A simple bijection between 312-avoiding permutations and triangulations

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Abstract

We present a simple bijection between the set of triangulations of a convex polygon and the set of 312-avoiding permutations.

Keywords: pattern-avoidance, bijection.

1 Introduction

The sequence $(C_n)_{n\geq 0}$ of Catalan numbers defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \prod_{i=2}^{n}\frac{n+i}{i}, \quad \text{for} \quad n \geq 0$$

is ubiquitous in enumerative combinatorics. We find a current list of more than 200 objects that are enumerated by Catalan numbers on the homepage of R.P. Stanley [9]. In this work we consider triangulations of convex polygons and 312–avoiding permutations. There are many bijections between Catalan families. (See for instance [3], [5], [10]). However we can not find any direct description of a bijection between triangulations and 312–avoiding permutations in the literature. Though our bijection can be constructed as a composition of known ones, we think that it is worthwhile to formulate this direct bijection. On the one hand it has a nice, simple description: we label the vertices of the underlying $(n+2)$–gon and project this labelling to the triangles. We define a special code word (permutation) for the labelled triangles. The way triangles are attached to each other (without any intersection in their interior) corresponds to the fact that the code permutation can not contain the pattern 312. On the other hand our bijection emphasizes the role of the inversion table of a 312–avoiding permutation, which we think is a new observation in this area. See Observations 1 and 4.

Another feature of our coding is that it can be applied to k–triangulations ([4], [8], [12]). In [8] a geometric view of k-triangulations is presented in which $(2k+1)$–stars are substituted for the triangles. The projection of the vertex labelling of the underlying polygon to a star labelling is natural. Also there are several natural methods of coding the labelled stars into words. To exploit this observation remains an open problem.

2 Previous results

The literature about Catalan families is very rich. We don't even attempt to give a complete picture of the previous work on the topic, but in this section we collect some related results.

Possibly the best known bijection involving triangulations is given in [9, Prop. 6.2.1 and Cor. 6.2.3.]. This translation of a triangulation to a binary tree is very natural and elegant.

In [5] one finds nice correspondences between several Catalan families, such as binary trees, nested parentheses, Dyck–paths, and certain integer sequences. Also included in Knuth's correspondences is a certain class of permutations which is characterized by conditions on its inversion table. This turns out to be the class of 312–avoiding permutations, though Knuth makes no note of that.

The bijection we present in our work is a composition of one given by Knuth and one given by Stanley. When we modify (by reflection) the binary tree that represents the colex forest in Knuth's work we obtain a binary tree that can be translated to a triangulation by the standard bijection given by Stanley. The bijection of our Theorem 1 is precisely the composition of these two bijections.

It is well known that the set of triangulations ordered by diagonal flips defines the Tamari lattice. In [2, Section 9] Björner and Wachs studied this lattice. They discuss – as they remark – a surprisingly close connection that exists between Tamari lattices and weak order on the symmetric group. By defining a map from permutations to binary trees they showed that the sublattice of the weak order consisting of 312–avoiding permutations (and thus the Tamari lattice) is a quotient of the weak order in the order theoretic sense.

We remark here that in view of our bijection this fact is a natural observation. It is interesting that though in [2] the Tamari lattice is actually defined on the set of inversion tables of 312–avoiding permutations, this simple fact is not mentioned there.

Reading [7] continued to study the Tamari lattice. He defined a direct map from permutations to triangulations that is identical to the one given in [2] up to the standard bijection from triangulations to binary trees. A generalization of this concept led to the introduction of Cambrian lattices.

3 312-avoiding permutations, inversion tables and triangulations

We denote by $\mathcal{T}_n$ the set of triangulations of a convex $(n+2)$-gon into n triangles by diagonals that do not intersect in their interior. The number of such triangulations is equal to the n-th Catalan number.

We denote by S_n the symmetric group of all permutations of the set $[n] := \{1, 2, \ldots, n\}$. We write a permutation $\pi \in S_n$ in one-line notation as a word $\pi = a_1 a_2 \cdots a_n$ of length n where $a_i = \pi(i)$. A *subword* of π is a subsequence $a_{i_1} a_{i_2} \cdots a_{i_k}$ of π with $i_1 < i_2 < \cdots < i_k$. Let $\pi \in S_n$ and $\tau \in S_k$; then π is called *τ-avoiding* if π does not contain a subword of length k having the same relative order as τ. In particular π is called *312-avoiding* if π does not contain a subword $a_i a_j a_k$ with $a_j < a_k < a_i$. We denote the set of 312-avoiding permutations by $S_n(312)$.

It is well known (see for example [1], [9]) that for any $\tau \in S_3$ the number of τ-avoiding permutations is equal to the n-th Catalan number.

A pair (a_i, a_j) is called an *inversion* of the permutation $\pi = a_1 a_2 \cdots a_n$ if $i < j$ and $a_i > a_j$. The *inversion table* of the permutation π is an n-tuple of integers $\underline{s} = (s_1, s_2, \ldots, s_n)$ where s_k is the number of elements that are greater than k and are to the left of it.

$$s_k = \left|\{a_i | a_i > k = a_j \text{ and } i < j\}\right|.$$

Clearly it is true that $0 \leq s_k \leq n - k$ for $1 \leq k \leq n$. The inversion table determines the permutation uniquely.

Observation 1 *The inversion table of a 312-avoiding permutation $\pi = a_1 a_2 \cdots a_n$ satisfies the following condition:*

$$s_{k+i} \leq s_k - i \quad \textit{for} \quad 1 \leq k \leq n-2 \quad \textit{and} \quad 1 \leq i \leq s_k.$$

Furthermore for an inversion table with this additional property the corresponding permutation is a 312-avoiding permutation.

Namely, when $1 \leq i \leq s_k$, then $k+i$ is to the left of k, but also the elements $k+1, \ldots, k+i, \ldots, (k+i) + s_{k+i}$ are to the left of k. We mention that this observation is very crucial for us. It is implicit in several of our references but we couldn't find it stated explicitly.

In this section we give a simple bijection between the sets $\mathcal{T}_n$ and $S_n(312)$. We label the vertices of the $(n+2)$-gon with the numbers $\{0, 1, \ldots, n, n+1\}$ in clockwise order. We mark the vertices of the triangle Q by A_Q, B_Q, C_Q so that $l(A_Q) < l(B_Q) < l(C_Q)$ where $l(P)$ denotes the label of the vertex P in the $(n+2)$-gon. We refer to these as the *first* (A_Q), *middle* (B_Q) and *last* (C_Q) vertices of the triangle.

Lemma 2 *In each triangulation, for every $i \in \{1, 2, \ldots, n\}$ there is exactly one triangle Q where the middle vertex is i ($l(B_Q) = i$).*

Proof. Assume that there exist two triangles P and Q with $l(B_P) = l(B_Q)$. Then without loss of generality

1. $l(A_P) = l(A_Q) < l(B_P) = l(B_Q) < l(C_P) < l(C_Q)$ or
2. $l(A_P) < l(A_Q) < l(B_P) = l(B_Q) < l(C_P), l(C_Q)$.

In the first case the sides $[A_P, C_P]$ and $[B_Q, C_Q]$ cross each other. In the second case the sides $[A_P, B_P]$ and $[A_Q, C_Q]$ cross each other. This is a contradiction to the fact that P and Q are triangles in a triangulation.

With the help of this observation we can define a map w from the set of triangulations to the set of 312-avoiding permutations.

Coding algorithm

Input: a triangulation T

Output: a permutation of $\{1, 2, \ldots, n\}$

1. Label the triangles according to their middle vertex
2. For $i = 2, \ldots, n + 1$ do the following:

   Consider the labels of the triangles such that the last vertex has label i. List these labels in decreasing order (this is the same as the counter clockwise order of the triangles meeting at vertex i).

The length of the listing increases as we process the vertices. The output ($w(T)$) is the list after examining the last vertex ($n + 1$). Then $w(T)$ contains the labels of all the triangles in some order. An example is given in Figure 1.

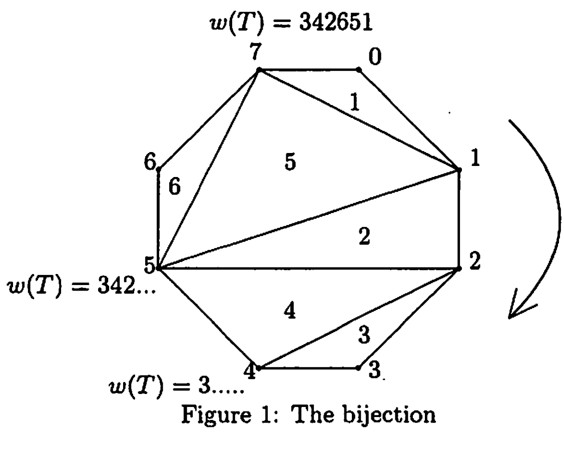

Figure 1: The bijection

Theorem 1 (Main theorem) *The map w is a bijection between the set of triangulations of a convex $(n+2)$-gon and the set of 312-avoiding permutations of $[n]$.*

4 Proofs and remarks

Lemma 3 *The word $w(T)$ is a 312-avoiding permutation of $[n]$ (we consider permutations as words).*

Proof. Note that for two triangles P and Q with $l(B_P) < l(B_Q)$ and $l(C_P) = l(C_Q)$ the algorithm records the label of Q before the label of P.
Assume that $w(T)$ is not 312-avoiding. Then there are three triangles P, Q, R with

$$l(B_P) < l(B_Q) < l(B_R) \text{ and } l(C_R) \leq l(C_P) < l(C_Q).$$

But then the sides $[B_P, C_P]$ and $[B_Q, C_Q]$ cross each other, a contradicton.

Observation 4 *Let T be a triangulation. Take the triangle labelled by i. Its $[B_i, C_i]$ side determines the i–th condition of the inversion table of the permutation $w(T)$:*

$$s_i = l(C_i) - l(B_i) - 1.$$

Proof of the main theorem. We prove the theorem by defining the inverse map of w. We use Observation 4 in constructing the decoding algorithm.

Decoding algorithm:

Input: a 312-avoiding permutation π

Output: a triangulation T

0. Compute the inversion table $(s_1, s_2, \ldots, s_n)$ of π.
1. Let the triangle with label 1 be: $l(A_1) = 0$, $l(B_1) = 1$, $l(C_1) = s_1 + 2$

i. $(i = 2, \ldots, n)$ Let the triangle with label i be $l(B_i) = i$, $l(C_i) = s_i + (i+1)$, and $l(A_i)$ minimal so that the $A_iB_iC_i$ can be a triangle of a triangulation ($[A_i, B_i]$ and $[A_i, C_i]$ do not create any new crossings).

From the properties of the inversion table of a 312-avoiding permutation it follows that the sides of the triangles do not cross each other and so the algorithm determines a unique triangulation.

Remark 1. Recall that for a permutation $\pi = a_1a_2\ldots a_n$ the *reverse* of π is defined as $\pi^r = a_na_{n-1}\ldots a_1$ and the *complement* of π as the permutation whose i-th entry is $n + 1 - a_i$.

Using these two operations we can modify the bijection w. We generate the reverse permutation when we read off the word moving around the polygon counter-clockwise, and we generate the complement permutation when we modify the labelling of the triangles so that the triangle Q gets the label $n + 1 - k$ when its middle vertex is k ($l(B_Q) = k$).

With these modifications we generate bijections between the set of triangulations and $S_n(213)$, $S_n(132)$, $S_n(231)$, since if a permutation avoids the pattern

312, then its reverse avoids the pattern 213, its complement avoids the pattern 132 and the reverse of its complement avoids the pattern 231 (see [1]).

With a further modification of the algorithm we can generate the inversion of a permutation. To this end we have to label the triangles according their last vertex and define the code word according the last vertices moving around the polygon clockwise.

Remark 2. As we already noted our coding algorithm can be used in several different ways to map k–triangulations to permutations. The most natural way is to label $(2k+1)$–stars by their middle vertex. The code words can be based on the $(k+2)$nd, $(k+3)$rd, ..., last vertices. This suggests two open problems. Is it true that the above defined map of k–triangulations to k–tuples of permutations is an injection? Can one characterize the image of this map?

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References

[1] M. Bóna. *Combinatorics of Permutations*, CRC, Boca Raton, 2004.

[2] A. Björner, M.Wachs. Shellable nonpure complexes and posets. II, *Transactions of the American Mathematical Society*, Vol. 349 (1997), 3945–3975.

[3] A. Claesson, S.Kitaev. Classification of bijections between 321- and 132–avoiding permutations, *Seminaire Lotharingien de Combinatoire*, Vol. B60d (2008), 30 pp.

[4] J. Jonsson. Generalized triangulations and diagonal–free subsets of stack polynominos, *Journal of Combinatorial Theory Series A*, Vol. 112 (2005), 117–142.

[5] D. Knuth. *The Art of Computer Programming, Vol. IV.*, Addison-Wesley, Reading, Massachusetts, 2004.

[6] C. Krattenthaler. Permutations with restricted patterns and Dyck paths, *Advances in Applied Mathematics*, Vol. 27 (2001), 510–530.

[7] N. Reading. From the Tamari lattice to Cambrian lattices and beyond, Associahedra, in *Tamari Lattices and Related Structures*, Tamari Memorial Festschrift, (Progress in Mathematics, vol. 299) Springer, Basel, 2012, 293-322.

[8] V. Pilaud, F. Santos. Multitriangulations as complexes of star polygons, *Discrete and Computational Geometry*, Vol. 41 (2009), 284–317.

[9] R. P. Stanley. http://www–math.mit.edu/∼ rstan/ec/catadd.pdf

[10] R. P. Stanley. *Enumerative Combinatorics, Vol. 1*, Cambridge University Press, New York/Cambridge, 1997.

[11] R. P. Stanley. *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, New York/Cambridge, 1999.

[12] C. Stump. A new perspective on k–triangulations, *Journal of Combinatorial Theory Series A*, Vol 118 (2011), 1794–1800.