_4$ exists. Without loss of generality, if the edge e_1 has frequency 3 in a graph (say, G_1) in this decomposition, then it must have frequency 1 in another graph (say, G_2) in the decomposition. Then edge e_2 has frequency 2 in G_2. There are two cases for the remaining frequencies on edge e_2: either e_2 has frequency 1 in each of two more graphs (say, G_3 and G_4) in the decomposition, or else e_2 has frequency 2 in another graph (say, G_3) in the decomposition. The former case gives edge e_1 frequency 2 in each of G_3 and G_4, for a total frequency of 8, which cannot occur. The latter case gives edge e_1 frequency 1 in G_3, for a total frequency of at least 5, which cannot occur. Hence, the minimal $\lambda \neq 4$. ♦

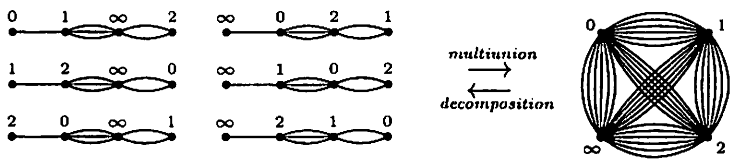
Suppose that an LEO–decomposition of $5K_4$ exists. Suppose that the edge e_1 has frequency 1 in two distinct graphs (say, G_1 and

G_2) in this decomposition. Then the edge e_2 has frequency 2 in each of G_1 and G_2. Then, e_2 must have frequency 1 in another graph (say, G_3) in the decomposition. But, e_1 will have frequency 2 in G_3, and thus must have frequency 1 in another graph (say, G_4) in the decomposition. This implies that e_2 has frequency 2 in G_4, giving e_2 frequency at least 7, which cannot occur. Thus, *we cannot have an edge of frequency 1 in two distinct graphs in this decomposition.* Now suppose that e_1 has frequency 2 in two distinct graphs (say, G_1 and G_2) in this decomposition. Then the edge e_2 has frequency 1 in each of G_1 and G_2. This situation cannot occur, as shown above. Thus, *we cannot have an edge of frequency 2 in two distinct graphs in this decomposition.* Of course, we cannot have an edge of frequency 3 in two distinct graphs in this decomposition, for then $\lambda \geq 6$. Hence, *no edge can occur more than once with the same multiplicity in this decomposition.* By simple counting, there must be at least 3 edges that occur once with multiplicity 1, once with multiplicity 2 and once with multiplicity 3, which cannot occur (since it forces $\lambda \geq 6$). Hence, the minimal $\lambda \neq 5$. ♦

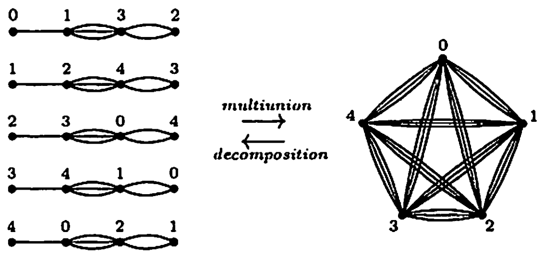
Thus, the minimal λ for LEO–decompositions of K_4 is 6, for which an LEO–decomposition exists (as given below). ■

The following examples illustrate the development of base graphs into decompositions of λK_n. We use difference family–type constructions to achieve decompositions of λK_n. In general, we exhibit the "base graphs", which can be developed (modulo either n or $n-1$) to obtain the decomposition. We note that special attention is needed with the base graphs containing the "dummy element" ∞; the non–∞ elements are developed, while ∞ is simply rewritten each time. We further note that the multiplicity of the edges is fixed by position (as per the LEO– or ELO–type of the graph). Lastly, we remark that LEO– and ELO–type graphs have 4 vertices; hence, $n \geq 4$.

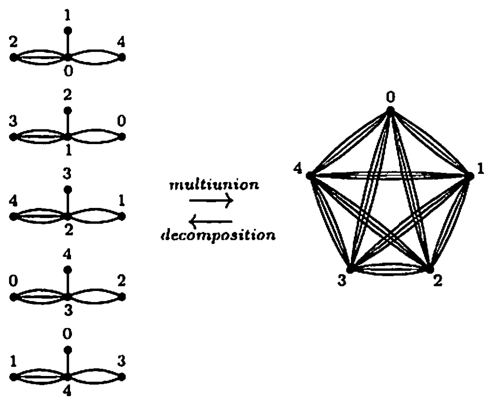
EXAMPLE 6. *Considering the set of points to be $V = \mathbb{Z}_3 \cup \{\infty\}$, the LEO–type base graphs $|0, 1, \infty, 2|$ and $|\infty, 0, 2, 1|$ (when developed modulo 3) constitute an LEO–decomposition of $6K_4$.*



EXAMPLE 7. *Considering the set of points to be $V = \mathbb{Z}_5$, the LEO–type base graph* $|0, 1, 3, 2|$ *(when developed modulo 5) constitutes an LEO–decomposition of* $3K_5$.



EXAMPLE 8. *Considering the set of points to be $V = \mathbb{Z}_5$, the ELO–type base graph* $(0, 1, 4, 2)$ *(when developed modulo 5) constitutes an ELO–decomposition of* $3K_5$.



3. Decompositions of λK_n

We are now in a position to prove the main results of the paper. We remind the reader that Theorem 1 gives the minimum number λ of copies of K_n under discussion in each case.

THEOREM 2. *The minimum number of copies of K_n can be decomposed into graphs of the LEO–type.*

PROOF. We begin with $\lambda = 3$. The values of n that correspond to this λ are $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ with the exception of $n = 4$. (We recall that for LEO–decompositions of λK_4, the minimal λ is 6. Such a decomposition is given in Example 6.) These cases are equivalent to $n \equiv 0, 1 \pmod 4$.

Case: $n \equiv 0 \pmod 4$

Let $n = 4t$ with $t \geq 2$. We consider the set V as $\mathbb{Z}_{4t-1} \cup \{\infty\}$. Then the differences we must achieve (modulo $4t-1$) are $1, 2, \ldots, 2t-1$. The number of graphs required is $\frac{3n(n-1)}{12} = \frac{(4t)(4t-1)}{4} = t(4t-1)$. Thus, we need t base graphs (modulo $4t-1$). For the first $t-2$ base graphs, we use $|2x-1, 0, 2x, 4x-1|$ for $x = 1, 2, \ldots, t-2$. For the last two base graphs, use $|\infty, 2t-3, 0, 2t-2|$ and $|4t-3, 2t-1, 0, \infty|$. Hence, in this case, an LEO–decomposition of $3K_n$ exists. ▲

Case: $n \equiv 1 \pmod 4$

Let $n = 4t+1$ with $t \geq 1$. We consider the set V as $\mathbb{Z}_{4t+1}$. Then the differences we must achieve (modulo $4t+1$) are $1, 2, \ldots, 2t$. The number of graphs required is $\frac{3n(n-1)}{12} = \frac{(4t+1)(4t)}{4} = t(4t+1)$. Thus, we need t base graphs (modulo $4t+1$). We use $|2x-1, 0, 2x, 4x-1|$ for $x = 1, 2, \ldots, t$. Hence, in this case, an LEO–decomposition of $3K_n$ exists. ♦

We now address $\lambda = 4$. The values of n that correspond to this λ are $n \equiv 3, 6, 7, 10 \pmod{12}$. These cases are equivalent to $n \equiv 0, 1 \pmod 3$.

Case: $n \equiv 0 \pmod 3$

First note that LEO–decompositions of $4K_3$ do not exist since LEO–type graphs have 4 vertices. We consider two subcases (t even and t odd) for $n = 3t$ with $t \geq 2$.

Let $n = 3t$ and $t = 2s$; that is, let $n = 6s$. We consider the set V as $\mathbb{Z}_{6s-1} \cup \{\infty\}$. Then the differences we must achieve (modulo $6s-1$) are $1, 2, \ldots, 3s-1$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s)(6s-1)}{3} = 2s(6s-1)$. Thus, we need $2s$ base graphs (modulo $6s-1$). For the first $2s-1$ base graphs, we use $|0, 1, 2s, 5s-1|$, $|0, 2, 2s, 5s-2|$, $|0, 3, 2s, 5s-2|$, $|0, 4, 2s, 5s-3|$, $|0, 5, 2s, 5s-3|, \ldots, |0, 2s-2, 2s, 4s|$ and $|0, 2s-1, 2s, 4s|$. For the last base graph, use $|1, \infty, 0, 3s-1|$. Hence, in this subcase, an LEO–decomposition of $4K_n$ exists. •

Let $n = 3t$ and $t = 2s+1$; that is, let $n = 6s+3$. We consider the set V as $\mathbb{Z}_{6s+2} \cup \{\infty\}$. Then the differences we must achieve (modulo $6s+2$) are $1, 2, \ldots, 3s+1$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+3)(6s+2)}{3} = (2s+1)(6s+2)$. Thus, we need $2s+1$ base graphs (modulo $6s+2$). For the first $2s$ base graphs, we use $|0,1,2s+1,5s+1|$, $|0,2,2s+1,5s+1|$, $|0,3,2s+1,5s|$, $|0,4,2s+1,5s|,\ldots,|0,2s-1,2s+1,4s+2|$ and $|0,2s,2s+1,4s+2|$. For the last base graph, we use $|1,\infty,0,3s+1|$. Hence, in this subcase, an LEO–decomposition of $4K_n$ exists. ▲

Case: $n \equiv 1 \pmod 3$

We again consider two subcases (t even and t odd) for $n = 3t+1$ with $t \geq 2$.

Let $n = 3t+1$ and $t = 2s$; that is, let $n = 6s+1$. We consider the set V as $\mathbb{Z}_{6s+1}$. Then the differences we must achieve (modulo $6s+1$) are $1, 2, \ldots, 3s$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+1)(6s)}{3} = 2s(6s+1)$. Thus, we need $2s$ base graphs (modulo $6s+1$). We use the base graphs $|0,x,2x+s,3x+3s|$ and $|0,x+s,2x+s,3x+3s|$ for $x = 1,2,\ldots,s$. Hence, in this subcase, an LEO–decomposition of $4K_n$ exists. •

For t odd, we note the following interesting lemma:

LEMMA 2. *Let $n = 3t+1$ with t even. If a specified cyclic LEO–decomposition of $4K_n$ exists, then a cyclic LEO–decomposition of $4K_{n+3}$ exists.*

PROOF. Notice that $n = 3t+1$ implies that $n+3 = 3(t+1)+1$, and that t even implies that $t+1$ is odd. Let an LEO–decomposition of $4K_n$ exist with specified base graphs (developed modulo n to give the decomposition) as given in the subcase above. Adjoin to these base graphs the base graph $|0,3s+1,6s+2,3s|$. Then, when all base graphs are developed modulo $n+3$, the aggregate is an LEO–decomposition of $4K_{n+3}$. ▲

Hence, in this subcase, an LEO–decomposition of $4K_n$ exists. ♦

Lastly, we address $\lambda = 6$. The values of n that correspond to this λ are $n \equiv 2, 11 \pmod{12}$.

Case: $n \equiv 2 \pmod{12}$

Let $n = 12t + 2$ with $t \geq 1$. We consider the set V as $\mathbb{Z}_{12t+1} \cup \{\infty\}$. Then the differences we must achieve (modulo $12t + 1$) are $1, 2, \ldots, 6t$. The number of graphs required is $\frac{6n(n-1)}{12} = \frac{(12t+2)(12t+1)}{2} = (6t+1)(12t+1)$. Thus, we need $6t+1$ base graphs (modulo $12t+1$). For the first $6t-1$ base graphs, we use $|0, 1, 6t, 12t-1|$, $|0, 2, 6t, 12t-2|$, $|0, 3, 6t, 12t-3|, \ldots, |0, 6t-2, 6t, 6t+2|$ and $|0, 6t-1, 6t, 6t+1|$. For the last two base graphs, use $|\infty, 0, 6t, 12t|$ and $|6t, 0, \infty, 1|$. Hence, in this case, an LEO–decomposition of $6K_n$ exists. ▲

Case: $n \equiv 11 \pmod{12}$

Let $n = 12t + 11$ with $t \geq 0$. We consider the set V as $\mathbb{Z}_{12t+11}$. Then the differences we must achieve (modulo $12t+11$) are $1, 2, \ldots, 6t + 5$. The number of graphs required is $\frac{6n(n-1)}{12} = \frac{(12t+11)(12t+10)}{2} = (6t+5)(12t+11)$. Thus, we need $6t+5$ base graphs (modulo $12t+11$). We use $|2, 1, 6t+6, 0|$, $|0, 2, 6t+6, 12t+10|$, $|0, 3, 6t+6, 12t+9|$, $|0, 4, 6t+6, 12t+8|, \ldots, |0, 6t+4, 6t+6, 6t+8|$, and $|0, 6t+5, 6t+6, 6t+7|$. Hence, in this case, an LEO–decomposition of $6K_n$ exists.

■

THEOREM 3. *The minimum number of copies of K_n can be decomposed into graphs of the ELO–type.*

PROOF. We begin with $\lambda = 3$. The values of n that correspond to this λ are $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$. These cases are equivalent to $n \equiv 0, 1 \pmod 4$.

Case: $n \equiv 0 \pmod 4$

Let $n = 4t$ with $t \geq 1$. We consider the set V as $\mathbb{Z}_{4t-1} \cup \{\infty\}$. Then the differences we must achieve (modulo $4t - 1$) are $1, 2, \ldots, 2t - 1$. The number of graphs required is $\frac{3n(n-1)}{12} = \frac{(4t)(4t-1)}{4} = t(4t-1)$. Thus, we need t base graphs (modulo $4t - 1$). If $t = 1$ (so that $n = 4$), we use the base graph $(0, 2, 1, \infty)$. If $t \geq 2$, for the first $t - 1$ base graphs, we use $(0, 2x - 1, 2x + 1, 2x)$ for $x = 1, \ldots, t - 1$. For the last base graph, we use $(0, 2t-1, 1, \infty)$. Hence, in this case, an ELO–decomposition of $3K_n$ exists. ▲

Case: $n \equiv 1 \pmod 4$

Let $n = 4t + 1$ with $t \geq 1$. We consider the set V as $\mathbb{Z}_{4t+1}$. Then the differences we must achieve (modulo $4t + 1$) are $1, 2, \ldots, 2t$. The number of graphs required is $\frac{3n(n-1)}{12} = \frac{(4t+1)(4t)}{4} = t(4t + 1)$. Thus, we need t base graphs (modulo $4t + 1$). If $t = 1$ (so that $n = 5$), we use the base graph $(0, 3, 2, 1)$. If $t \geq 2$, for the first $t - 1$ base graphs, we use $(0, 2x, 2x + 2, 2x - 1)$ for $x = 1, \ldots, t - 1$. For the last base graph, we use $(0, 2t, 2, 2t - 1)$. Hence, in this case, an ELO–decomposition of $3K_n$ exists. ♦

We now address $\lambda = 4$. The values of n that correspond to this λ are $n \equiv 3, 6, 7, 10 \pmod{12}$. These cases are equivalent to $n \equiv 0, 1 \pmod 3$.

Case: $n \equiv 0 \pmod 3$

First note that ELO–decompositions of $4K_3$ do not exist since ELO–type graphs have 4 vertices. We consider two subcases (t even and t odd) for $n = 3t$ with $t \geq 2$.

Let $n = 3t$ and $t = 2s$; that is, let $n = 6s$. We consider the set V as $\mathbb{Z}_{6s-1} \cup \{\infty\}$. Then the differences we must achieve (modulo $6s - 1$) are $1, 2, \ldots, 3s - 1$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{6s(6s-1)}{3} = 2s(6s - 1)$. Thus, we need $2s$ base graphs (modulo $6s - 1$). For the first $2s - 2$ base graphs, we use $(0, 3x - 2, 3x - 1, 3x)$ and $(0, 3x, 3x - 1, 3x - 2)$ for $x = 1, 2, \ldots, s - 1$. For the last two base graphs, use $(0, \infty, 3s - 1, 3s - 2)$ and $(0, 3s - 2, 3s - 1, \infty)$. Hence, in this subcase, an ELO–decomposition of $4K_n$ exists. •

Let $n = 3t$ and $t = 2s + 1$; that is, let $n = 6s + 3$. We consider the set V as $\mathbb{Z}_{6s+2} \cup \{\infty\}$. Then the differences we must achieve (modulo $6s + 2$) are $1, 2, \ldots, 3s + 1$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+3)(6s+2)}{3} = (2s + 1)(6s + 2)$. Thus, we need $2s + 1$ base graphs (modulo $6s + 2$). For the first three base graphs, we use $(0, \infty, 3s + 1, 1)$, $(0, 1, 2, \infty)$ and $(0, 3, 2, 6s - 1)$. For the last $2s - 2$ base graphs, we use $(0, 3x - 2, 3x - 1, 3x)$ and $(0, 3x, 3x - 1, 3x - 2)$ for $x = 2, 3, \ldots, s$. Hence, in this subcase, an ELO–decomposition of $4K_n$ exists. ▲

Case: $n \equiv 1 \pmod 3$

We note the following interesting lemma:

LEMMA 3. *If an ELO-decomposition of $4K_m$ exists for $m = 3t$ with $t \geq 2$, then an ELO-decomposition of $4K_{m+1}$ exists.*

PROOF. Let $m+1 = 3t+1$ and $V = \{0, 1, \ldots, 3t\}$. Let an ELO-decomposition of $4K_m$ exist on the $3t$ points $V - \{0\}$. (Simply relabel points in the construction in the above case as $x \mapsto x+1$ and $\infty \mapsto 3t$.) Adjoin to this decomposition the graphs $(0, 3x-2, 3x-1, 3x)$ and $(0, 3x, 3x-1, 3x-2)$ for $x = 1, 2, \ldots, t$. Then, the aggregate is an ELO-decomposition of $4K_{m+1}$ on V. ▲

Let $n = 3t+1$ with $t \geq 1$. If $t = 1$ (so that $n = 4$), we use the base graph $(0, 1, 2, 3)$ developed modulo 4. If $t \geq 2$, use Lemma 3. Hence, in this case, an ELO-decomposition of $4K_n$ exists. ◆

Lastly, we address $\lambda = 6$. The values of n that correspond to this λ are $n \equiv 2, 11 \pmod{12}$.

Case: $n \equiv 2 \pmod{12}$

Let $n = 12t+2$ with $t \geq 1$. We consider the set V as $\mathbb{Z}_{12t+1} \cup \{\infty\}$. Then the differences we must achieve (modulo $12t+1$) are $1, 2, \ldots, 6t$. The number of graphs required is $\frac{6n(n-1)}{12} = \frac{(12t+2)(12t+1)}{2} = (6t+1)(12t+1)$. Thus, we need $6t+1$ base graphs (modulo $12t+1$). For the first $6t-3$ base graphs, we use $(0, x, 12t+1-x, x+1)$ for $x = 1, \ldots, 6t-3$. For the last four base graphs, we use $(0, 6t-2, 6t+3, 1)$, $(0, 6t, 6t-1, \infty)$, $(0, \infty, 6t, 6t-1)$ and $(0, 6t-1, \infty, 6t)$. Hence, in this case, an ELO-decomposition of $6K_n$ exists. ▲

Case: $n \equiv 11 \pmod{12}$

Let $n = 12t+11$ with $t \geq 0$. We consider the set V as $\mathbb{Z}_{12t+11}$. Then the differences we must achieve (modulo $12t+11$) are $1, 2, \ldots, 6t+5$. The number of graphs required is $\frac{6n(n-1)}{12} = \frac{(12t+11)(12t+10)}{2} = (6t+5)(12t+11)$. Thus, we need

$6t+5$ base graphs (modulo $12t+11$). For the first $6t+4$ base graphs, we use $(0, x, 12t+11-x, x+1)$ for $x = 1, \ldots, 6t+4$. For the last base graph, we use $(0, 6t+5, 6t+6, 1)$. Hence, in this case, an ELO–decomposition of $6K_n$ exists.

■

4. Conclusion

We have completed the problem of finding cyclic decompositions of λK_n (for the appropriate minimal λ) using the two types of the four nonisomorphic $S(4,3)$'s not considered in [4]; namely, the LEO– and ELO–type graph decompositions. This completes the result for all nonisomorphic $S(4,3)$'s.

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