Independent 2-rainbow domination in graphs

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Abstract

A 2-rainbow dominating function of a graph G is a function g that assigns to each vertex a set of colors chosen from the set $\{1,2\}$ so that for each vertex v with $g(v)=\emptyset$ we have $\cup_{u\in N(v)} g(u)=\{1,2\}$. The minimum of $g(V(G))=\sum_{v\in V(G)}|g(v)|$ over all such functions is called the 2-rainbow domination number $\gamma_{r2}(G)$. A 2-rainbow dominating function g of a graph G is independent if no two vertices assigned non empty sets are adjacent. The independent 2-rainbow domination number $i_{r2}(G)$ is the minimum weight of an independent 2-rainbow dominating function of G. In this paper, we study independent 2-rainbow domination in graphs. We present some bounds and relations with other domination parameters.

Keywords: 2-rainbow domination, independent 2-rainbow domination, Roman domination, independent Roman domination.

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1 Introduction

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *degree* of v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. The *maximum* and *minimum degrees* of a graph G are denoted by Δ and δ, respectively. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$.

A set $D \subseteq V(G)$ is a *dominating set* if every vertex of $V(G) - D$ has a neighbor in D. The *independent domination number* $i(G)$ is the minimum cardinality of a set that is both independent and dominating. The concept of domination in graphs and its many variations are now well studied in graph theory (see for example [6]). A set $S \subseteq V(G)$ is a *packing set* of G if $N[x] \cap N[y] = \emptyset$ for all pairs of distinct vertices x and y in S. The *packing number* $\rho(G)$ is the maximum cardinality of a packing set in G.

For a graph G, let $f : V(G) \to \{0,1,2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V = V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0,1,2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0,1,2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \to \{0,1,2\}$ is a *Roman dominating function* (**RDF**) on G if every vertex u of G for which $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne et al. [4]. A function $f = (V_0; V_1; V_2)$ is called an *independent Roman dominating function* (**IRDF**) on G if f is a RDF and no two vertices in $V_1 \cup V_2$ are adjacent. The *independent Roman domination number* $i_R(G)$ is the minimum weight of an independent Roman dominating function of G. Independent Roman domination was studied in [1, 5, 7].

Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1,2\}$; that is $f : V(G) \to \mathcal{P}(\{1,2\})$, where $\mathcal{P}(\{1,2\})$ is the power set of $\{1,2\}$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have $\bigcup_{u\in N(v)} f(u) = \{1,2\}$, then f is called a *2-rainbow dominating function* (2RDF) of G. The weight of a 2RDF f is defined as $w(f) = \sum_{v\in V(G)} |f(v)|$. The minimum weight of a 2-rainbow dominating function is called the *2-rainbow domination number* of G, denoted by $\gamma_{r2}(G)$. We say that a function f is a *$\gamma_{r2}(G)$-function* if it is a 2RDF and $w(f) = \gamma_{r2}(G)$. For a 2RDF f we let $V_1^f = \{v : f(v) = \{1\}\}$. V_2^f, V_{12}^f, and V_0^f are similarly defined. 2-rainbow domination was introduced by Brešar et al. [3], and further studied for example in [8, 9].

Relations between Roman domination and 2-rainbow domination were studied for example in [2, 9]. In this paper we initiate the study of 2-rainbow dominating functions f such that $V_1^f \cup V_2^f \cup V_{12}^f$ is independent. A function $f : V(G) \to \mathcal{P}(\{1,2\})$ is called an *independent 2-rainbow dominating function* (I2RDF) of G, if f is a 2RDF and no two vertices in $V(G)-V_0^f$ are adjacent. The *independent 2-rainbow domination number* $i_{r2}(G)$ is the minimum weight of an independent 2-rainbow dominating function of G. Clearly for any graph G, $i_{r2}(G) \geq \gamma_{r2}(G)$. Thus if G has a $\gamma_{r2}(G)$-function such that $V_1^f \cup V_2^f \cup V_{12}^f$ is independent, then $i_{r2}(G) = \gamma_{r2}(G)$. We investigate independent 2-rainbow domination in graphs. We present some bounds, characterizations and relations with other domination parameters

2 Bounds and extremal graphs

We begin with the following which shows that $i_{r2}(G)$ is well-defined in a graph G.

Proposition 1 *Any graph G of order n admits an I2RDF of weight at most n.*

Proof. If $\Delta(G) = 0$, then f defined on $V(G)$ by $f(x) = \{1\}$ is an I2RDF with weight n. Thus assume that $\Delta(G) > 0$. Without loss of generality assume that G is connected. Let x_1 be a vertex of maximum degree in G, and $G_1 = G - N[x_1]$. If $\Delta(G_1) = 0$, then f_1 defined on $V(G)$ by $f_1(x_1) = \{1,2\}$, $f_1(u) = \emptyset$ if $u \in N(x_1)$, and $f_1(u) = \{1\}$ if $u \notin N[x_1]$ is an I2RDF for G, and clearly $w(f_1) \leq n$. Thus assume that $\Delta(G_1) > 0$.

Let x_2 be a vertex of maximum degree in G_1, and $G_2 = G_1 - N[x_2]$. If $\Delta(G_2) = 0$, then f_2 defined on $V(G)$ by $f_2(x_1) = f_2(x_2) = \{1,2\}$, $f_2(u) = \emptyset$ if $u \in N(x_1) \cup N(x_2)$, and $f_2(u) = \{1\}$ if $u \notin N[x_1] \cup N[x_2]$ is an I2RDF for G, and clearly $w(f_2) \leq n$. By continuing this process we obtain an I2RDF of weight at most n. ■

According to Proposition 1, we have the following two observations.

Observation 2 *If G is a disconnected graph with components $G_1, G_2, ..., G_n$, then $i_{r2}(G) = i_{r2}(G_1) + i_{r2}(G_2) + ... + i_{r2}(G_n)$.*

Observation 3 *If G is a graph of order $n \geq 2$, then $2 \leq i_{r2}(G) \leq n$.*

Proposition 4 *Let $X \subseteq V(G)$ be an independent set of a graph G and let $G' = G - N[X]$. Then $i_{r2}(G) \leq 2|X| + n - |N[X]|$.*

Proof. Let us define for any i_{r2}-function f of G', a function g on $V(G)$ by $g(v) = \{1,2\}$ if $v \in X$, $g(u) = \emptyset$ if $v \in N(X)$, and $g(u) = f(u)$ if $v \notin N[X]$. By Observation 3, $i_{r2}(G') \leq n - |N[X]|$ and so $i_{r2}(G) \leq 2|X| + n - |N[X]|$. ■

According to Proposition 4, if we choose $X = \{v\}$, where v is a vertex of maximum degree, we obtain:

Corollary 5 *For any graph G, $i_{r2}(G) \leq n - \Delta + 1$.*

Moreover, since $|N[X]| \geqslant (\delta+1)\,|X|$ for every packing set X, Proposition 4 gives a relationship between the independent 2-rainbow domination number and the packing number of a graph:

Corollary 6 *For every graph G, $i_{r2}(G) \leq n - (\delta - 1)\rho(G)$.*

Next we give extremal graphs attaining each bound in Observation 3. Let $K^*_{2,m}$, where $m \geq 2$, be the graph obtained from a complete bipartite graph $K_{2,m}$ by adding edges (possibly none) between vertices belonging to partite set of size m.

Proposition 7 *For a graph G of order $n \geq 2$, $i_{r2}(G) = 2$ if and only if $G = 2K_1$, $G = K^*_{2,m}$, where $m \geq 2$, or $\Delta(G) = n - 1$.*

Proof. Let $i_{r2}(G) = 2$. If G is disconnected, then clearly $G = 2K_1$. Thus assume that G is connected. Let f be an $i_{r2}(G)$-function such that $|V^f_{12}|$ is maximum. If $|V^f_{12}| = 0$, then clearly $|V^f_1| = |V^f_2| = 1$. Since vertices of V^f_0 may be adjacent, we deduce that $G = K^*_{2,m}$ for some integer $m \geq 2$. Now assume that $|V^f_{12}| = 1$. Since V^f_{12} is a dominating set for G, we find that $\Delta(G) = n - 1$.

The converse is obvious. ■

Proposition 8 *For any graph G of order n, $i_{r2}(G) = n$ if and only if $G = mK_2 \cup lK_1$ for some integers m, l with $n = 2m + l$.*

Proof. It is obvious that if $G = mK_2 \cup lK_1$ for some integers m, l with $n = 2m + l$, then $i_{r2}(G) = n$.

Conversely, assume that G is a graph of order n with $i_{r2}(G) = n$. For $n = 1, 2$ the statement is obviously true. Thus assume that $n > 2$. By Corollary 5, G has maximum degree at most one, and so the result follows. ■

Proposition 9 *For any graph G of order n, $i_{r2}(G) = n - 1$ if and only if $G = H \cup pK_2 \cup qK_1$, where $H = P_3$, C_3 or P_4 and $p, q \geq 0$.*

Proof. It is easy to see that if $G = H \cup pK_2 \cup qK_1$, where $H = P_3$, C_3 or P_4 and $p, q \geq 0$, then $i_{r2}(G) = n - 1$.

Conversely, assume that G is a graph of order n with $i_{r2}(G) = n - 1$. Clearly since $i_{r2}(G) \geq 2$, $n \geq 3$. Also, if each component of G has order one or two, then by Proposition 8, $i_{r2}(G) = n$, a contradiction. Hence G contains at least one component with three vertices or more. Now let v be a vertex of maximum degree in G. By Corollary 5, $d_G(v) \leq 2$. So $d_G(v) = 2$ since v belongs to a component of order at least three. Therefore every component of G is either a path (possibly trivial) or a cycle. Also if G has two components each order at least three, then $i_{r2}(G) < n - 1$, a contradiction. Thus G has one component, say H, of order at least three. If $|V(H)| \geq 5$ or $H = C_4$, then it is easy to show that $i_{r2}(H) \leq |V(H)| - 2$, implying that $i_{r2}(G) \leq n-2$, a contradiction. Now we observe that $H = P_3$, C_3 or P_4. ■

We close this section with the following problem.

Problem 10 *Find a sharp bound for i_{r2} in terms of the order for trees.*

3 Relations with independent domination

Now we turn our attention to establish some results relating the independent 2-rainbow domination number to the independent domination number. The following is easily verified.

Observation 11 *For any graph G, $i(G) \leq i_{r2}(G) \leq 2i(G)$. These bounds are sharp.*

The lower bound is attained for the cycle C_4 and the upper bound is attained for the path P_3.

Proposition 12 *For a graph G, if $i_{r2}(G) = i(G)$, then for any $i_{r2}(G)$-function f, $V_{12}^f = \emptyset$.*

Proof. Let $i_{r2}(G) = i(G)$, and let f be an $i_{r2}(G)$-function. Then

$$i(G) \leq |V_1^f| + |V_2^f| + |V_{12}^f| \leq |V_1^f| + |V_2^f| + 2|V_{12}^f| = i_{r2}(G).$$

Thus $|V_{12}^f| = 0$. ■

Corollary 13 *For a graph G of order n, if $i_{r2}(G) = i(G)$ then $i(G) + \frac{\gamma_R(G)}{2} \leq n$.*

Proof. Let G be a graph of order n with $i_{r2}(G) = i(G)$. Let f be an $i_{r2}(G)$-function. By Proposition 12, $V_{12}^f = \emptyset$. Now g defined on $V(G)$ by $g(u) = 2$ if $|f(u)| = 0$, and $g(u) = 0$ if $|f(u)| = 1$, is an RDF for G, implying that $\gamma_R(G) \leq 2(n - |V_1^f| - |V_2^f|) = 2(n - i(G))$, and thus the result follows. ■

Note that the converse of Proposition 12 does not hold. To see this consider the cycle C_8. One can easily see that for any $i_{r2}(C_8)$-function f, $V_{12}^f = \emptyset$ but $i_{r2}(C_8) > i(C_8)$.

Proposition 14 *Any graph G is an induced subgraph of a graph H with $i_{r2}(H) = i(H)$.*

Proof. Let G be a graph of order n. Let H be obtained from G and n copies of $K_{2,k}$, where $k \geq 3$, by identifying a vertex of degree two of each copy of $K_{2,k}$ with a vertex of G. Then $i_{r2}(H) = i(H) = 2n$. ■

Proposition 15 *For a tree T, $i_{r2}(T) = i(T)$ if and only if $T = K_1$.*

Proof. Assume that T is a tree of order n with $i_{r2}(T) = i(T)$. Suppose that $n > 1$ and let f be an $i_{r2}(T)$-function. By Proposition 12, $V_{12}^f = \emptyset$. Since $n > 1$, there is a path xyz with $f(x) = \{1\}$, $f(y) = \emptyset$ and $f(z) = \{2\}$. Then $\{u : f(u) \neq \emptyset \text{ and } u \notin N(y)\} \cup \{y\}$ is an independent dominating set for T of cardinality less than $i(T)$, a contradiction. Hence $n = 1$. ■

Proposition 16 *For a unicyclic graph G, $i_{r2}(G) = i(G)$ if and only if $G = C_4$.*

Proof. Clearly, $i_{r2}(C_4) = 2 = i(C_4)$. Let G be a unicyclic graph with $i_{r2}(G) = i(G)$. Let f be an $i_{r2}(G)$-function. By Proposition 12, $V_{12}^f = \emptyset$. Let z be a vertex with $f(z) = \emptyset$. There are two vertices $x, y \in N(z)$ such that $f(x) = \{1\}$ and $f(y) = \{2\}$. If $N(x) \cap N(y) = \{z\}$, then $\{u : f(u) \neq \emptyset$ and $u \notin N(z)\} \cup \{z\}$ is an independent dominating set for G of cardinality less than $i(G)$, a contradiction. Thus $N(x) \cap N(y) \neq \{z\}$. Since G is unicyclic, we have $|N(x) \cap N(y) - \{z\}| = 1$. Let $N(x) \cap N(y) - \{z\} = \{w\}$. Consequently, any vertex u of degree at least two with $f(u) = \emptyset$ is contained in a 4-cycle. Thus we obtain that $d_G(x) = d_G(y) = 2$, and so G is a graph formed by a cycle C_4 with possibly leaves attached at z or w. Hence $\{z, w\}$ is an independent dominating set for G, that is $i(G) = 2$. Now if $d_G(z) \geq 3$ or $d_G(w) \geq 3$, then every leaf $u \in N(z) \cup N(w)$ has $|f(u)| = 1$, implying that $i_{r2}(G) > 2$, a contradiction. We conclude that $d_G(z) = d_G(w) = 2$, and consequently $G = C_4$. ■

If a graph G does not contain an induced subgraph that is isomorphic to some graph F, then we say that G is *F-free.*

Theorem 17 *If G is a C_4-free graph without isolated vertices, then $i_{r2}(G) \geq i(G) + 1$.*

Proof. Assume to the contrary that $i_{r2}(G) < i(G) + 1$. By Observation 11, $i_{r2}(G) = i(G)$. Let f be any $i_{r2}(G)$-function. By Proposition 12, $V_{12}^f = \emptyset$, and so every vertex of V_0^f has at least two neighbors in $V_1^f \cup V_2^f$. Recall that $V_1^f \cup V_2^f$ is an independent set. Let w be any vertex of V_0^f and let us denote by $N^*(w)$ the set of neighbors of w in $V_1^f \cup V_2^f$. Note that $|N^*(w)| \geq 2$. If every vertex u in V_0^f different from w has a neighbor in $V_1^f \cup V_2^f - N^*(w)$, then $\left(V_1^f \cup V_2^f - N^*(w)\right) \cup \{w\}$ is a maximal independent set for G, implying that $i(G) \leq \left|\left(V_1^f \cup V_2^f - N^*(w)\right) \cup \{w\}\right| \leq i_{r2}(G) - 1$, a contradiction. Hence there is at least one vertex in V_0^f different from w such that its neighborhood in $V_1^f \cup V_2^f$ is contained in $N^*(w)$. Let A be the set of all such vertices u of V_0^f such that $N^*(u) \subseteq N^*(w)$. If w is not adjacent to a vertex w_1 of A, then w, w_1 and $N^*(w)$ induce a cycle C_4, a contradiction. Hence w is adjacent to all vertices of A, in particular w dominates A. But then $\left(V_1^f \cup V_2^f - N^*(w)\right) \cup \{w\}$ is a maximal independent set for G, implying that $i(G) \leq \left|\left(V_1^f \cup V_2^f - N^*(w)\right) \cup \{w\}\right| \leq i_{r2}(G) - 1$, a contradiction too. Therefore $i_{r2}(G) \geq i(G) + 1$. ■

Recall that a graph is *chordal* if it contains no chordless cycle of length at least four as an induced subgraph. Thus we have the following corollary to Theorem 17, and we note that trees belong to the class of chordal graphs.

Corollary 18 *If G is a chordal graph without isolated vertices, then $i_{r2}(G) \geq i(G) + 1$.*

The next observation will be useful.

Observation 19 *For a graph G of order n, if $i_{r2}(G) = i(G) + 1$ then for any $i_{r2}(G)$-function f, $|V_{12}^f| \leq 1$.*

Proof. Assume that $i_{r2}(G) = i(G) + 1$. Let f be an i_{r2}-function such that $|V_{12}^f|$ is maximum. Since $V_1^f \cup V_2^f \cup V_{12}^f$ is an independent dominating set, we deduce that $|V_{12}^f| \leq 1$. ∎

In the next we give a characterization of trees T with $i_{r2}(T) = i(T) + 1$. The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. Let $\mathcal{T}$ be the class of trees T such that T is a star, a double-star, $S(S_{p,q})$, $S(K_{1,n})$ for some $n \geq 2$, or T is obtained from $S(K_{1,n})$ for some $n \geq 2$ by adding at least one leaf to the central vertex of $S(K_{1,n})$ and possibly some leaves to the support vertices of $S(K_{1,n})$ such that the number of leaves at distance two from the center of $S(K_{1,n})$ is at most the degree of the center of $S(K_{1,n})$ minus one.

Theorem 20 *For a tree T, $i_{r2}(T) = i(T) + 1$ if and only if $T \in \mathcal{T}$.*

Proof. First it is routine to check that for any tree $T \in \mathcal{T}$, $i_{r2}(T) = i(T) + 1$. Let T be a tree with $i_{r2}(T) = i(T) + 1$. Let x-x_1-x_2...x_{d-1}-y be a diametrical path of T, where $d = \text{diam}(T)$. Let f be an $i_{r2}(T)$-function such that V_{12}^f is maximum. By Observation 19, $|V_{12}^f| \leq 1$. Assume that $\text{diam}(T) \geq 5$. We shall show that $T = S(S_{p,q})$.

First, suppose that $|V_{12}^f| = 0$. Clearly $f(x_1) = f(x_{d-1}) = \emptyset$. If $d_T(x_{d-1}) \geq 3$, then

$$\sum_{v \in N(x_{d-1})} |f(v)| \geq 2.$$

Now if $|f(x_{d-2})| = 0$, then we change $f(x_{d-1})$ to $\{1,2\}$ and $f(v)$ to $\emptyset$ for each leaf neighbor of x_{d-1}, contradicting our choice of f. Thus $|f(x_{d-2})| = 1$. Then $\{u : f(u) \neq \emptyset, u \notin N(x_{d-1})\} \cup \{x_{d-1}\}$ is an independent dominating set for T of cardinality less than $i(T)$, a contradiction. Hence $d_T(x_{d-1}) = 2$ and likewise $d_T(x_1) = 2$. Thus $|f(x_2)| = |f(x_{d-2})| = 1$, and so $\text{diam}(T) \geq$

6. Now if $\text{diam}(T) \geq 7$, then $\{u : f(u) \neq \emptyset, u \neq x, x_2, x_{d-2}, y\} \cup \{x_1, x_{d-1}\}$ is an independent dominating set of cardinality less than $i(T)$, a contradiction. Thus $\text{diam}(T) = 6$, and so $f(x_3) = \emptyset$. Note that since $\text{diam}(T) = 6$, every leaf of T is at distance at most three from x_3. Now suppose that $d_T(x_3) \geq 3$. If some neighbor of x_3 different to x_2 and x_4 is assigned non-empty set, then $\{u : f(u) \neq \emptyset, u \neq x, x_2, x_4, y\} \cup \{x_1, x_5\}$ is an independent dominating set of cardinality less than $i(T)$, a contradiction. Thus we may assume that $f(t) = \emptyset$ for every $t \in N(x_3) - \{x_2, x_4\}$. Since $d_T(x_3) \geq 3$, every vertex in $N(x_3) - \{x_2, x_4\}$ is a support vertex adjacent to at least two leaves, and having x_3 as the unique non-leaf neighbor. Let z be any vertex of $N(x_3) - \{x_2, x_4\}$, and let L_z be the set of leaves adjacent to z. In this case $\{u : f(u) \neq \emptyset, u \neq x, x_2, x_4, y \text{ and } u \notin L_z\} \cup \{x_1, x_5, z\}$ is an independent dominating set of cardinality less than $i(T)$, a contradiction. Therefore $d_T(x_3) = 2$. Now if $d_T(x_2) \geq 3$, then every neighbor of x_2 different from x_3 is a support vertex playing the same role as x_1, that its degree is also two. Likewise, if $d_T(x_4) \geq 3$, then every neighbor of x_4 different from x_3 is a support vertex of degree two. Therefore T is the subdivision graph of a double star that belongs to $\mathcal{T}$.

Next assume that $|V_{12}^f| = 1$. Clearly $0 \in \{|f(x_1)|, |f(x_{d-1})|\}$. Without loss of generality, assume that $|f(x_{d-1})| = 0$. Assume that $f(x_{d-2}) \neq \{1,2\}$. If $d_T(x_{d-1}) \geq 3$, then $\sum_{v \in N(x_{d-1})} |f(v)| \geq 2$. If $|f(x_{d-2})| = 0$, then we change $f(x_{d-1})$ to $\{1,2\}$ and $f(v)$ to $\emptyset$ for each leaf neighbor of x_{d-1}, contradicting our choice of f. Hence $|f(x_{d-2})| = 1$. Then $\{u : f(u) \neq \emptyset, u \notin N(x_{d-1})\} \cup \{x_{d-1}\}$ is an independent dominating set for T of cardinality less than $i(T)$, a contradiction. Thus $d_T(x_{d-1}) = 2$, and so $|f(x_{d-2})| = 1$. Now $\{u : f(u) \neq \emptyset, u \neq x_{d-2}, y\} \cup \{x_{d-1}\}$ is an independent dominating set of cardinality less than $i(T)$, a contradiction. Thus we can assume that $f(x_{d-2}) = \{1,2\}$. If $d = 5$, then $f(x_2) = \emptyset$, and thus we may have that $f(x_1) = \{1,2\}$, a contradiction. Thus $d \geq 6$. Now similar to the above discussion we may assume that $f(x_1) = \emptyset$, $d_T(x_1) = 2$, and $|f(x)| = |f(x_2)| = 1$. Now $\{u : f(u) \neq \emptyset, u \neq x, x_2\} \cup \{x_1\}$ is an independent dominating set of cardinality less than $i(T)$, a contradiction.

From now on we assume that T has diameter at most 4. We first assume that $\text{diam}(T) = 4$. If $|f(x_3)| = 2$, then $f(x_2) = \emptyset$, and we can find an $i_{r2}(T)$-function g with $|V_{12}^g| > 2$, contradicting Observation 19. Thus $|f(x_3)| \neq 2$, and likewise $|f(x_1)| \neq 2$. This implies that $f(x_1) = f(x_3) = \emptyset$. Thus $f(u) = \emptyset$, for any vertex $u \in N(x_2)$ with $d_T(u) > 1$.

Now let us suppose that $|V_{12}^f| = 0$. If $|f(x_2)| = 0$, then $d_T(x_1) \geq 3$, $d_T(x_3) \geq 3$ and so $\{u : f(u) \neq \emptyset, u \notin N(x_3) \cup N(x_1)\} \cup \{x_1, x_3\}$ is an independent dominating set of cardinality less than $i(T)$, a contradiction. Thus $|f(x_2)| = 1$, implying that x_2 is not a support vertex. If $d_T(x_3) > 2$ or $d_T(x_1) > 2$, then $\{u : f(u) \neq \emptyset, u \notin N(x_3) \cup N(x_1)\} \cup \{x_1, x_3\}$ is an independent dominating set of cardinality less than $i(T)$, a contradiction. Thus $d_T(x_1) = d_T(x_3) = 2$, and so any vertex in $N(x_2)$ is of degree two. Consequently, $T = S(K_{1,d_T(x_2)}) \in \mathcal{T}$.

Assume now that $|V_{12}^f| = 1$. If $f(x_2) = \emptyset$, then $d_T(x_3) > 2$ and we can change $f(x_3)$ to $\{1,2\}$ and $f(u)$ to $\emptyset$ for any $u \in N(x_3) - \{x_2\}$, contradicting our choice of f. Thus $f(x_2) \neq \emptyset$. Suppose that $|f(x_2)| = 1$. Then the V_{12}^f contains a vertex at distance two from x_2. Without loss of generality, assume that $V_{12}^f = \{x\}$. Then $\{u : f(u) \neq \emptyset, u \notin N[x_1]\} \cup \{x_1\}$ is an independent dominating set for T, a contradiction. Thus $f(x_2) = \{1,2\}$. This implies that x_2 is a support vertex, for otherwise we can decrease $w(f)$ by changing $f(x_2)$ to $\{1\}$ and every leaf v at distance two from x_2 with $f(v) = \{1\}$ to $f(v) = \{2\}$. Now let l be the number of leaves at distance two from x_2. Then $w(f) = i_{r2}(T) = 2 + l$. If $l \geq d_T(x_2)$, then $N(x_2)$ is an independent dominating set for T of cardinality less than $i(T)$, a contradiction. Thus $d_T(x_2) \geq l + 1$, and therefore $T \in \mathcal{T}$.

Finally, if $\mathrm{diam}(T) = 3$, then T is a double-star and if $\mathrm{diam}(T) \leq 2$, then T is a star. Hence $T \in \mathcal{T}$. ■

Proposition 21 *For a graph G, $i_{r2}(G) = 2i(G)$ if and only if there is an $i_{r2}(G)$-function f such that $|V_1^f| = |V_2^f| = 0$.*

Proof. Assume that $i_{r2}(G) = 2i(G)$ and let S be an $i(G)$-set. Then f defined on $V(G)$ by $f(u) = \{1,2\}$ if $u \in S$, and $f(u) = \emptyset$ if $u \notin S$, is an I2RDF for G. Since $i_{r2}(G) = 2i(G)$, f is the desired function.

Conversely, assume that there is an $i_{r2}(G)$-function f such that $|V_1^f| = |V_2^f| = 0$. Then V_{12}^f is an independent dominating set for G, implying that $i(G) \leq \frac{i_{r2}(G)}{2}$, and thus the result follows. ■

4 Relations with independent Roman domination

In this section we present some relations between the independent 2-rainbow domination and independent Roman domination numbers of a graph G.

Theorem 22 *For any graph G, $i_{r2}(G) \leq i_R(G) \leq \frac{3}{2} i_{r2}(G)$.*

Proof. Let $f = (V_0; V_1; V_2)$ be an IRDF on G. Clearly every vertex of V_0 has a neighbor in V_2. We define a function g on G by $g(x) = \emptyset$ if $x \in V_0$, $g(x) = \{1,2\}$ if $x \in V_2$ and $g(x) = \{1\}$ or $\{2\}$ if $x \in V_1$. Clearly g is an I2RDF on G and so $i_{r2}(G) \leq |V_1| + 2|V_2| = i_R(G)$.

Now to show the upper bound, let f be an $i_{r2}(G)$-function. Let k_i be the number of vertices u for which $i \in f(u)$, for $i = 1, 2$. Then $i_{r2}(G) = k_1 + k_2$. Without loss of generality, suppose that $k_1 \leq k_2$. Hence $k_1 \leq (k_1 + k_2)/2 = i_{r2}(G)/2$. Now we define $g : V(G) \longrightarrow \{0,1,2\}$ such that $g(x) = 0$ if $f(x) = \emptyset$, $g(x) = 1$ if $f(x) = \{2\}$, and $g(x) = 2$ if $1 \in f(x)$. Since f is a I2RDF for G, we obtain that g is an IRDF for G, implying that $i_R(G) \leq w(g) = 2k_1 + k_2$. Therefore,

$$i_R(G) \leq 2k_1 + k_2 = k_1 + k_1 + k_2 \leq i_{r2}(G)/2 + i_{r2}(G) = \frac{3}{2} i_{r2}(G).$$

Note that the other case if $k_2 \leq k_1$ provides the same result. ■

The upper bound is attained for the cycle C_4, where $i_R(C_4) = 3$ and $i_{r2}(C_4) = 2$. Also the lower bound is attained for the path P_4.

Next we give a necessary and sufficient condition for graphs G with $i_{r2}(G) = i_R(G)$.

Proposition 23 *For a graph G, $i_{r2}(G) = i_R(G)$ if and only if there is a $i_{r2}(G)$-function f such that any vertex x with $|f(x)| = 0$ is adjacent to a vertex y with $|f(y)| = 2$.*

Proof. Assume that there is a $i_{r2}(G)$-function f such that any vertex x with $|f(x)| = 0$ is adjacent to a vertex y with $|f(y)| = 2$. Let g be defined on $V(G)$ by $g(u) = |f(u)|$. Then g is an IRDF for G implying that $i_R(G) \le i_{r2}(G)$, and thus $i_{r2}(G) = i_R(G)$.

Conversely, assume that $i_{r2}(G) = i_R(G)$ and let f be an $i_R(G)$-function. Then g defined by $g(u) = \emptyset$ if $f(u) = 0$, $g(u) = \{1\}$ if $f(u) = 1$, and $g(u) = \{1, 2\}$ if $f(u) = 2$, is an $i_{r2}(G)$-function such that any vertex x with $|g(x)| = 0$ is adjacent to a vertex y with $|g(y)| = 2$. ■

Proposition 24 *If $i_R(G) = \frac{3}{2}i_{r2}(G)$, then for any $i_{r2}(G)$-function f, $|V_{12}^f| = 0$, and $|V_1^f| = |V_2^f|$.*

Proof. Let G be a graph with $i_R(G) = \frac{3}{2}i_{r2}(G)$, and let f be a $i_{r2}(G)$-function. Assume that $|V_{12}^f| > 0$. Without loss of generality, assume that $|V_1^f| \le |V_2^f|$. We define $g : V(G) \longrightarrow \{0, 1, 2\}$ such that $g(x) = 0$ if $f(x) = \emptyset$, $g(x) = 1$ if $f(x) = \{2\}$, and $g(x) = 2$ if $1 \in f(x)$. Since f is an I2RDF for G, we obtain that g is an IRDF for G, implying that $i_R(G) \le w(g) = 2|V_{12}^f| + 2|V_1^f| + |V_2^f| = 2|V_{12}^f| + |V_1^f| + |V_2^f| + |V_1^f| < i_{r2}(G) + i_{r2}(G)/2 = 3i_{r2}(G)/2$, a contradiction. Thus $|V_{12}^f| = 0$. Similarly, $|V_1^f| = |V_2^f|$. ■

Corollary 25 *If $i_R(G) = \frac{3}{2}i_{r2}(G)$, then $\delta(G) \ge 2$.*

Proof. Let G be a graph with $i_R(G) = \frac{3}{2}i_{r2}(G)$. Assume that G has a vertex x of degree one. Let y be the neighbor of x, and let f be a $i_{r2}(G)$-function. By Proposition 24, $|V_{12}^f| = 0$, and $|V_1^f| = |V_2^f|$, say k. Thus $i_{r2}(G) = 2k$. Clearly, $f(y) = \emptyset$. Now, without loss of generality, assume that $f(x) = \{1\}$. Note that at least some vertex $z \in N(y) - \{x\}$ has $f(z) = \{2\}$. We define $g : V(G) \longrightarrow \{0, 1, 2\}$ such that $g(y) = 2$, $g(u) = 0$ if $u \in N(y)$, and for $u \notin N[y]$, $g(u) = 0$ if $f(u) = \emptyset$, $g(u) = 1$ if $f(u) = \{2\}$, and $g(u) = 2$ if $f(u) = \{1\}$. Since f is an I2RDF for G, we obtain that g is an IRDF for G, implying that $i_R(G) \le w(g) \le 2(|V_1^f| - 1) + (|V_2^f| - 1) + 2 = 3i_{r2}(G)/2 - 1$, a contradiction. ■

Thus according to Corollary 25, for every tree T, $i_R(T) < \frac{3}{2}i_{r2}(T)$. Also, using Proposition 24 and Corollary 25, we obtain the following result. We omit the proof.

Proposition 26 *For a unicyclic graph G, $i_R(G) = \frac{3}{2}i_{r2}(G)$ if and only if $G = C_4$.*

Proposition 27 *For every even integer k there is a graph G with $i_{r2}(G) = k$ and $i_R(G) = \frac{3}{2}i_{r2}(G)$.*

Proof. If $k = 2$, then C_4 is the desired graph. For $k \geq 4$, let $V(P_{2k-1}) = \{v_1, v_2, ..., v_{2k-1}\}$, where v_i is adjacent to v_{i+1} for $i = 1, 2, ..., 2k-2$. For any odd integer $i = 1, 3, ..., 2k-3$, we add two ears $v_i w_{i1} v_{i+2}$ and $v_i w_{i2} v_{i+2}$ to obtain a graph G. It is straightforward to see that $i_{r2}(G) = k$ and $i_R(G) = \frac{3}{2}i_{r2}(G)$. ■

5 Nordhaus-Gaddum-type results

The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$ and with exactly the edges that do not belong to G. We now study some Nordhaus-Gaddum-type results. One can easily see for any graph G that $i_{r2}(\overline{G}) \leqslant \delta + 2$.

Proposition 28 *For any graph G of order $n \geq 3$, $5 \leq i_{r2}(G) + i_{r2}(\overline{G}) \leq n - \Delta + \delta + 3$.*
Equality holds in the lower bound if and only if one of the following holds:
(1) G or $\overline{G}$ is K_3,
(2) G is obtained from the complement graph of $K^*_{2,m}$, for some $m \geq 2$, by adding a new vertex and joining it to every vertex of $\overline{K^*_{2,m}}$.
(3) $\overline{G}$ is obtained from $K^*_{2,m}$, for some $m \geq 2$, by adding a new vertex and joining it to every vertex of $K^*_{2,m}$.
(4) $\delta(G) = 1$ and $\Delta(G) = n - 1$.
(5) $\delta(\overline{G}) = 1$ and $\Delta(\overline{G}) = n - 1$.

Proof. The upper bound follows from Corollary 5 and the fact that $i_{r2}(\overline{G}) \leqslant \delta + 2$.

We next prove the lower bound. Since $n \geq 3$, we have $i_{r2}(G) \geq 2$ and $i_{r2}(\overline{G}) \geq 2$. If $i_{r2}(G) = 2$, then by Proposition 7, $G = K^*_{2,m}$, where $m \geq 2$, or $\Delta(G) = n - 1$. If $G = K^*_{2,m}$ for some $m \geq 2$, then $\overline{G}$ is not connected and so $i_{r2}(\overline{G}) \geq 4$, implying that $i_{r2}(G) + i_{r2}(\overline{G}) \geq 6$. Thus assume that $\Delta(G) = n - 1$. Let x be a vertex of maximum degree in G. Then x is an isolated vertex in $\overline{G}$, and thus $i_{r2}(\overline{G}) \geq 3$, since $n \geq 3$. We deduce that $i_{r2}(G) + i_{r2}(\overline{G}) \geq 5$.

Assume now that $i_{r2}(G) + i_{r2}(\overline{G}) = 5$. Without loss of generality assume that $i_{r2}(G) = 2$ and $i_{r2}(\overline{G}) = 3$. By Proposition 7, $G = K^*_{2,m}$, where $m \geq 2$, or $\Delta(G) = n - 1$. If $G = K^*_{2,m}$ for some $m \geq 2$, then $i_{r2}(\overline{G}) \geq 4$, a contradiction. Thus assume that $\Delta(G) = n - 1$. Let x be a vertex of G of maximum degree. Then x is an isolated vertex in $\overline{G}$. Since $i_{r2}(\overline{G}) = 3$, we have $i_{r2}(\overline{G} - x) = 2$. By Proposition 7, $\overline{G} - x = 2K_1$, $\overline{G} - x = K^*_{2,m}$, where $m \geq 2$, or $\Delta(\overline{G} - x) = n - 2$. If $\overline{G} - x = 2K_1$, then $G = K_3$. If $\overline{G} - x = K^*_{2,m}$, where $m \geq 2$, then G is a graph as described in (2). We next assume that $\Delta(\overline{G} - x) = n - 2$. Let y be a vertex of maximum degree in $\overline{G}$. Then $d_G(y) = 1$, and thus $\delta(G) = 1$. The converse is easily verified. ■

We turn our attention to the product of $i_{r2}(G)$ and $i_{r2}(\overline{G})$.

Theorem 29 *For any graph G with $\rho(G) = \rho$,*

$$i_{r2}(G)i_{r2}(\overline{G}) \leqslant \frac{(n+3\rho)(n+2\rho+1)}{4\rho}$$

Proof. Since $i_{r2}(\overline{G}) \leqslant \delta + 2$, and according to Corollary 6, we conclude that

$$\begin{aligned} i_{r2}(G)i_{r2}(\overline{G}) &\leqslant (n - (\rho - 1)\delta)(\delta + 2) \\ &\leq \max_{\delta}(n - (\rho - 1)\delta)(\delta + 2) = \frac{(n+3\rho)(n+2\rho+1)}{4\rho}. \end{aligned}$$

■

We finish by mentioning some open questions.

1. Characterize graphs achieving equality in bounds of Theorem 22.
2. Is there a polynomial algorithm for computing i_{r2} for trees?

3. Determine $i_{r2}(G)$ for every grid graph $G = P_m \Box P_n$?

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